Magnetic reconnection in three dimensions

Allen H. Boozer
Columbia University, New York, NY 10027
ahb17@columbia.edu
(Dated: May 8, 2017)

Many generic features of magnetic reconnection are not generic to the literature on reconnection. This anomaly arises from the focus of reconnection studies on magnetic fields that depend on only two spatial coordinates. When a magnetic field depends on all three spatial coordinates, reconnection is enhanced by a mechanism that cannot occur when the field depends on only two—the exponentially increasing separation of neighboring magnetic lines with the distance ℓ along a line. The dependence of the theory on the number of coordinates is explained in a model that could address many unanswered questions in three-dimensional reconnection. The computational difficulty of studying reconnection in three spatial dimensions increases as $e^{5\sigma}$, where σ is the number of exponentiations in the field line separation. This appears to limit direct simulations to $\sigma_{max} \approx 10$, which is consistent with a $\sigma \approx 8$ required to understand fast reconnection in fusion plasmas but much smaller than $\sigma \approx 20$ required to understand reconnection in the solar corona.

I. INTRODUCTION

A. Ubiquity of magnetic reconnection

An evolving magnetic field that is embedded in a highly conducting plasma generically undergoes fast magnetic reconnections [1, 2], which means with a reconnection speed determined not by resistivity but by the Alfvén velocity or the plasma viscosity.

Evolution generally causes neighboring magnetic field lines to develop [2, 3] an exponentially increasing spatial separation $\delta_\perp$ with distance ℓ along a line, $\delta_\perp(\ell) = \delta_0 e^{\sigma(\ell)}$. Neighboring field lines are defined in the limit as their $\ell = 0$ separation δ_0 goes to zero.

Even when the plasma resistivity vanishes, electron inertia causes a breaking of the magnetic field lines on the c/ω_{pe} spatial scale, which can scramble the magnetic field lines over a distance scale as large as the characteristic spatial scale a of the drive for the evolution [2, 3]. This occurs when $(c/\omega_{pe})e^{\sigma(\ell)} \approx a$.

In the solar corona, $c/\omega_{pe} \sim 10$ cm and the radius of the sun is about ten orders of magnitude greater, $R_\odot \approx 7\times10^5$ km, so an exponentiation $\sigma \sim 23$ could be responsible for the observed reconnection events. An even larger σ may arise in astrophysical reconnection. A much smaller σ is required to explain fast reconnections in fusion experiments. The minor radius of ITER is $a = 2.0$ m, and the standard operating density is 10^{20}electrons/m^3, which makes $a/(c/\omega_{pe}) = e^{8.2}$.

B. Two-coordinate reconnection theory

Many generic features of magnetic reconnection are not generic to the literature on magnetic reconnection. The reason is that much of the reconnection literature considers magnetic fields that depend on only two rather than three spatial coordinates [1]. In two-coordinate systems, magnetic field lines can exponentially separate only in an exponentially small fraction of space [2], while in three-coordinate systems exponential separation can occur throughout a large volume.

Remarkably, Alfvénic reconnection can be obtained in two-coordinate systems [1] though generally with an initial Harris-sheet profile [4]. In (x, y, z) Cartesian coordinates, a Harris sheet has the form $B_y = B_h \tanh(x/\delta_h)$, where B_h and δ_h are constants. A Harris sheet does not have a static equilibrium when distorted in the y direction [3] and consequently does not represent naturally arising magnetic fields.

A reconnection trigger is difficult to obtain in two-coordinate magnetic fields that evolve from a simple initial state. In evolving magnetic fields that depend on all three-coordinates, magnetic reconnection naturally arises no matter how simple the initial state of the magnetic field. The trigger time for reconnection is the exponential separation of neighboring magnetic field lines σ reaching a sufficiently large value.

C. Simple reconnection model

Features of reconnection that arise in magnetic field structures that depend on all three spatial coordinates can be studied in highly simplified models since many features are generic. Explanations will be given using what is called a reduced MHD model [5, 6] in which the magnetic field has the form [7, 8]

$$\vec{B} = B_g(\hat{z} + \vec{\nabla}_\perp H \times \hat{z}) \tag{1}$$

in Cartesian coordinates, where B_g is a constant guide field and $\vec{\nabla}_\perp = \hat{x}\partial/\partial x + \hat{y}\partial/\partial y$.

Van Ballegooijen [7] used the magnetic field of Equation (1) in a simple evolution model to study whether current singularities would develop but did not directly study magnetic reconnection.

In van Ballegooijen's model, the system extends from a wall at $z = 0$ to a wall at $z = L$ with $L|\vec{\nabla}_\perp H|$ remaining finite as $L \to \infty$. The $z = 0$ wall is a rigid perfect conductor, but the $z = L$ wall is a flowing perfect conductor that has a velocity $\vec{v}_w = \vec{\nabla}_\perp \phi_w \times \hat{z}$. The guide field B_g is too strong to be compressed, so the velocity of the plasma that lies in the region $0 < z < L$ can be assumed to have the velocity

$$\vec{v} = \vec{\nabla}_\perp \phi \times \hat{z}, \tag{2}$$

where $\phi(x, y, z, L, t) = \phi_w(x, y, t)$, the stream function in the flowing wall. Energy is put into the system by the moving wall and must be removed by dissipation, Appendix A.

In addition to $B_g H$ being the $\hat{z}$ component of the vector potential, $H(x, y, z, t)$ is the Hamiltonian for the magnetic field lines with t a parameter,

$$\frac{dx}{dz} = -\frac{\partial H}{\partial y} \tag{3}$$

$$\frac{dy}{dz} = \frac{\partial H}{\partial x}. \tag{4}$$

Magnetic field line trajectories are given by a Hamiltonian of the same type as $H(x, y, z, t)$ in far more general representations of the magnetic field than that of Equation (1). The magnetic field in a stellarator or tokamak can always be represented as [9, 10]

$$2\pi\vec{B} = \vec{\nabla}\psi_t \times \vec{\nabla}\theta + \vec{\nabla}\varphi \times \vec{\nabla}\psi_p(\psi_t, \theta, \varphi, t), \tag{5}$$

where the poloidal flux ψ_p is the field line Hamiltonian: $d\theta/d\varphi = \partial\psi_p/\partial\psi_t$ and $d\psi_t/d\varphi = -\partial\psi_p/\partial\theta$. The toroidal magnetic flux is ψ_t, the poloidal angle is θ, and the toroidal angle is φ.

The stream function $\phi(x, y, z, t)$ is the Hamiltonian that describes the motion of plasma points in a constant-z plane,

$$\frac{dx}{dt} = -\frac{\partial \phi}{\partial y} \tag{6}$$

$$\frac{dy}{dt} = \frac{\partial \phi}{\partial x}. \tag{7}$$

That is, z is a parameter in the Hamiltonian $\phi(x, y, z, t)$ and not one of the canonical variables.

D. Reconnection is two versus three coordinate systems

The fundamental difference in magnetic reconnection in two versus three coordinate systems is contained in the Hamiltonian description of magnetic field line trajectories. At each point in time, the Hamiltonian for magnetic field line trajectories, Equations (3) and (4), is a one-degree-of-freedom Hamiltonian, $H(x, y)$, for two-coordinate systems and a one-and-a-half-degree-of-freedom Hamiltonian, $H(x, y, z)$, for three-coordinate systems. In 1986 the qualitative differences between the trajectories given by these two types of Hamiltonians was so exciting that Sir James Lighthill titled an article [11] in the *Proceedings of the Royal Society* "*The Recently Recognized Failure of Predictability in Newtonian Dynamics*."

