CMSO/CMPD Winter School 2006.

Lecture # 1. Frozen Field Lines and Diffusion.

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These notes contain many details that cannot be treated in the lecture.

1 Introduction.

In this lecture I will introduce the concept of frozen field lines. I will also discuss in general terms the conditions for field lines to be frozen and how they "break" or "reconnect". We cannot really start talking about reconnection until we understand when lines don't reconnect. Many of you know this material but you may find a review helpful. I have gone beyond the usual material treated in MHD text books – as you will see. I will not treat the Lagrangian description of field line motion (see *Plasma Astrophysics* by Russell Kulsrud, Princeton, 2004.), the Hamiltonian description of field structure (see JR Cary and RG Littlejohn, Ann. Phys. (NY) 151, 1 (1982)) or the description of magnetic fields in terms of potentials.

First consider how we define a field line *at a fixed instant of time.* The unit vector $\mathbf{b}(\mathbf{r},t) = \mathbf{B}/|\mathbf{B}|$ is tangent to the field line at a point $\mathbf{r}$. Thus a small vector along the field line $\delta\mathbf{r}$ is parallel to $\mathbf{b}(\mathbf{r},t)$ and we can follow the field line by taking small steps in the direction of $\mathbf{b}(\mathbf{r},t)$. Mathematically we can find the field line passing through some point $\mathbf{r}_0$ by integrating the equation:

$$\frac{d\mathbf{r}}{dl} = \mathbf{b}(\mathbf{r},t)$$

where $dl = |d\mathbf{r}|$ and t is fixed.

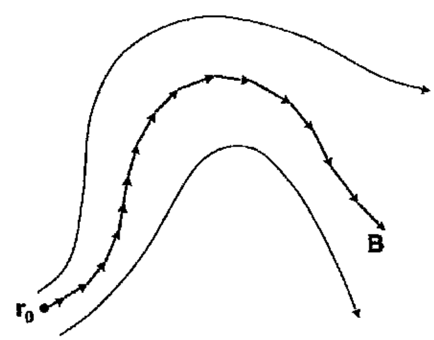

Figure 1: Field line integration passing through $\mathbf{r}_0$.

The evolution of the magnetic field comes, of course, from Faraday's law:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}. \tag{1}$$

While we can always define field lines at a fixed time, it is not possible to define how they *move* in general. However in many situations it *is* possible to define field line motion because the behavior of electrons constrains the form of the electric field – this is the subject of this lecture. First we examine the electron equations and how they constrain the form of **E**.

2 Electron Dynamics – Ohm's Law.

The electron fluid momentum equation (the **v** moment of the electron Fokker-Planck equation) can be rearranged as an equation for the electric field – this is usually referred to as the generalized Ohm's law.

$$\mathbf{E} + \mathbf{v_e} \times \mathbf{B} = \mathbf{R} - \frac{\nabla \cdot \mathbf{P_e}}{en_e} - \frac{m_e}{e}\left(\frac{\partial \mathbf{v_e}}{\partial t} + \mathbf{v_e} \cdot \nabla \mathbf{v_e}\right). \tag{2}$$

where

- **R** is the collisional momentum exchange between the electrons and the ions. In the collisional limit it is usually replaced by $\eta \mathbf{J}$ where the resistivity is $\eta \sim m_e \nu_{ei}/(n_e e^2)$ and ν_{ei} is the electron ion collision rate.
- **P** is the electron pressure tensor. In the collisional limit it becomes isotropic ($\mathbf{P} = p_e \mathbf{I} = n_e T_e \mathbf{I}$) and $\frac{\nabla \cdot \mathbf{P_e}}{en_e} = \frac{\nabla p_e}{en_e}$.

