Fast Magnetic Reconnection in 3-D

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Fast magnetic reconnection implies magnetic field lines are broken at a rate determined by the Alfvén speed not by the resistivity η.

Fast magnetic reconnection is sufficiently common that it must arise naturally in evolving magnetic fields no matter how simple the initial state of the field.

Will show why this is essentially universally true when the evolution involves all three spatial dimensions and not just two.

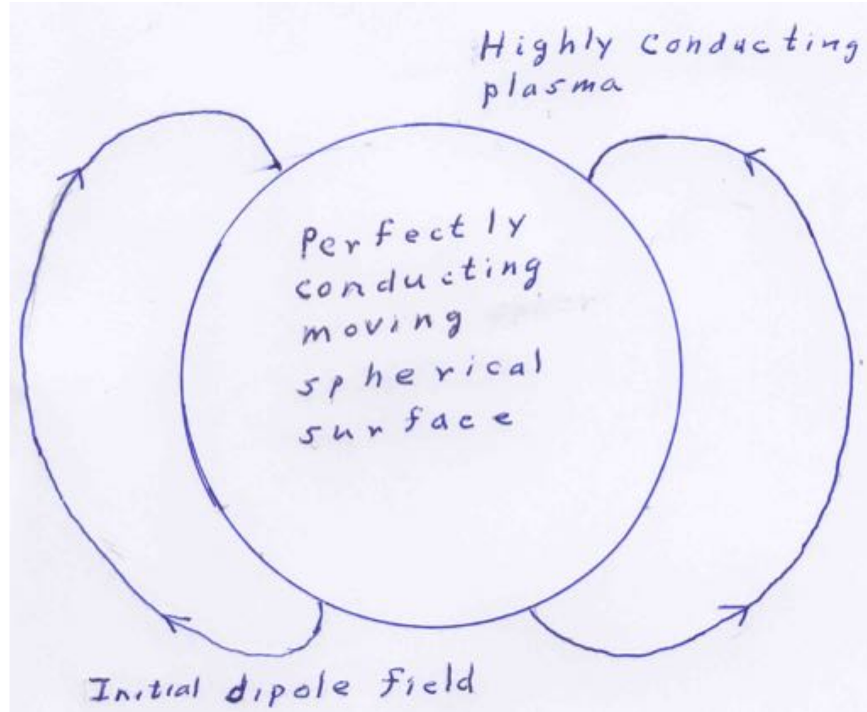
Solar-Relevant Example

Spherical surface at *r=a* moves with a divergence-free velocity

$$\vec{v}_s = \vec{\nabla}\times\left(\phi\vec{\nabla}r\right).$$

Stream function (in spherical harmonics)

$$\phi = \sum \phi_{\ell n}(t) Y_\ell^n(\theta,\varphi).$$



An initial dipole field develops structures on a scale that becomes exponentially small, $e^{-\sigma}$ with $\sigma \propto v_s t/a$, even when $\phi_{\ell n}$ is time independent and only a few low order terms are nonzero.

Fast reconnection occurs when small spatial scales reach c/ω_{pe}.

On sun, electric field will exceed Dreicer field giving a corona.

Reason for Alfvénic reconnection in the solar problem is obvious.

Requires only that a reduced MHD model in Cartesian coordinates be understood (my Spineto contribution).

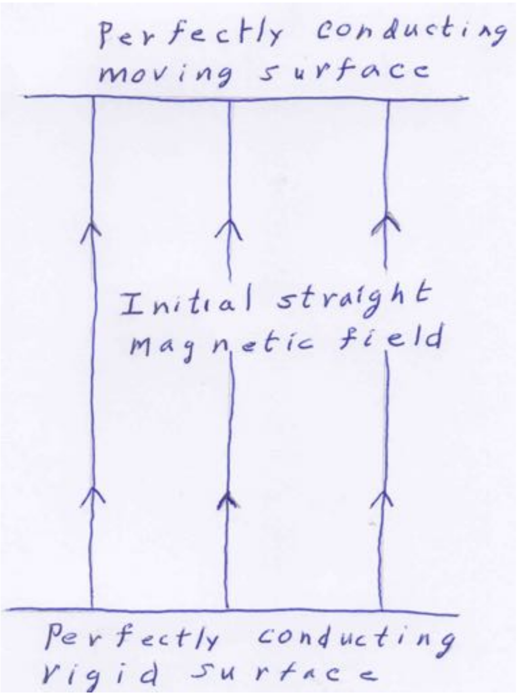
Need for a solar corona is even more obvious.

Although the parallel electric field need not be large in 3-D reconnection, the scale height of the density in the photosphere is less than a 100 km, so $n(r) = n(a)e^{-(r-a)/h}$ with $h \lesssim 100$ km.

The loops coming out of the sun have a scale of ~10^4 km, so there are far too few electrons to carry the current through the height of the loop without electron runaway, RMP **76**, 1071 (2004) see page 1092.