When the magnetic field line Hamiltonian has the form $H(x, y)$ at a given point in time, $\vec{B} \cdot \vec{\nabla} H = 0$, so the lines must stay on constant-H surfaces. The implication is that only an $\approx e^{-2\sigma}$ fraction of the area of the (x, y) plane is occupied by field lines that can exponentiate apart by σ e-folds. The only places at which magnetic field lines can exponentiate apart are at saddle points, where the Hamiltonian has the Taylor expansion $H = H_{sp} + (\partial^2 H/\partial x \partial y)_{sp}(x - x_{sp})(y - y_{sp}) + \cdots$. When a is the characteristic spatial scale of the function $H(x, y)$, only lines started within a radius $ae^{-\sigma}$ of the point (x_{sp}, y_{sp}) can exponentiate σ times before being too far from the saddle point to be influenced by it. The fraction of the area in the (x, y) plane in which σ exponentiations can take place is therefore $\approx e^{-2\sigma}$.

When the magnetic field line Hamiltonian has the form $H(x, y, z)$ at a given point in time, the fraction of the area in a constant-z plane occupied by lines that exponentiate apart is generically of order unity.

The trajectory of a magnetic field line can be written as $\vec{x}(x_s, y_s, \ell)$, where (x_s, y_s) is the starting point of the trajectory. The distance along the line is ℓ, which in the reduced MHD model is indistinguishable from z. That is,

$$\vec{x}(x_s, y_s, \ell) = \vec{x}_\perp(x_s, y_s, \ell) + \ell\hat{z}, \text{ where} \tag{8}$$
$$\vec{x}_\perp(x_s, y_s, \ell) \equiv x(x_s, y_s, \ell)\hat{x} + y(x_s, y_s, \ell)\hat{y}; \tag{9}$$
$$x(x_s, y_s, \ell = 0) = x_s \text{ and } y(x_s, y_s, \ell = 0) = y_s. \tag{10}$$

It is useful to use ℓ to indicate one is using (x_s, y_s, ℓ) as the spatial coordinates rather than the Cartesian coordinates (x, y, z). In other words, $\vec{x}(x_s, y_s, \ell)$ defines a coordinate system in which the magnetic field lines are trivial.

The separation between neighboring trajectories is

$$\vec{\delta}_\perp \equiv \frac{\partial \vec{x}_\perp}{\partial x_s}\delta x_s + \frac{\partial \vec{x}_\perp}{\partial y_s}\delta y_s, \text{ so} \tag{11}$$

$$\begin{pmatrix} \delta_x \\ \delta_y \end{pmatrix} = \begin{pmatrix} \frac{\partial x}{\partial x_s} & \frac{\partial x}{\partial y_s} \\ \frac{\partial y}{\partial x_s} & \frac{\partial y}{\partial y_s} \end{pmatrix} \begin{pmatrix} \delta x_s \\ \delta y_s \end{pmatrix}, \tag{12}$$

where $\vec{\delta}_\perp = \delta_x\hat{x} + \delta_y\hat{y}$. The Jacobian matrix is

$$\overleftrightarrow{\mathcal{J}} \equiv \begin{pmatrix} \frac{\partial x}{\partial x_s} & \frac{\partial x}{\partial y_s} \\ \frac{\partial y}{\partial x_s} & \frac{\partial y}{\partial y_s} \end{pmatrix}, \tag{13}$$

and the determinant of the Jacobian matrix is the Jacobian $\mathcal{J}$ of the (x_s, y_s) coordinates. Since the area element in the $\hat{z}$ direction is $d\vec{a} = \hat{z}\mathcal{J}dx_s dy_s$, magnetic flux conservation implies $\mathcal{J} = 1$. The mathematical implication of a unit Jacobian is that the singular value decomposition of the Jacobian matrix has the form

$$\overleftrightarrow{\mathcal{J}}(x_s, y_s, \ell) = \overleftrightarrow{U} \cdot \begin{pmatrix} e^{\sigma} & 0 \\ 0 & e^{-\sigma} \end{pmatrix} \cdot \overleftrightarrow{V}^\dagger, \tag{14}$$

where $\overleftrightarrow{U}$ and $\overleftrightarrow{V}$ are orthogonal matrices, $\overleftrightarrow{U}^\dagger \cdot \overleftrightarrow{U} = \overleftrightarrow{1}$. Each of the two-by-two orthogonal matrices gives a rotation through an angle α as is clear from the general form that goes to the unit matrix at $\alpha = 0$,

$$\overleftrightarrow{U} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix}. \tag{15}$$

There is a direction in which the magnetic field lines approach each other exponentially, but when integrating lines, numerical errors will quickly cause the the exponential separation to overwhelm the exponential convergence.

Equation (14) demonstrates that neighboring magnetic field lines have a separation that depends exponentially on $\sigma(x_s, y_s, \ell)$. Neighboring magnetic fields lines characteristically exponentiate apart with the distance ℓ along a line unless there is a constraint that prevents the exponentiation, as when the magnetic field depends on only two spatial coordinates.

E. Drive and plasma equations

The drive for reconnection in the simple model of Section I C is the flow $\vec{v}_w = \vec{\nabla}\phi_w \times \hat{z}$ in the wall, which is often assumed to be very slow

$$M_{eff} \equiv \frac{v_w L}{V_A a} << 1, \tag{16}$$

where a is the characteristic spatial scale for variations in ϕ_w. The energy input and dissipation are discussed in Appendix A.

Three plasma properties are of importance. The first plasma property is the smallness of the Debye length, which implies the current density is divergence free [10]. That and the expression for the electromagnetic or Lorentz force $\vec{f} = \vec{j} \times \vec{B}$ implies

$$\vec{B} \cdot \vec{\nabla}\frac{j_{||}}{B} = \vec{B} \cdot \vec{\nabla} \times \frac{\vec{f}}{B^2}. \tag{17}$$

Using Equation (1) for $\vec{B}$, this is equivalent to

$$\vec{B} \cdot \vec{\nabla}K = \frac{1}{V_A^2}\vec{B} \cdot \vec{\nabla} \times \frac{\vec{f}}{\rho_0}; \tag{18}$$
$$K \equiv \frac{\mu_0 j_{||}}{B}; \tag{19}$$
$$V_A^2 \equiv \frac{B_g^2}{\mu_0 \rho_0}, \tag{20}$$

where ρ_0 is the plasma density, which is assumed to be a constant, and V_A is the Alfvén velocity.

Ampere's law, $\vec{\nabla} \times \vec{B} = \mu_0\vec{j}$, implies

$$\nabla_\perp^2 H = -K. \tag{21}$$

The second plasma property is the force exerted by the plasma, which is taken to have the form

$$\frac{\vec{f}}{\rho_0} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{v} - \frac{\vec{\nabla}p}{\rho_0} - \nu_v \nabla_\perp^2 \vec{v}. \tag{22}$$

The term $\vec{v}\cdot\vec{\nabla}\vec{v} = \vec{\nabla}(v^2/2) - \vec{v}\times\vec{\nabla}\times\vec{v}$. The vorticity, $\vec{\Omega} \equiv \vec{\nabla} \times \vec{v}$ has only a $\hat{z}$ component with

$$\nabla_\perp^2 \phi = -\Omega, \text{ so} \tag{23}$$
$$\frac{\vec{B}}{B_g} \cdot \vec{\nabla} \times \frac{\vec{f}}{\rho_0} = \frac{\partial \Omega}{\partial t} + \vec{v} \cdot \vec{\nabla}\Omega - \nu_v \nabla_\perp^2 \Omega. \tag{24}$$

The third plasma property involves the electric field,

$$\vec{E} + \vec{v} \times \vec{B} = \hat{z} \left(\frac{c}{\omega_{pe}} \right)^2 \left(\frac{\partial}{\partial t} + \nu_c \right) \mu_0 j_{||}. \tag{25}$$

The resistivity η is related to the electron collision frequency ν_c by $\eta/\mu_0 = (c/\omega_{pe})^2 \nu_c$. The term involving the time derivative of the parallel current density is due to the electron inertia and is always present since the electron is the lightest charged particle.