In principle we substitute E from Eq. (2) into Faraday's law to find the evolution of **B** – in many applications some of the terms in Ohm's law are small and can be ignored. For definiteness let us consider a solar flare of size L and timescale τ with parameters:

$$B \sim 10^{2-3}G, \quad T_e \sim T_i \sim 100eV, \quad L \sim 10^7 m, \quad \tau \sim 100s \quad n_e \sim 2 \times 10^{14} m^{-3}.$$

Thus assuming the plasma moves roughly the size L in time τ the typical velocity is $v \sim v_e \sim L/\tau \sim 10^5 ms^{-1}$ which is also typical of the ion thermal velocity, v_{thi}. The mean free path is $\sim 6 \times 10^5 m$ and the ion larmor radius $\sim \rho_i \sim 0.1m$ and the plasma beta $\beta \sim 10^{-2}$. Let's estimate the size of various terms relative to the $\mathbf{v_e} \times \mathbf{B}$ term:

- $\frac{|\eta \mathbf{J}|}{|\mathbf{v_e} \times \mathbf{B}|} \sim \frac{\mu_0 \eta B/L}{vB} \sim 10^{-12}$ where we have used Ampere's law $\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$.
- $\frac{\nabla p_e}{en_e|\mathbf{v_e} \times \mathbf{B}|} \sim \frac{v_{thi}}{v_e}\frac{\rho_i}{L} \sim 10^{-8}$ the plasma is collisional enough that the pressure is isotropic.
- $m_e\left(\frac{\partial \mathbf{v_e}}{\partial t} + \mathbf{v_e} \cdot \nabla \mathbf{v_e}\right)/(e|\mathbf{v_e} \times \mathbf{B}|) \sim \frac{m_e}{m_i}\frac{\rho_i}{L} \sim 5 \times 10^{-12}$.

Thus to a very good approximation we have $\mathbf{E} + \mathbf{v_e} \times \mathbf{B} = 0$ – as we shall see this implies the field is *frozen to the electrons*. The plasma is roughly charge neutral since L is much larger than the Debye length – *i.e.* using Poisson's equation we have $\epsilon_0 \nabla \cdot \mathbf{E}/n_e \sim (n_i - n_e)/n_e < 10^{-13}$. Thus $\mathbf{v_e} = \mathbf{v_i} - \mathbf{J}/(en_e)$ and we can write the collisional Ohm's law as:

$$\mathbf{E} + \mathbf{v_i} \times \mathbf{B} = \eta \mathbf{J} - \frac{\nabla p_e}{en_e} + \frac{\mathbf{J} \times \mathbf{B}}{en_e} - \frac{m_e}{e}\left(\frac{\partial \mathbf{v_e}}{\partial t} + \mathbf{v_e} \cdot \nabla \mathbf{v_e}\right). \tag{3}$$

The extra term, $\frac{\mathbf{J} \times \mathbf{B}}{en_e}$, is usually referred to as the *Hall Term* and it is a noniminal factor β^{-1} bigger than the electron pressure term but still much smaller (by a factor of 10^{-6}) than the $\mathbf{v_i} \times \mathbf{B}$ term. In MHD we denote $\mathbf{v_i}$, the mass flow, by **v**. Thus we arrive at the conclusion that to a very good approximation:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0. \tag{4}$$

This equation is often called the *Ideal Ohms Law.* It is easily shown from relativity that in this approximation the electric field in the frame moving with the plasma (*i.e.* at velocity $\mathbf{v}$) is zero – in this sense the plasma is a perfect conductor. Substituting Eq. (4) into Faraday's law Eq. (1) we obtain an evolution equation for the magnetic field:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{5}$$

This is the ideal MHD equation for the evolution of magnetic field. Since Eq. (22) is approximately correct for many plasmas we need to understand it's consequences in detail.

3 Flux Freezing and Line Freezing.

In this section we examine the consequences of Eq. (4). Suppose we consider a plasma moving at a constant velocity $\mathbf{v}$. In the frame moving with the plasma the electric field is zero and therefore by Faraday's law (Eq. (1) the magnetic field is constant in this frame. This is trivial and obvious but we can generalize this for velocities that are not constant in time or space – *i.e.* $\mathbf{v}(\mathbf{r},t)$. There are two useful theorems.

3.1 Frozen Flux.

The first theorem we want to prove is:

Flux Freezing. If $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ the magnetic flux through a loop that moves with the plasma flow, $\mathbf{v}(\mathbf{r}, t)$, is constant in time.