Reduced MHD Model

$$\vec{B} = B_g\left(\hat{z} + \vec{\nabla}\times(H\hat{z})\right)$$



Perfectly conducting walls at *z*=0,*L*; periodic in *x* and *y* with periods 2*πa*. As $L/a \to \infty$, both B_g/L and $\left|L\vec{\nabla}_\perp H\right|$ remain finite.

Velocity of top wall must be divergence free $\vec{v}_w = \vec{\nabla}\phi_w(x,y,z,t)\times\hat{z}$ as must the plama flow, $\vec{v} = \vec{\nabla}\phi(x,y,z,t)\times\hat{z}$. The bottom wall is rigid and stationary.

$H(x,y,z,t)$ is the magnetic field line Hamiltonian with *t* a paramter:

$$\frac{dx}{dz} = -\frac{\partial H}{\partial y} \text{ and } \frac{dy}{dz} = \frac{\partial H}{\partial x}, \text{ and } \vec{E} = -B_g\frac{\partial H}{\partial t}\hat{z} - \vec{\nabla}\Phi.$$

Trajectory Differences between *H*(*x,y*) and *H*(*x,y,z*)

The fundamental differences between the trajectories of $H(x, y)$ *and* $H(x, y, z)$ *has only been widely appreciated for about 50 years.*

$\vec{B} \cdot \vec{\nabla} H = B_g \frac{\partial H}{\partial z}$. When $H(x, y)$, the field lines must follow constant-*H* contours. Neighboring lines can exponentiate apart $\sigma(z) >> 1$ times only within a distance $ae^{-\sigma}$ of saddle points of $H(x, y)$.

Field lines are $\vec{x}(x_0, y_0, z) = x(x_0, y_0, z)\hat{x} + y(x_0, y_0, z)\hat{y} + z\hat{z}$, where (x_0, y_0) is a starting point on the rigid wall at *z*=0.

Separation of neighboring trajectories is $\vec{\delta}_\perp \equiv \frac{\partial \vec{x}}{\partial x_0}\delta x_0 + \frac{\partial \vec{x}}{\partial y_0}\delta y_0$.

Exponential Property of the Separation $\vec{\boldsymbol{\delta}}_{\perp} = \boldsymbol{\delta}_x \hat{\boldsymbol{x}} + \boldsymbol{\delta}_y \hat{\boldsymbol{y}}$

$\begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \overleftrightarrow{\mathfrak{J}} \cdot \begin{pmatrix} \delta x_0 \\ \delta y_0 \end{pmatrix}$. The Jacobian matrix $\overleftrightarrow{\mathfrak{J}} \equiv \begin{pmatrix} \frac{\partial x}{\partial x_0} & \frac{\partial x}{\partial y_0} \\ \frac{\partial y}{\partial x_0} & \frac{\partial y}{\partial y_0} \end{pmatrix}$.

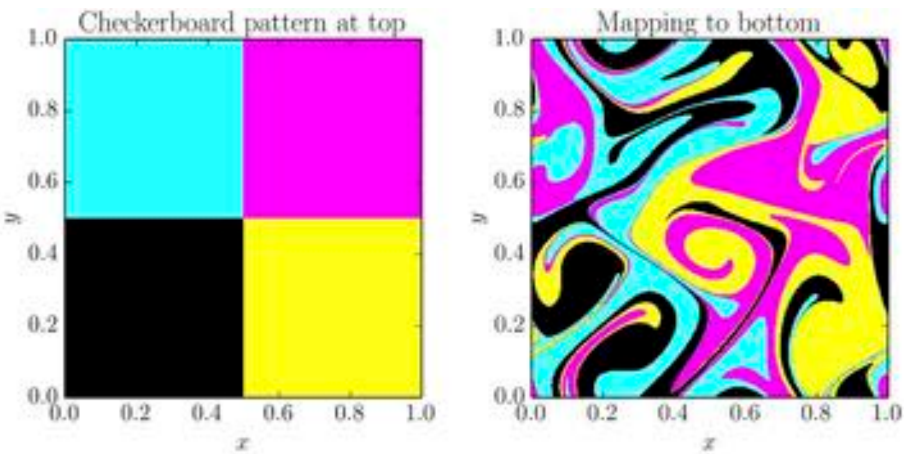
The deteminant $\mathfrak{J}$ of $\overleftrightarrow{\mathfrak{J}}$ is unity since the magnetic flux in a tube $\int B_g \ \mathfrak{J} dx_0 dy_0$ must be independent of *z* and $\mathfrak{J} = 1$ at *z*=0.