By definition, an ideal magnetic evolution can move the magnetic field lines but cannot break them. The condition that must be satisfied for an ideal magnetic evolution is derived in Appendix B. Two terms in the expression for the electric field break the ideal evolution of the magnetic field. The dimensionless coefficients that give the strength of this breaking are the magnetic Reynolds number,

$$R_m \equiv \frac{a v_w}{\eta/\mu_0}, \text{ and} \tag{26}$$

$$\frac{a}{c/\omega_{pe}} \tag{27}$$

Both dimensionless coefficients must be very large compared to unity to have a non-trivial reconnection problem.

The viscosity ν_v breaks the ideal equation for the plasma flow and has the Reynolds number as its dimensionless coefficient,

$$R_e \equiv \frac{a v_w}{\nu_v}. \tag{28}$$

A non-trivial problem in magnetic reconnection exists for any value of R_e. The strength of the viscosity is sometimes given by the magnetic Prandtl number $P_m \equiv \mu_0 \nu_v / \eta$, which can be very large compared to unity in plasmas. A more important parameter in measuring the relative importance of viscous to resistive dissipation will be found to be the Alfvén-weighted Prandtl number,

$$\mathcal{P}_A \equiv M_{eff}^2 P_m$$

$$= \left(\frac{v_w L}{a V_A} \right)^2 \frac{\mu_0 \nu_v}{\eta}. \tag{29}$$

F. Unanswered questions

Many features of the evolving magnetic fields are not known when the plasma dissipation is small. Many questions on driven reconnection that remain unanswered could be answered in the simple reconnection model associated with Equations (1) and (2).

Unfortunately, when the dependence of the magnetic field on all three spatial coordinates is retained, the difficulty of computations increases exponentially. A simple estimate, Sec. III, is as $e^{5\sigma}$, which appears to place an upper limit on what is computationally achievable, $\sigma_{max} \approx 10$.

The limit $\sigma_{max} \approx 10$ is consistent with the $\sigma \approx 8$ required to understand fast reconnections in ITER but far below $\sigma \approx 20$, which is important for reconnection in the solar corona. Nevertheless, the simplicity of these evolution equations may permit approximations that allow studies of reconnection at larger values of σ. More complete models are presumably even more challenging for numerical simulations.

The evolution equations associated with Equations (1) and (2) were derived in 1985 by van Ballegooijen [7] and have been studied by a number of authors, notably [8] and [12]. The study [12] by Yi-Min Huang et al is closely related to the model discussed here but was limited to exponentiations $\sigma \lesssim 6$. Daughten et al [13] have done extensive kinetic calculations of magnetic reconnection in a more complete magnetic field model retaining all three spatial coordinates. Nevertheless, a Harris sheet was used as an initial condition, and the number of exponentiations was $\sigma \lesssim 8$.

(1) How does the reconnection depend on the three dimensionless parameters R_m, $\mathcal{P}_A$, and M_{eff}?

Reconnection is trivial unless the magnetic Reynolds number R_m is very large compared to unity, but the complexity of the behavior as $R_m \to \infty$ is poorly understood. Unless the plasma has significant dissipation, the energy released by the reconnection will drive strong Alfvén waves. These waves propagate along the magnetic field lines and develop an extremely complicated spatial dependence and enhanced dissipation from exponentially increasing separation of the lines [3, 14].

Significant dissipation with a very large magnetic Reynolds number requires the Alfvén-weighted Prantl number $\mathcal{P}_A$ be large. The dependence of reconnection phenomena on $\mathcal{P}_A$ is not understood. Although the Alfvénic Mach number M_{eff} is generally assumed to be small compared to unity, the actual plasma flow can become large due to the large spatial excursions made by magnetic field lines. Indeed, the plasma may become unstable [12] and make a transition to a different state on an Alfvénic time scale.

Because of the computational limit on the number of exponentiations σ that can be resolved, it is particularly important that the maximum naturally occurring number of exponentiations σ_{max} be understood.

(2) How does the reconnection depend on the complexity of the drive $\phi_w(x, y, t)$?

An important but poorly understood question is when the drive ϕ_w continues for a long time does the reconnection settle into a quasi-steady-state or is it episodic. A finite time, the trigger time, is clearly required to obtain the first reconnection event when the initial condition is $H = 0$. The dependence of the trigger time on either the dimensionless parameters or the complexity of ϕ_w is essentially unknown.

Is the reconnection behavior qualitatively similar for all functions $\phi_w(x, y, t)$ that have a similar characteristic spatial scale? When $\phi_w(x, y, t)$ depends on time, neighboring points in the flowing wall generically separate exponentially in time; when ϕ_w has no time dependence neighboring wall points do not exponentiate apart. Does this produce a qualitative difference in the reconnection?

When the dominant spatial scale a of ϕ_w is small compared to the size b of the reconnecting region, which may be a periodicity length $2\pi b$, the reconnection has a relatively slow diffusive nature, but this is not well understood.

(3) How quickly do plasma elements that originally lay along one field line spread across the reconnecting volume?

The rapid spreading of impurities across a tokamak plasma during a disruption is poorly understood but may be a byproduct of the fast magnetic reconnection that is associated with the current spike [15].

(4) Under what conditions does c/ω_{pe} dominate the resistivity η for causing magnetic field line breaking?

The large magnetic field line excursions that occur when σ is large can give large flow velocities and make the time derivative term involving c/ω_{pe} far more important than it appears to be.

(5) When ϕ_w has a form that gives episodic rather than quasi-steady-state reconnection, how quickly is equilibrium, $\partial K/\partial \ell = 0$, reestablished?

This question is of particular interest in tokamak disruptions because it presumably determines the length of the current spike [15].

(6) How is the plasma heating spread over the plasma?

When the magnetic Reynolds number R_m is very large, reconnection can occur with little dissipation of energy; most of the energy goes first into Alfvén waves. The rapidity with which the Alfvén waves are damped is a complicated issue [3, 14] but defines the spatial region over which the plasma heating occurs.

II. MODEL EQUATIONS

A. Relation between $\partial K/\partial \ell$ and $d\Omega/dt$

$K \equiv \mu_0 j_{||}/B_0$ is constant along magnetic field lines in a static reduced-MHD equilibrium. The variation in K along the magnetic field, $\partial K/\partial \ell$, Eq. (C3), is related to the vorticity $\Omega = \hat{z} \cdot \vec{\nabla} \times \vec{v}$. The relation between $\partial K/\partial \ell$ and Ω is given by the divergence-free constraint on the current, Eq. (18), the parallel component of the curl of the force, Eq. (24), and the definition of total time derivative, Eq. (C1):

$$\frac{\partial K}{\partial \ell} = \frac{1}{V_A^2}\left(\frac{d\Omega}{dt} - \nu_v \nabla_\perp^2 \Omega\right). \tag{30}$$

When the Alfvén speed goes to infinity, $\partial K/\partial \ell \to 0$. Writing the magnetic field line trajectories as $\vec{x}(x_s, y_s, \ell)$, the equation $\partial K/\partial \ell = 0$ implies $K(\vec{x}(x_s, y_s, \ell)) = K(x_s, y_s)$.