We should clarify what we mean by a loop moving with the plasma flow. Every point on the loop $\mathbf{r}(t)$ satisfies the dynamical equation $\frac{d\mathbf{r}}{dt} = \mathbf{v}(\mathbf{r}, t)$. You know what this means intuitively if you imagine a thin loop of cotton thread in a stream moving and being deformed by the flow.

Proof of Flux Freezing. Consider the magnetic flux through a loop L spanned by a surface S at time t:

$$\Phi(t) = \int_S d\mathbf{A} \cdot \mathbf{B}(\mathbf{r}, t). \tag{6}$$

Where $d\mathbf{A}$ is an element of the surface. This is illustrated in Fig. (2). Note that because $\nabla \cdot \mathbf{B} = 0$ this flux is independent of the choice of surface spanning the loop. Now consider the loop an infinitesimal time δt later after it has been moved with the flow. Each point on the loop has moved a distance $\mathbf{v}(\mathbf{r}, t)\delta t$. The moving loop traces out a tube joining the original loop to the current position of the loop. The flux through the loop at time $t + \delta t$ can be calculated through a surface which consists of the original surface S plus the surface of the tube which we call δS. Thus the flux through the loop at time $t + \delta t$ can be written:

$$\Phi(t + \delta t) = \int_S d\mathbf{A} \cdot \mathbf{B}(\mathbf{r}, t + \delta t) + \int_{\delta S} d\mathbf{A} \cdot \mathbf{B}(\mathbf{r}, t + \delta t). \tag{7}$$

See Fig. (3). To first order in δt an element of surface area of the tube can be written as $d\mathbf{A} = \mathbf{v}(\mathbf{r}, t)\delta t \times d\mathbf{l}$ where $d\mathbf{l}$ is an element of length around the original loop (see Fig. (3)). Thus expanding Eq. (7) to first order in δt and using Eq. (6) we obtain the rate of change of flux through the moving loop:

$$\frac{d\Phi}{dt} = \int_S d\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial t} - \oint d\mathbf{l} \cdot [\mathbf{v} \times \mathbf{B}], \tag{8}$$

where the second integral is taken around the loop. The final step is to use Fraraday's law (Eq. (1)) to eliminate $\frac{\partial \mathbf{B}}{\partial t}$ and Stoke's theorem to write the surface integral of $\nabla \times \mathbf{E}$ as a line integral. Thus,

$$\frac{d\Phi}{dt} = -\oint d\mathbf{l} \cdot [\mathbf{E} + \mathbf{v} \times \mathbf{B}]. \tag{9}$$

Clearly when Eq. (4) holds the flux through the loop remains constant – thus our proof is complete. An important point (that we return to later) is that we have taken the velocity of the loop to be the plasma velocity $\mathbf{v}$; however Eq. (9) holds for any velocity of the loop if we replace $\mathbf{v}$ with the loop velocity.

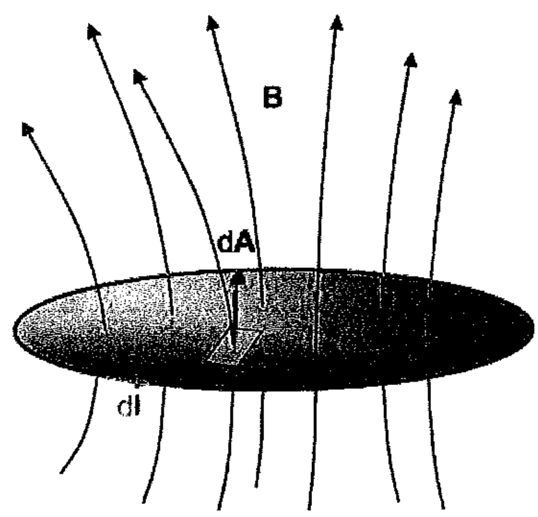

Figure 2: Diagram illustrating flux through in loop in plasma. See text for explanation.

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[1]The concept of frozen flux in MHD is due to Hannes Alfvén in 1945. The proof is the same as the much earlier proof by Cauchy and Kelvin that vorticity is frozen into Euler flows – see Horace Lamb's book *Hydrodynamics* Dover.