Singular-Value decomposition requires $\overleftrightarrow{\boldsymbol{\mathfrak{J}}} \equiv \overleftrightarrow{\boldsymbol{U}} \cdot \begin{pmatrix} \boldsymbol{e^{\sigma}} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{e^{-\sigma}} \end{pmatrix} \cdot \overleftrightarrow{\boldsymbol{V}}^{\dagger}$;

Unitary matrices $\overleftrightarrow{U}$ and $\overleftrightarrow{V}$ give rotations and $\sigma(x_0, y_0, z)$ gives exponentiation.

Trajectories of $H(x, y, z)$ exponentiate apart unless there is a prohibiting constraint, as when $H(x, y)$.

Why Does Exponentiation Give Alfvénic Reconnection?



Electon inertia implies free reconnection on the $\frac{c}{\omega_{pe}}$ scale, which allows reconnection over the scale $\frac{c}{\omega_{pe}} e^{\sigma}$.

In solar corona, two lines separated by $\frac{c}{\omega_{pe}}$ at one point can be on opposite sides of the sun when $\sigma = 22$.

Will show entire evolution is intrisically Alfvénic.

Relation of a Slow Evolution to Alfvén Velocity V_A Reconnection

When $\vec{E} + \vec{v}\times\vec{B} = \eta\vec{j} + \left(\frac{c}{\omega_{pe}}\right)^2 \frac{\partial\vec{j}}{\partial t}$ and $\frac{\vec{f}}{\rho_0} = \frac{d\vec{v}}{dt} - \upsilon_v \nabla_\perp^2 v$, Maxwell's equations and the Lorentz force $\vec{f} = \vec{j}\times\vec{B}$ imply (my Spineto paper)

$\frac{d^2\Omega}{dz^2} = \frac{1}{V_A^2}\frac{d^2\Omega}{dt^2} + \mathbb{N}_A$. Vorticity $\Omega \equiv -\nabla_\perp^2\phi$ with $\vec{v} = \vec{\nabla}\phi\times\hat{z}$.

$\mathbb{N}_A$ gives the non-ideal terms in the equation.

The total derivatives mean along the magnetic field lines.

$\frac{\partial H}{\partial t} = \frac{d\phi}{dz} - \mathbb{N}_B$ where $\mathbb{N}_B$ gives the non-ideal terms.

Partial derivative ∂/∂t means holding (x,y,z) constant.

Cause of Alfvénic Reconnection is in Ideal Solution

$\frac{d^2\Omega}{dz^2} = \frac{1}{V_A^2}\frac{d^2\Omega}{dt^2}$ solved by $\Omega = \Omega_u(t - z/V_A) + \Omega_d(t + z/V_A)$

Arguments mean along field lines $\vec{x}(x_0, y_0, z, t)$.

$\Omega_u(t - z/V_A)$ is an Alfvén wave propagating upward along the field and $\Omega_d(t + z/V_A)$ is one propagating downward.

At *z*=0, rigid wall, $\Omega = 0$. At *z*=L, moving wall, $\Omega = \Omega_w \equiv -\nabla_\perp^2\phi_w$, so

$\Omega_u(t) + \Omega_d(t) = 0$ and $\Omega_u(t - L/V_A) + \Omega_u(t + L/V_A) = \Omega_w(x_0, y_0, z, t)$.

Even a simple Ω_w in Cartesian coordinates gives a very complicated $\Omega_w(x_0, y_0, t)$ due to magnetic field line exponentiation.

When motion is slow $\Omega = \Omega'(x_0, y_0, t)z$, but field line exponentiation causes flows and disipation to increase exponetially with time.

Questions on 3-D Reconnection

1. How does reconnection depend on the terms that break the ideal evolution of the magnetic field, $\boldsymbol{\eta}$ and $\boldsymbol{c/\omega_{pe}}$?

Reconnection is trivial unless $R_m \equiv \frac{av_w}{\eta/\mu_0} >> 1$ and $\frac{a}{c/\omega_{pe}} >> 1.$

2. How does reconnection depend on viscosity $\boldsymbol{v_v}$?

Non-trivial reconnection exists for essentially any $R_e \equiv \frac{av_w}{v_v}$.

As $R_m \to \infty$, energy released by reconnection goes to Alfvén waves with damping that depends qualitatively on the magnitude of R_e.

3. Is reconnection quasi-continuous or episodic?

Trigger time obvious, but for continuous drive what happens?

4. How does reconnection depend on complexity of $\boldsymbol{\phi_w(x, y, t)}$?

When ϕ_w depends on t, points in wall separate exponentially in t.

When ϕ_w has a spatial scale small compared to the periodicity length, reconnection is slowed to a diffusive rate.

5. For slow drive $\boldsymbol{v_w/V_A << 1}$, how long is required for equilibrium to be reestablished after a fast reconnection?

Alfvénic, but with waves that are generally heavily damped.

> Important for understanding tokamak current spikes and their association with the production of relativistic electrons.

6. How large a $\boldsymbol{\sigma}$ can simulations follow?

Numerical difficulty increases as $e^{5\sigma}$; only $\sigma \lesssim 10$ clearly accessible.