B. Relation between $\partial \phi/\partial \ell$ and $\partial H/\partial t$

The evolution of the magnetic field line Hamiltonian, $\partial H/\partial t$, and the variation in the stream function along the magnetic field lines, $\partial \phi/\partial \ell$, are related by the electric field, Equation (25). Equations (1) for $\vec{B}$ and (2) for $\vec{v}$ imply $\vec{v} \times \vec{B}/B_g = -\vec{\nabla}_\perp \phi + \{\hat{z} \cdot (\vec{\nabla}_\perp H \times \vec{\nabla}_\perp \phi)\}\hat{z}$. Faraday's law can be written as $\vec{E} = -B_g(\partial H/\partial t)\hat{z} - \vec{\nabla}\Phi$. Two equations are obtained $\phi = -\Phi/B_g$ and

$$\frac{\partial \phi}{\partial \ell} = \frac{\partial H}{\partial t} + \mathcal{N}_B, \text{ where} \tag{31}$$

$$\mathcal{N}_B \equiv \left(\frac{c}{\omega_{pe}}\right)^2 \left(\frac{\partial}{\partial t} + \nu_c\right) K \tag{32}$$

breaks the ideal evolution of the magnetic field, which means the maintenance of unbroken magnetic field lines, Appendix B.

C. Combined equations

1. Relation between $\partial\Omega/\partial\ell$ and dK/dt

The commutator for $\partial/\partial\ell$ and $\nabla_\perp^2$, Eq. (C4), implies

$$\nabla_\perp^2\frac{\partial\phi}{\partial\ell}=\frac{\partial}{\partial\ell}\nabla_\perp^2\phi+\hat{z}\cdot(\vec{\nabla}_\perp K\times\vec{\nabla}_\perp\phi); \quad (33)$$

$\partial\phi/\partial\ell=\partial H/\partial t+\mathcal{N}_B$ and $\nabla_\perp^2 H=-K$ imply

$$\nabla_\perp^2\frac{\partial\phi}{\partial\ell}=-\frac{\partial K}{\partial t}+\nabla_\perp^2\mathcal{N}_B. \quad (34)$$

Therefore the current K and the vorticity $\Omega=-\nabla_\perp^2\phi$ are related by

$$\frac{dK}{dt}=\frac{\partial\Omega}{\partial\ell}+\nabla_\perp^2\mathcal{N}_B. \quad (35)$$

2. Expression for $\frac{d}{dt}\frac{\partial K}{\partial\ell}$

Using the commutator relation between $\partial/\partial\ell$ and d/dt, Eq. (C8), one finds that

$$\frac{d}{dt}\frac{\partial K}{\partial\ell}=\frac{\partial}{\partial\ell}\frac{dK}{dt}-\hat{z}\cdot(\vec{\nabla}_\perp K\times\vec{\nabla}_\perp\mathcal{N}_B). \quad (36)$$

3. Alfvén wave equation

Equations (30), (35), and (36) give an Alfvénic wave equation,

$$\frac{\partial^2\Omega}{\partial\ell^2} = \frac{1}{V_A^2}\frac{d^2\Omega}{dt^2}+\mathcal{N}_A \quad (37)$$

$$\mathcal{N}_A \equiv -\frac{\nu_v}{V_A^2}\frac{d}{dt}\nabla_\perp^2\,\Omega -\frac{\partial\nabla_\perp^2\mathcal{N}_B}{\partial\ell}+\hat{z}\cdot(\vec{\nabla}_\perp K\times\vec{\nabla}_\perp\mathcal{N}_B). \quad (38)$$

The two dissipative terms, one proportional to ν_v and the other proportional to η/μ_0, have a ratio of the Alfvén-weighted Prandtl number $\mathcal{P}_A$, Eq. (29), which follows from $H\sim a^2/L$ and $\Omega\sim v_w/a$. The term $d^2\Omega/dt^2$ has a relative size compared to the viscosity term of the Reynold number R_e.

The term $\partial^2\Omega/\partial\ell^2$ in (x_s,y_s,ℓ) coordinates is more complicated in Cartesian coordinates, Equation (C12).

III. SOLUTIONS

A. Difficulty of obtaining solutions

The evolution of the simple model for a driven magnetic field can be obtained by integration until the number of exponentiations σ in the separation between neighboring magnetic field lines becomes large. Unfortunately, the difficulty of following the evolution increases as $e^{5\sigma}$, so a hard cutoff exists in the maximum value of σ that can be resolved. A petascale computer can perform 10^{15} operations a second or $\approx 10^{26}$ per day. When $\sigma=10$, the number of operations is increased by approximately 10^{22} times over the case when $\sigma\lesssim 1$, so $\sigma\approx 10$ appears to be an upper limit on what can be computed. Increasing the computer power by a thousand increases the maximum computable σ from $\sigma\approx 10$ to $\sigma\approx 11.4$. Reaching values of $\sigma\approx 8$ for studying magnetic reconnection in fusion devices is credible, but $\sigma\approx 20$, which is required to understand reconnection in the corona, appears impossible. Simulations using modest values of σ must be sufficiently well understood to devise extrapolations or reliable approximations.

Once σ becomes large, distances in the (x,y) plane of order $ae^{-\sigma}$ must be resolved as must distances of order $Le^{-\sigma}$ in the z direction. The speed with which the magnetic field lines move is of order v_we^{σ}, so the time required for a field line to move over a spatial scale $ae^{-\sigma}$ is $(a/v_w)e^{-2\sigma}$. Assuming the computational difficulty scales as the number of spatial grid cells times the number of time steps, the difficulty scales as $e^{5\sigma}$.

B. Solution when viscosity is dominant

When the viscosity dominates the damping of the vorticity, Equation (37) can be written as

$$\frac{d^2\Omega}{dt^2}-V_A^2\frac{\partial^2\Omega}{\partial\ell^2}=-\frac{d\mathcal{D}}{dt}; \quad (39)$$

$$\mathcal{D}=-\nu_v\nabla_\perp^2\Omega. \quad (40)$$

When the dissipative function $\mathcal{D}$ is negligible, the general solution to Equation (39) is

$$\Omega=\Omega_d(x_s,y_s,t+\ell/V_A)+\Omega_u(x_s,y_s,t-\ell/V_A), \quad (41)$$

where $\Omega_u(x_s,y_s,t-\ell/V_A)$ is an upward-going Alfvén wave propagating along a field line defined by its $\ell=0$ starting point (x_s,y_s) and Ω_d is a downward-going Alfvén wave. Note d/dt means moving with the plasma velocity. As shown in Appendix B, in the ideal case this means moving with the magnetic field lines.

The two functions Ω_u and Ω_d can be found using the boundary conditions, $\Omega = 0$ at $\ell = 0$ and $\Omega = \Omega_L$ at $\ell = L$. Letting $\vec{x}_\perp(x_s, y_s, \ell, t)$ be the trajectories of the magnetic field lines with $\ell = 0$ starting points (x_s, y_s), the vorticity at $\Omega_L(x_s, y_s, t) = \Omega_w\big(\vec{x}_\perp(x_s, y_s, L, t), t\big)$, where the vorticity in the wall $\Omega_w(x, y, t)$ is known in Cartesian coordinates, $\Omega_w = -\nabla_\perp^2 \phi_w$.

When the evolution of Ω is slow compared to the Alfvén transit time L/V_A, the dissipationless solution is $\Omega = (\Omega'_d - \Omega'_u)\ell/V_A$, where the prime means a differentiation in $t \pm \ell/V_A$ argument. The two boundary conditions are then $\Omega_d + \Omega_u = 0$ and $(\Omega'_d - \Omega'_u)L/V_A = \Omega_w$.