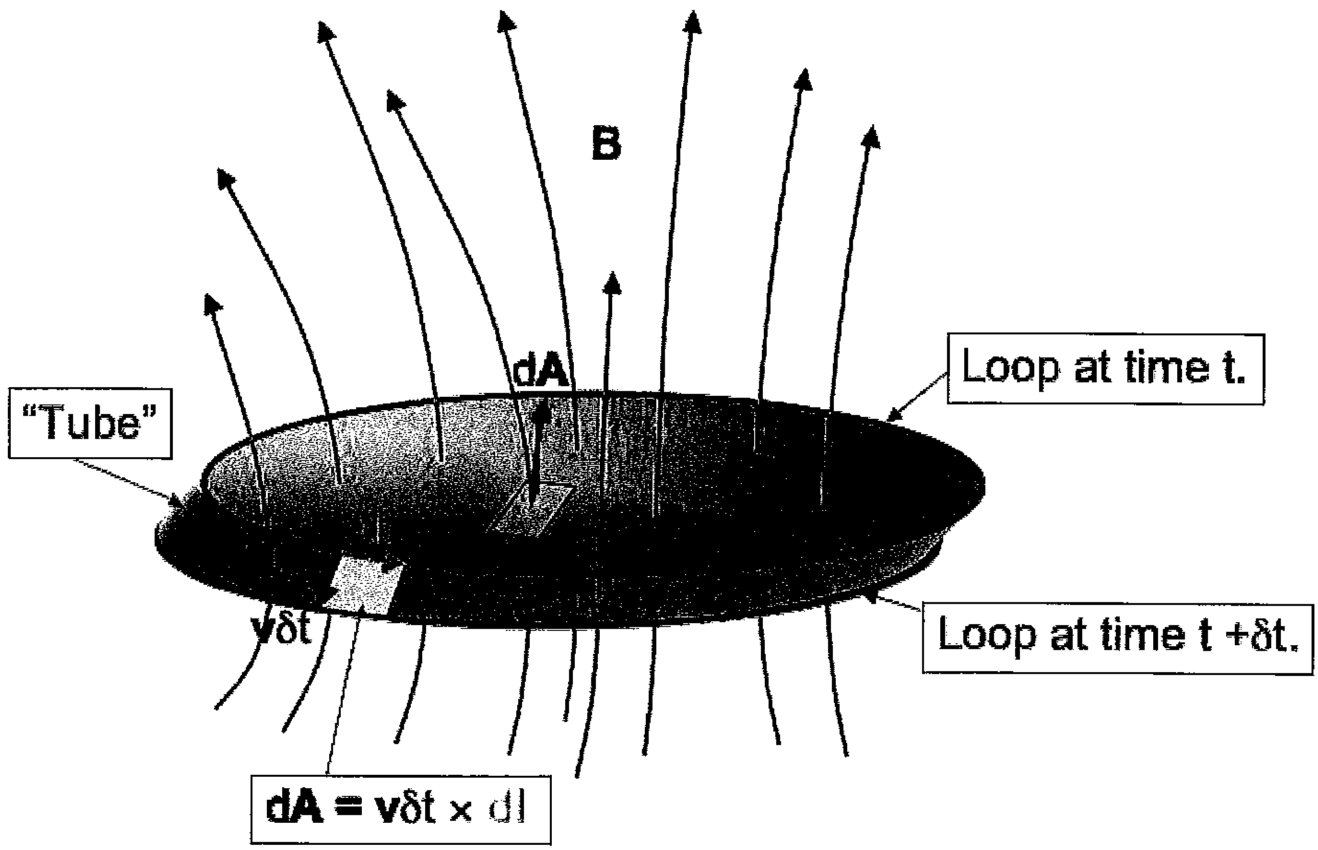

Figure 3: Diagram illustrating new spanning surface for loop at time $t + \delta t$. See text for details.

3.2 Frozen Field Lines.

A second theorem follows directly from the frozen in flux theorem. This is:

Frozen Field Lines. If $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ the magnetic field lines change as though they are simply convected with velocity $\mathbf{v}$. Thus we say that the field lines are frozen to the plasma flow.

Proof Consider a narrow tube surrounding a field line at time t – we call such a tube a *flux tube* (see Fig. (4). The sides of the tube are parallel to the field and therefore no flux passes through any patch on the side of the tube. We follow the tube of plasma until a time T later. Since the flux through any patch on the surface is zero at all times from the *Frozen Flux Theorem* the field inside the tube cannot leave the tube through the sides. The flux through the end of the tube stays constant. Thus we can say that the field line is always inside the tube moving with the plasma. Shrinking the tube to zero thickness we can say that the field line is moving with the plasma flow.

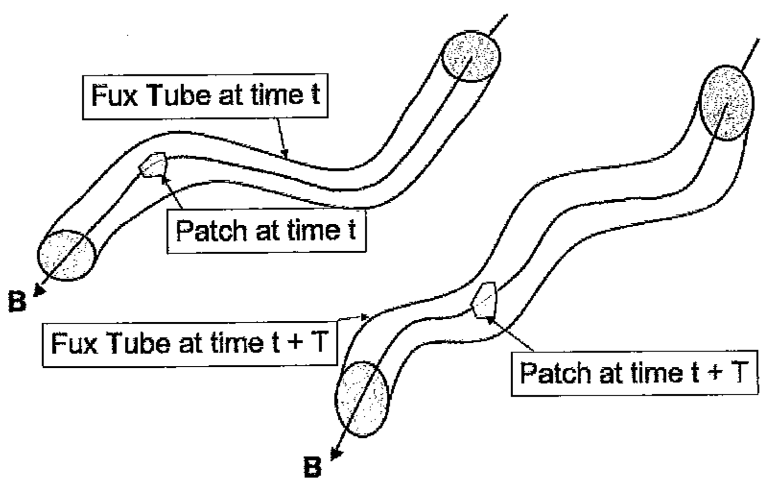

Figure 4: A tube of plasma with field line inside moving with plasma. The field line cannot leave the tube as there can be no flux through any of the moving "patches" on the surface. Thus we say that the field line is frozen into the plasma.

The flux freezing and frozen field line concepts are enormously helpful to intuition – for example if a flux tube narrows in time the field must be getting stronger to preserve flux through the ends. They also show that as long as the plasma flow $\mathbf{v}$ is single valued, field lines cannot pass through each other, join or break when $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$. *To reconnect field lines we must include at least one of the terms in the Ohms law that we neglected.*

4 Resistive Evolution.

Perhaps the simplest non ideal effect that allows the reconnection of field lines is the resistive term. It was also the first to be considered in the 1950s by Dungey, Parker, Sweet and others. The resistive Ohms law is:

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}. \tag{10}$$

In most cases reconnection is non-relativistic and one can ignore the displacement current. Thus,

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{11}$$

Substituting for $\mathbf{E}$ in Faraday's law (Eq. (1) we obtain:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) - \eta \nabla \times \mathbf{J} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \mu_0 \eta \nabla^2 \mathbf{B}. \tag{12}$$

As we have discussed the $\mathbf{v} \times \mathbf{B}$ term convects the field lines. The resistive term, $\mu_0 \eta \nabla^2 \mathbf{B}$, diffuses the field lines – without the $\mathbf{v} \times \mathbf{B}$ term the equation becomes the diffusion equation. [2] It is common to define a dimensionless number that expresses the relative size of the resistive term. Actually there are two such numbers:

$$S = \frac{V_A L}{\mu_0 \eta} \qquad \textit{Lunquist Number}, \tag{13}$$

$$R_M = \frac{vL}{\mu_0 \eta} \qquad \textit{Magnetic Reynolds Number}. \tag{14}$$

Where $V_A = \sqrt{B^2/(\mu_0 \rho)}$ is the Alfvén velocity, v is a typical flow velocity and L a typical length. In many cases both dimensionless numbers are large – in the solar corona case (see above) $R_M \sim 10^{12}$ and $S \sim 10^{13}$. Some papers normalize Eq. (12) so that lengths are measured in units of L, velocity in units of V_A and time in units of l/V_A. Then