A solution to Equation (39) with dissipation can be found by treating the dissipation $d\mathcal{D}/dt$ as an inhomogeneous term. The solution is then $\Omega = \Omega_h + \Omega_\mathcal{D}$, where $\Omega_h = \Omega_d(x_s, y_s, t + \ell/V_A) + \Omega_u(x_s, y_s, t - \ell/V_A)$ and $\Omega_\mathcal{D}$ is a particular solution to Equation (39). Suppressing the (x_s, y_s) arguments to simplify the notation, the viscous dissipation function $\mathcal{D}$ can be written using the step function $\sigma_{st}(\ell)$, where $\sigma_{st}(\ell < 0) = 0$ and $\sigma_{st}(\ell > 0) = 1$:

$$\begin{aligned}\mathcal{D}(\ell, t) &= \mathcal{D}_0(t)\sigma_{st}(\ell) + \mathcal{D}_L(t)\sigma_{st}(\ell - L) \\ &\quad + \sum_n \mathcal{D}_n(t) \sin\left(\frac{n\pi\ell}{L}\right). \end{aligned} \tag{42}$$

The two boundary conditions, which determine the functions $\Omega_d(x_s, y_s, t + \ell/V_A)$ and $\Omega_u(x_s, y_s, t - \ell/V_A)$, are

$$\Omega_d(t) + \Omega_u(t) - \int_0^t \mathcal{D}_0(t) dt = 0 \text{ at } \ell = 0. \tag{43}$$

$$\Omega_d\left(t + \frac{L}{V_A}\right) + \Omega_u\left(t - \frac{L}{V_A}\right) - \int_0^t \mathcal{D}_L(t) dt = \Omega_L \text{ at } \ell = L. \tag{44}$$

The particular solution $\Omega_\mathcal{D}$ is zero at $\ell = 0$ and $\ell = L$ and is given by

$$\Omega_\mathcal{D} = \sum_n \Omega_n(t) \sin\left(\frac{n\pi\ell}{L}\right); \tag{45}$$

$$\frac{d^2\Omega_n}{dt^2} + \omega_n^2\Omega_n = -\frac{d\mathcal{D}_n}{dt}; \tag{46}$$

$$\omega_n \equiv n\frac{\pi V_A}{L}. \tag{47}$$

Since $\mathcal{D}(t)$ is zero for $t \leq 0$, Equation (46) for Ω_n can be solved by introducing a constant γ_n and letting

$$\Omega_n(t) = e^{\gamma_n t}\int \tilde{\Omega}_n e^{-i\omega t} d\omega, \text{ so} \tag{48}$$

$$\int \big((\gamma_n - i\omega)^2 + \omega_n^2\big)\tilde{\Omega}_n e^{(\gamma_n - i\omega)t} d\omega = -\frac{d\mathcal{D}_n}{dt}. \tag{49}$$

The function $\mathcal{D}_n(t)$ is zero for $t \leq 0$, and can be written as

$$\mathcal{D}(t) = \int \tilde{\mathcal{D}}_n e^{(\gamma_n - i\omega)t} d\omega, \text{ where} \tag{50}$$

$$\tilde{\mathcal{D}}_n = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\gamma_n t}\mathcal{D}(t) e^{i\omega t} dt, \text{ so} \tag{51}$$

$$\tilde{\Omega}_n = -\frac{\gamma_n - i\omega}{(\gamma_n - i\omega)^2 + \omega_n^2}\tilde{\mathcal{D}}_n. \tag{52}$$

The constant γ_n must be chosen to avoid a singularity in the solution at $\omega^2 = \omega_n^2$, which presumably implies that γ_n should be a fraction of ω_n.

The dissipative function $\mathcal{D}(x_s, y_s, \ell, t)$ can be obtained using the unity of the Jacobian between (x, y) and (x_s, y_s) coordinates

$$\begin{aligned}\mathcal{D}(x_s, y_s, \ell, t) &= \int \tilde{\mathcal{D}}(\vec{k}_\perp, z, t) e^{i\vec{k}_\perp\cdot\vec{x}_\perp(x_s, y_s, \ell, t)} dk_x dk_y \\ &= \nu_v \int \vec{k}_\perp \cdot \vec{k}_\perp \tilde{\Omega}(\vec{k}_\perp, \ell, t) e^{i\vec{k}_\perp\cdot\vec{x}_\perp} dk_x dk_y \end{aligned} \tag{53}$$

$$\tilde{\Omega}(\vec{k}_\perp, \ell, t) = \frac{\int \Omega(x_s, y_s, \ell, t) e^{-i\vec{k}_\perp\cdot\vec{x}_\perp(x_s, y_s, \ell, t)} dx_s dy_s}{(2\pi)^2}, \tag{54}$$

where $\vec{k}_\perp = k_x\hat{x} + k_y\hat{y}$.

The remainder of the calculation is carried out in Cartesian coordinates. The vorticity in Cartesian coordinates $\Omega(x, y, z, t)$ is determined using the equality of z and ℓ:

$$\Omega(x, y, z, t) = \int \tilde{\Omega}(\vec{k}_\perp, z, t) e^{i(k_x x + k_y y)} dk_x dk_y \tag{55}$$

The stream function $\phi(x, y, z, t)$ is obtained from $\nabla_\perp^2 \phi = -\Omega(x, y, z, t)$. A common assumption is that the system is periodic in x and y; then $\phi(x, y, z, t)$ can be particularly easily determined as a Fourier series. As time advances the change in the Hamiltonian $H(x, y, z, t)$ is given by $\partial H/\partial t = \partial\phi/\partial\ell - \mathcal{N}_B$, which has the explicit form

$$\frac{\partial H}{\partial t} = \frac{\partial\phi}{\partial z} + \left(\frac{\partial\phi}{\partial x}\frac{\partial H}{\partial y} - \frac{\partial\phi}{\partial y}\frac{\partial H}{\partial x}\right) - \mathcal{N}_B. \tag{56}$$

The non-ideal term depends on K through $\mathcal{N}_B$, which can be found from $K = -\nabla_\perp^2 H$.

The Hamiltonian $H(x, y, z, t)$ at each instant of time gives the magnetic field line trajectories $\vec{x}(x_s, y_s, \ell, t)$, so a solution can be followed as the system evolves from an initial state in which $H = 0$.

Acknowledgements

This material is based upon work supported by the U.S. Department of Energy, Office of Science, Office of Fusion Energy Sciences under Award Numbers DE-FG02-95ER54333 and De-FG02-03ER54696.

Appendix A: Power flow and dissipation

1. Wall equations

When the upper wall is a thin shell, its current density has the form $\vec{j}_w = (\vec{\nabla}_\perp g)\delta(z - L)$. In principle, $\vec{j}_w = \left(\vec{\nabla}_\perp g + \vec{\nabla}_\perp \kappa_w \times \hat{z}\right)\delta(z - L)$. The current potential in the wall κ_w is given by $\vec{\nabla} \times \vec{j}_w$, but $\vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = B_g \vec{\nabla} K \times \hat{z}$. The current density in the $\hat{z}$ direction is assumed to vanish at the $z = L$ boundary so $\vec{\nabla} \times \vec{j}_w = 0$ and $\kappa_w = 0$.

The divergence-free property of the current implies $\partial j_z/\partial z + \vec{\nabla}_\perp \cdot \vec{j}_w = 0$. Integrating in the z direction across the wall,

$$\nabla_\perp^2 g = \frac{B_g}{\mu_0} K_L, \tag{A1}$$

where $j_z = (B_g/\mu_0)K_L$ is the current flowing from the plasma into the wall at $z = L$.

The power required to drive the flow in the wall is

$$\begin{aligned}
P_w^{(d)} &= -\int_w \vec{v}_w \cdot \vec{f} d^3x \\
&= -\int_w (\vec{\nabla}_\perp \phi_w \times \hat{z}) \cdot (\vec{j}_w \times \vec{B}) d^3x \\
&= -B_g \int (\vec{\nabla}_\perp \phi_w \cdot \vec{\nabla}_\perp g) da_w \\
&= B_g \int (\phi_w \nabla_\perp^2 g) da_w \\
&= \frac{B_g^2}{\mu_0} \int (\phi_w K_L) da_w, \text{ or} \qquad \text{(A2)} \\
&= \frac{B_g^2}{\mu_0} \int (\Omega_w H_L) da_w, \qquad \text{(A3)}
\end{aligned}$$

where H_L is the field line Hamiltonian at $\ell = L$.