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{S} \nabla^2 \mathbf{B}. \tag{15}$$

Consider a simple problem where $\mathbf{v} = 0$ and $\mathbf{B} = \mathbf{B}_0(t) \exp i\mathbf{k} \cdot \mathbf{r}$. Substituting in Eq. (12) we obtain:

$$\mathbf{B} = \hat{\mathbf{B}}_0 \exp(i\mathbf{k} \cdot \mathbf{r} - \mu_0 \eta k^2 t). \tag{16}$$

The field decays due to resistive diffusion – slowly unless k is large and the field is at very small scale. *To make magnetic field lines reconnect when resistivity is small we must create a small scale variation in* $\mathbf{B}$ *so that the diffusion term is enhanced.* All reconnection mechanisms have a narrow layer where the actual breaking and joining of lines takes place.

[2] For a hydrogen plasma (with the Coulomb logarithm taken to be 15) $\mu_0 \eta \sim 10^3 T_e^{-3/2} m^2 s^{-1}$ where T_e is the electron temperature in electron volts.

5 Hall and Pressure Terms.

In most tokamak turbulence and some astrophysical turbulence the turbulent scales become of order the ion larmor radius. Do we expect the field lines in this turbulence to reconnect? In this section I will show you that although the Ohms law must include the Hall and electron pressure terms the field lines do not reconnect. This is also a useful result to keep in mind when you hear about Hall reconnection later in the week. Taking typical scales for *Ion Temperature Gradient Turbulence* in the JET tokamak we can simplify the electron equation, Eq. (2), to:

$$\mathbf{E} + \mathbf{v_e} \times \mathbf{B} = -\frac{\nabla p_e}{e n_e}. \tag{17}$$

We can also use the fact that the conduction of electron heat along the field line is so fast that:

$$\mathbf{B} \cdot \nabla T_e = 0. \tag{18}$$

We expand the pressure term as:

$$\begin{aligned} \frac{\nabla p_e}{e n_e} &= \frac{\nabla T_e}{e} + \frac{\nabla (T_e \ln n_e)}{e} - \ln n_e \frac{\nabla T_e}{e} \\ &= \frac{\nabla T_e}{e} + \frac{\nabla (T_e \ln n_e)}{e} + \frac{\ln n_e}{e B^2} (\nabla T_e \times \mathbf{B}) \times \mathbf{B}. \end{aligned} \tag{19}$$

Where we have used Eq. (18) to write the last term as a cross product with $\mathbf{B}$. We can write Eq. (17) as:

$$\mathbf{E} + \tilde{\mathbf{v}} \times \mathbf{B} = -\nabla \chi. \tag{20}$$

where we have defined the *field line velocity* $\tilde{\mathbf{v}}$ as:

$$\tilde{\mathbf{v}} = \mathbf{v_e} + \frac{\ln n_e}{e B^2} (\nabla T_e \times \mathbf{B}), \tag{21}$$

and $\chi = T_e (1 + \ln n_e)/e$. Substituting Eq. (20) into Eq. (1) we obtain,

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\tilde{\mathbf{v}} \times \mathbf{B}). \tag{22}$$

Thus the flux and field lines are frozen to a fictitious substance moving with the velocity $\tilde{\mathbf{v}}$ – which is neither the plasma nor electron velocity. Clearly in this approximation, where we include both the Hall and pressure terms, reconnection is impossible. Note this conclusion is independent of the ion dynamics, which for tokamak turbulence can be quite complicated. Of, course when we include the other terms in Ohms law reconnection is allowed – but these terms are generally small in tokamaks.

6 Conclusions

In this lecture we have looked at the consequences of simplified electron dynamics on the evolution of the magnetic field. We have proved the frozen in flux and field line theorems for the simple ideal Ohms law. The importance of including resistivity at small scales is also discussed. Finally in Section. (5) we have shown that when the pressure and Hall terms are included (and $\mathbf{B} \cdot \nabla T_e = 0$) the field is frozen into a fictitious substance moving with a velocity $\tilde{\mathbf{v}}$ defined in Eq. (21).