When the upper wall has a finite resistivity, the Ohm's law in the wall is $\vec{E} + \vec{v}_w \times \vec{B} = \eta_w (\vec{\nabla}_\perp g)\delta(z - L)$. Crossing this expression with $\hat{z}$ and averaging over the wall of thickness Δ_w,

$$\hat{z} \times \vec{E}_w + B_g \vec{v}_w = \frac{\eta_w}{\Delta_w} \hat{z} \times \vec{\nabla}_\perp g. \tag{A4}$$

The electric field has the form $\vec{E}_w = -\partial\vec{A}/\partial t - \vec{\nabla}\Phi$. Since the time derivative of the vector potential is in the $\hat{z}$ direction, $\hat{z} \times \vec{E}_w = -\hat{z} \times \vec{\nabla}_\perp \Phi$, so $(\eta/\Delta_w)\vec{\nabla}_\perp g = B_g \vec{\nabla}_\perp(\phi_L - \phi_w)$ with the boundary condition that the tangential electric field is continuous. This boundary condition is required because otherwise the curl of the electric field and the time derivative of the magnetic field would be infinite. The stream function ϕ in the plasma equals $-\Phi/B_g$ with ϕ_L the value of ϕ at $z = L$.

The divergence of the equation relating $\vec{\nabla}_\perp g$ to $\vec{\nabla}_\perp(\phi_L - \phi_w)$ implies

$$\Omega_L = \Omega_w - \frac{\eta_w}{\mu_0} \frac{K_L}{\Delta_w}, \tag{A5}$$

where the vorticity in the wall is $\Omega_w = -\nabla_\perp^2 \phi_w$ and the vorticity in the plasma next to the wall is $\Omega_L = -\nabla_\perp^2 \phi_L$. When the resistivity of the wall is zero, $\Omega_L = \Omega_w$ but when the resistivity in wall is non-zero there is a slippage between the plasma flow and the wall flow.

The power required to drive the flow in the wall is the sum of two parts, $P_w^{(d)} = P_w + P_\eta$, where P_w is the power transmitted to the plasma and P_η is the power dissipated in the wall:

$$\begin{aligned}
P_w &= \frac{B_g^2}{\mu_0} \int (\phi_L K_L) da_w; \qquad \text{(A6)} \\
P_\eta &= \frac{B_g^2}{\mu_0} \frac{\eta_w}{\mu_0 \Delta_w} \int K_L H_L da_w \\
&= \frac{B_g^2}{\mu_0} \frac{\eta_w}{\mu_0 \Delta_w} \int \vec{\nabla}_\perp H_L \cdot \vec{\nabla}_\perp H_L da_w \\
&= \frac{\eta_w}{\mu_0 \Delta_w} \int \frac{B_\perp^2}{\mu_0} da_w, \qquad \text{(A7)}
\end{aligned}$$

where $\vec{B}_\perp = B_g \vec{\nabla} H \times \hat{z}$ is the magnetic field tangential to the wall at $z = L$.

2. Energy and power

The energy density is $w = B^2/2\mu_0 + \rho_0 v^2/2$, so

$$w = \frac{B_g^2}{2\mu_0}\left(1 + (\vec{\nabla}_\perp H)^2\right) + \frac{\rho_0}{2}(\vec{\nabla}_\perp \phi)^2 \tag{A8}$$

$$\int wda = \int \left(\frac{B_g^2}{2\mu_0}(1+HK) + \frac{\rho_0}{2}\phi\Omega \right) da; \quad \text{(A9)}$$

$$W \equiv \int wdad\ell = \int wd^3x \quad \text{(A10)}$$

is the energy.

The energy consists of three terms,

$$W = W_{vac} + \frac{B_g^2}{2\mu_0}\int HKd^3x + \frac{\rho_0}{2}\int \phi\Omega d^3x; \quad \text{(A11)}$$

$$\text{where} \quad W_{vac} = \frac{B_g^2}{2\mu_0}\int d^3x \quad \text{(A12)}$$

is the vacuum energy, which is the energy in the absence of plasma current or motion.

The power input per unit length along the field line is

$$\frac{\partial \int wda}{\partial t} = \int \left(\frac{B_g^2}{\mu_0}K\frac{\partial H}{\partial t} + \rho_0\phi\frac{\partial \Omega}{\partial t} \right) da; \quad \text{(A13)}$$

$$\int K\frac{\partial H}{\partial t}da = \frac{\partial \int K\phi da}{\partial \ell} - \int \phi\frac{\partial K}{\partial \ell}da$$
$$- \left(\frac{c}{\omega_{pe}}\right)^2 \left(\frac{1}{2}\frac{\partial}{\partial t} + \nu_c\right)\int K^2 da; \quad \text{(A14)}$$

$$\int \phi\frac{\partial K}{\partial \ell}da = \frac{1}{V_A^2}\int \phi\left(\frac{d\Omega}{dt} - \nu_v\vec{\nabla}_\perp^2\Omega\right)da \quad \text{(A15)}$$

The energy is W obeys

$$\frac{dW}{dt} = P_w - \mathcal{N}_W \quad \text{(A16)}$$

$$\mathcal{N}_W = \rho_0\nu_v\int \Omega^2 d^3x + \frac{B_g^2}{2\mu_0}\left(\frac{c}{\omega_{pe}}\right)^2\left(\frac{1}{2}\frac{\partial}{\partial t} + \nu_c\right)\int K^2 d^3x \quad \text{(A17)}$$

since $\int \phi\vec{\nabla}_\perp^2\Omega d^3x = \int \Omega^2 d^3x$. P_w is the power entering through the wall, and $\mathcal{N}_W$ is the power loss. The term $\partial K^2/\partial t$ in $\mathcal{N}_W$ is technically not a loss of energy; the energy goes into the kinetic energy of the electrons.

The typical value for Ω/K is v_wL/a, so the ratio of the kinetic energy to the magnetic energy associated with the plasma current is M_{eff}^2. The ratio of the viscous to the resistive dissipation of energy is the Alfvén-weighted Prandtl number $\mathcal{P}_A$, Eq. (29).

Appendix B: Preservation of magnetic field lines

Magnetic field lines move but are unbroken when a potential $\Phi_\mathcal{E}$ exists such the general form for the electric field, $\vec{E} + \vec{v}\times\vec{B} = \vec{\mathcal{E}}$, is consistent with $\vec{B}\cdot\vec{\mathcal{E}} = -\vec{B}\cdot\vec{\nabla}\Phi_\mathcal{E}$. This condition is always satisfied in a spatially local region in which $\vec{B} \neq 0$, but the resulting potential $\Phi_\mathcal{E}$ may not satisfy the boundary conditions. When $\Phi_\mathcal{E}$ does not satisfy the boundary conditions, field lines must break.

The condition for the preservation of magnetic field lines in the model field of Equation (1) can be obtained using the freedom of canonical transformations of the magnetic field line Hamiltonian, which in Cartesian coordinates is $H(x,y,z,t)$.

Canonical coordinates (ξ,η,z) are defined by their relation to Cartesian coordinates and have a velocity through space of $\vec{u}$:

$$\vec{x}(\xi,\eta,z,t) \equiv x(\xi,\eta,z,t)\hat{x} + y(\xi,\eta,z,t)\hat{y} + z\hat{z}; \quad \text{(B1)}$$

$$\vec{u} \equiv \frac{\partial \vec{x}(\xi,\eta,z,t)}{\partial t}. \quad \text{(B2)}$$

Derivatives with respect to time will be denoted as $(\partial f/\partial t)_c$ when holding the canonical coordinates constant and as $(\partial f/\partial t)_{\vec{x}}$ when holding the Cartesian coordinates constant. The magnetic field line Hamiltonian in general canonical coordinates will be denoted by $\mathcal{H}(\xi,\eta,z,t)$.

The vector potential $\vec{A}$ for the model magnetic field, Eq. (1), in an arbitrary gauge g and its time derivative at a fixed point in Cartesian coordinates are

$$\frac{\vec{A}}{B_g} = \xi\vec{\nabla}\eta + \mathcal{H}(\xi,\eta,z,t)\hat{z} + \vec{\nabla}g; \quad \text{(B3)}$$

$$\frac{\left(\frac{\partial \vec{A}}{\partial t}\right)_{\vec{x}}}{B_g} = \left(\frac{\partial \xi}{\partial t}\right)_{\vec{x}}\vec{\nabla}\eta - \left(\frac{\partial \eta}{\partial t}\right)_{\vec{x}}\vec{\nabla}\xi + \left(\frac{\partial \mathcal{H}}{\partial t}\right)_{\vec{x}}\hat{z} + \vec{\nabla}\left(\left(\frac{\partial g}{\partial t}\right)_{\vec{x}} + \xi\left(\frac{\partial \eta}{\partial t}\right)_{\vec{x}}\right); \quad \text{(B4)}$$

Time derivatives holding the canonical coordinates fixed are related to time derivatives holding the Cartesian coordinates fixed by $(\partial f/\partial t)_c = (\partial f/\partial t)_{\vec{x}} + \vec{u}\cdot\vec{\nabla}f$. Consequently, $(\partial\xi/\partial t)_{\vec{x}} = -\vec{u}\cdot\vec{\nabla}\xi$, $(\partial\eta/\partial t)_{\vec{x}} = -\vec{u}\cdot\vec{\nabla}\eta$, and $(\partial\mathcal{H}/\partial t)_{\vec{x}} = (\partial\mathcal{H}/\partial t)_c - \vec{u}\cdot\vec{\nabla}\mathcal{H}$. The expression for $\vec{u}\times\vec{B} = \vec{u}\times(\vec{\nabla}\times\vec{A})$ is

$$\vec{u}\times\frac{\vec{B}}{B_g} = (\vec{u}\cdot\vec{\nabla}\eta)\vec{\nabla}\xi - (\vec{u}\cdot\vec{\nabla}\xi)\vec{\nabla}\eta - (\vec{u}\cdot\vec{\nabla}\mathcal{H})\hat{z} \quad \text{(B5)}$$

using $\vec{u}\cdot\vec{\nabla}\hat{z} = 0$ since $\hat{z}$ is a constant unit vector.

The electric field

$$\vec{E} = -\left(\frac{\partial \vec{A}}{\partial t}\right)_{\vec{x}} - \vec{\nabla}\Phi \quad \text{(B6)}$$

$$= -\vec{u}\times\vec{B} - B_g\left(\frac{\partial\mathcal{H}}{\partial t}\right)_c\hat{z} - \vec{\nabla}\Phi_c \quad \text{(B7)}$$

$$\Phi_c \equiv \Phi + \left(\left(\frac{\partial g}{\partial t}\right)_{\vec{x}} + \xi\left(\frac{\partial\eta}{\partial t}\right)_{\vec{x}}\right)B_g. \quad \text{(B8)}$$

When the electric field has the form $\vec{E}+\vec{v}\times\vec{B} = -\vec{\nabla}\Phi_v$, the choice of canonical coordinates $\vec{u}=\vec{v}$ gives $B_g(\partial\mathcal{H}/\partial t)_c\hat{z} = \vec{\nabla}(\Phi_v-\Phi_c)$, which has the solution $\Phi_c=\Phi_v$ and $(\partial\mathcal{H}/\partial t)_c = 0$. Since the Hamiltonian $\mathcal{H}$ does not change in canonical coordinates, the magnetic field lines remain the same in canonical coordinates. The magnetic field lines do move with the velocity $\vec{u}=\vec{v}$ through Cartesian coordinates, which means the lines are attached perfectly to the fluid and do not break.

A solution with $(\partial\mathcal{H}/\partial t)_c = 0$ does not exist when $\vec{E}+\vec{v}\times\vec{B} = \vec{\mathcal{E}}$ unless one can represent $\vec{B}\cdot\vec{\mathcal{E}} = -\vec{B}\cdot\vec{\nabla}\Phi_{\mathcal{E}}$. When such a potential $\Phi_{\mathcal{E}}$ does exist, a new velocity can be defined, $\vec{v}_n = \vec{v} - \vec{B}\times(\vec{\mathcal{E}}+\vec{\nabla}\Phi_{\mathcal{E}})$, so that the electric field has the form $\vec{E}+\vec{v}_n\times\vec{B} = -\vec{\nabla}\Phi_{\mathcal{E}}$ and the magnetic field lines are unbroken moving with the velocity $\vec{v}_n$.

When the Cartesian coordinates are chosen as the canonical coordinates $\vec{u}=0$. Assuming $\vec{E}=-\vec{v}\times\vec{B}$, one finds $\vec{v}\times\vec{B} = B_g(\partial\mathcal{H}/\partial t)_x\hat{z} + \vec{\nabla}\Phi_c$, which for $\vec{v}=\vec{\nabla}\phi\times\hat{z}$ gives $\phi=-\Phi_c$ and $(\partial\mathcal{H}/\partial t)_x = \partial\phi/\partial\ell$.

Appendix C: Derivatives

1. Definition of $\frac{df}{dt}$ and $\frac{\partial f}{\partial \ell}$

The total derivative with respect to time df/dt means moving with the plasma,

$$\frac{df}{dt} \equiv \left(\frac{\partial f}{\partial t}\right)_{(x,y,z)} + \vec{v}\cdot\vec{\nabla}f$$
$$= \frac{\partial f}{\partial t} + \hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp\phi). \quad \text{(C1)}$$

The partial derivative with respect to distance along the magnetic field lines $\partial f/\partial\ell$ is defined by

$$\frac{\partial f}{\partial\ell} \equiv \frac{\vec{B}}{B_0}\cdot\vec{\nabla}f \quad \text{(C2)}$$
$$= \frac{\partial f}{\partial z} + \hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp H) \quad \text{(C3)}$$

2. Commutator of $\nabla_\perp^2$ with $\frac{df}{dt}$ and $\frac{\partial f}{\partial\ell}$

$$\nabla_\perp^2\frac{\partial f}{\partial\ell} = \nabla_\perp^2\left(\frac{\partial f}{\partial z} + \hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp H)\right)$$
$$= \frac{\partial\nabla_\perp^2 f}{\partial z} + \hat{z}\cdot(\vec{\nabla}_\perp\nabla_\perp^2 f\times\vec{\nabla}_\perp H)$$
$$+\hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}\nabla_\perp^2 H)$$
$$= \frac{\partial}{\partial\ell}\nabla_\perp^2 f + \hat{z}\cdot(\vec{\nabla}_\perp K\times\vec{\nabla}_\perp f). \quad \text{(C4)}$$

$$\nabla_\perp^2\frac{df}{dt} = \nabla_\perp^2\left(\frac{\partial f}{\partial t} + \hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp\phi)\right)$$
$$= \frac{\partial\nabla_\perp^2 f}{\partial z} + \hat{z}\cdot(\vec{\nabla}_\perp\nabla_\perp^2 f\times\vec{\nabla}_\perp\phi)$$
$$+\hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp\nabla_\perp^2\phi)$$
$$= \frac{d}{dt}\nabla_\perp^2 f + \hat{z}\cdot(\vec{\nabla}_\perp\Omega\times\vec{\nabla}_\perp f). \quad \text{(C5)}$$

3. Commutator of $\frac{\partial}{\partial\ell}$ and $\frac{\partial}{\partial t}$

$$\frac{\partial}{\partial t}\frac{\partial f}{\partial\ell} = \frac{\partial}{\partial z}\frac{\partial f}{\partial t} + \hat{z}\cdot(\vec{\nabla}_\perp\frac{\partial f}{\partial t}\times\vec{\nabla}_\perp H)$$
$$+\hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp\frac{\partial H}{\partial t})$$
$$= \frac{\partial}{\partial\ell}\frac{\partial f}{\partial t} + \hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp\frac{\partial H}{\partial t}). \quad \text{(C6)}$$

4. Commutator of $\frac{\partial}{\partial\ell}$ and $\frac{d}{dt}$

This derivation uses

$$\frac{\partial\hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp g)}{\partial\ell} = \hat{z}\cdot(\vec{\nabla}_\perp\frac{\partial f}{\partial\ell}\times\vec{\nabla}_\perp g)$$
$$+\hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp\frac{\partial g}{\partial\ell}), \quad \text{(C7)}$$

which follows from the coordinate invariance of $\hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp g)$, Appendix C 6.

$$\frac{\partial}{\partial\ell}\frac{df}{dt} = \frac{\partial}{\partial\ell}\left(\frac{\partial f}{\partial t} + \hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp\phi)\right)$$
$$= \left(\frac{\partial}{\partial t}\frac{\partial f}{\partial\ell} - \hat{z}\cdot\left(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp\frac{\partial H}{\partial t}\right)\right)$$
$$+\frac{\partial}{\partial\ell}\hat{z}\cdot(\vec{\nabla}_\perp f\times\vec{\nabla}_\perp\phi)$$

$$
\begin{aligned}
&= \frac{d}{dt}\frac{\partial f}{\partial \ell} - \hat{z}\cdot(\vec{\nabla}_\perp f \times \vec{\nabla}_\perp(\frac{\partial H}{\partial t} - \frac{\partial \phi}{\partial \ell}) \\
&= \frac{d}{dt}\frac{\partial f}{\partial \ell} + \hat{z}\cdot(\vec{\nabla}_\perp f \times \vec{\nabla}_\perp \mathcal{N}_B). \qquad \text{(C8)}
\end{aligned}
$$

5. Form for $\hat{z}\cdot(\vec{\nabla}_\perp f \times \vec{\nabla}_\perp g)$ useful in integrations

$$
\hat{z}\cdot(\vec{\nabla}_\perp f \times \vec{\nabla}_\perp g) = \frac{\partial}{\partial x}\left(f\frac{\partial g}{\partial y}\right) - \frac{\partial}{\partial y}\left(f\frac{\partial g}{\partial x}\right). \quad \text{(C9)}
$$

6. Coordinate invariance properties of $\hat{z}\cdot(\vec{\nabla}_\perp f \times \vec{\nabla}_\perp g)$

In arbitrary coordinates $x_a(x,y)$ and $y_a(x,y)$, the gradient $\vec{\nabla}_\perp f = (\partial f/\partial x_a)\vec{\nabla}x_a + (\partial f/\partial y_a)\vec{\nabla}y_a$, which implies

$$
\begin{aligned}
\hat{z}\cdot(\vec{\nabla}_\perp f \times \vec{\nabla}_\perp g) &= \frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x} \\
&= \left(\frac{\partial f}{\partial x_a}\frac{\partial g}{\partial y_a} - \frac{\partial f}{\partial y_a}\frac{\partial g}{\partial x_a}\right)\mathcal{J}_a; \text{(C10)}
\end{aligned}
$$

$$
\mathcal{J}_a \equiv \hat{z}\cdot(\vec{\nabla}x_a \times \vec{\nabla}y_a). \qquad \text{(C11)}
$$

The Jacobian of the (x_a, y_b) coordinates $\mathcal{J}_a$ is unity when the coordinates are the starting points (x_s, y_s) of magnetic field line trajectories.

7. Cartesian coordinate expression for $\frac{\partial^2 f}{\partial \ell^2}$

$$
\begin{aligned}
\frac{\partial^2 f}{\partial \ell^2} &= \frac{\partial}{\partial z}\frac{\partial f}{\partial \ell} + \hat{z}\cdot(\vec{\nabla}_\perp \frac{\partial f}{\partial \ell} \times \vec{\nabla}_\perp H) \\
&= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial z} + \hat{z}\cdot(\vec{\nabla}_\perp f \times \vec{\nabla}_\perp H)\right) \\
&\quad + \left\{\hat{z}\cdot(\vec{\nabla}_\perp\left(\frac{\partial f}{\partial z} + \hat{z}\cdot(\vec{\nabla}_\perp f \times \vec{\nabla}_\perp H)\right) \times \vec{\nabla}_\perp H)\right\} \\
&\quad = \frac{\partial^2 f}{\partial z^2} + 2\hat{z}\cdot(\vec{\nabla}_\perp \frac{\partial f}{\partial z} \times \vec{\nabla}_\perp H) \\
&\qquad + \hat{z}\cdot(\vec{\nabla}_\perp f \times \vec{\nabla}_\perp \frac{\partial H}{\partial z}) \qquad \text{(C12)}
\end{aligned}
$$

[1] N. F. Loureiro and D. A. Uzdensky, *Magnetic reconnection: from the Sweet-Parker model to stochastic plasmoid chains*, Plasma Phys. and Control. Fusion **58** 014021 (2016).

[2] A. H. Boozer, *Model of magnetic reconnection in space and astrophysical plasmas*, Phys. Plasmas **20**, 032903 (2013).

[3] A. H. Boozer, *Formation of current sheets in magnetic reconnection*, Phys. Plasmas **21**, 072907 (2014).

[4] E. G. Harris, *On a plasma sheath separating regions of oppositely directed magnetic fields*, Nuovo Cimento, **23**, 115-121, (1962).

[5] B. B. Kadomtsev and O. P. Pogutse, *Nonlinear helical perturbations of a plasma in the tokamak*, Sov. Phys.-JETP, **38**, 283-290 (1974).

[6] H. R. Strauss, *Nonlinear, 3-dimensional magnetohydrodnamics of noncircular tokamaks*, Phys. Fluids **19**, 134-140 (1976).

[7] A. A. Ballegooijn, *Electric currents in the solar corona and the existence of magnetostatic equilibrium*, Ap. J. **298**, 421-430 (1985).

[8] C. S. Ng, L. Lin, and A. Bhattacharjee, *High-Lundquist number scaling in three-dimensional simulations of Parker's model of coronal heating*, Astrophys. J., **747**, 109 (2012).

[9] A. H. Boozer, *Evaluation of the structure of ergodic Fields*, Phys. Fluids **26**, 1288-1291 (1983).

[10] A. H. Boozer, *Non-axisymmetric magnetic fields and toroidal plasma confinement*, Nucl. Fusion **55**, 025001 (2015).

[11] J. Lighthill, *The recently recognized failure of predictability in Newtonian dynamics*, Proceedings of the Royal Society, Series A, **407**, 35-50 (1986).

[12] Yi-Min Huang, A. Bhattacharjee, and A. H. Boozer, *Rapid change in field line connectivity and reconnection in stochastic magnetic fields*, Ap. J. **793**, 106 (2014).

[13] W. Daughton, T. K. M. Nakamura, H. Karimabadi, V. Roytershteyn, and B. Loring,*Computing the reconnection rate in turbulent kinetic layers by using electron mixing to identify topology*, Phys. Plasmas **21**, 052307 (2014).

[14] P.L. Similon and R. N. Sudan, *Energy-dissipation of Alfvén-wave packets deformed by irregular magnetic-fields in solar-coronal arches*, Ap. J. **336**, 442-453 (1989).

[15] A. H. Boozer, *Runaway electrons and ITER*, Nucl. Fusion **57**, 056018 (2017).