7 Homework

Question 1.: 2D Reconnection. Suppose we have a 2D resistive case with $\mathbf{E}+\mathbf{v}\times\mathbf{B} = \eta\mathbf{J}$. We take $\mathbf{v} = (v_x(x,y,t), v_y(x,y,t), 0)$ and $\mathbf{B} = (B_x(x,y,t), B_y(x,y,t), 0)$. Use Ampere's law to show that $\mathbf{J} = (0,0,J_z(x,y,t))$. Show that we can find a $\hat{\mathbf{v}}$ such that: $\mathbf{E}+\hat{\mathbf{v}}\times\mathbf{B} = 0$ – give an expression for $\hat{\mathbf{v}}$. Does this mean there is no reconnection in two dimensions? If not why not.

Question 2.: Line Conservation. Does line conservation imply flux conservation? If not, construct a counter example.

Question 3.: Helicity Conservation. Show that the helicity $\int d^3\mathbf{r}(\mathbf{A}\cdot\mathbf{B})$ (where the vector potential is defined by $\nabla\times\mathbf{A} = \mathbf{B}$) is constant if $\mathbf{E}+\mathbf{v}\times\mathbf{B} = 0$ and the integration is over a domain such that $\mathbf{B}$ is tangent to the solid boundary.

Lecture # 3. Introduction to Gyrokinetics.

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This lecture is meant to introduce the simplest ideas in gyro-kinetics. It would take at least 5 lectures to develop the theory in all its detail, but hopefully the key ideas can be communicated in 1 hour. We shall be examining the approximation in a non-uniform slab *in the electrostatic approximation* since the full electromagnetic gyro-kinetic treatment in general geometry is far too complicated. Hence I will derive the equation in a non-uniform equilibrium with straight field lines – *i.e.* we take the slowly evolving equilibrium to be $F_0 = F_0(x, \mathbf{v}, t)$ and $\mathbf{B}_0 = B_0(x,t)\mathbf{z}$. We shall take the plasma to be in a periodic box in the y and z planes and effectively infinite in x. Even this simplified geometry is algebraically complex and I shall skip some details where the derivation is obvious but tediously long. **These notes provide considerable detail – a lot of this detail is impossible to convey in the lecture but I hope you will read the notes at your leisure.**

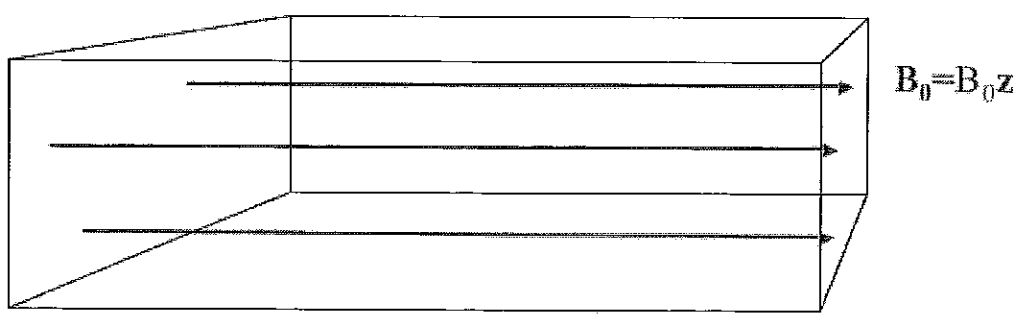

Figure 1: Slab equilibrium with straight field. Periodicity in y and z is assumed. The plasma is non-uniform in the x direction.

1 Gyro-kinetic Ordering

Length Scales. There are two basic length scales:

- Macroscopic length L – might be size of plasma, or the density gradient length $(n/|\nabla n|)$ *etc.*.
- Microscopic length, the larmor radius ρ – usually the ion larmor radius ρ_i.

For example in ITER these lengths are approximately: $L \sim n/|\nabla n| \sim 2m$ and $\rho_i \sim \times 10^{-3} m$. We use these length scales to define the fundamental small parameter of the theory:

$$\epsilon = \frac{\rho}{L} \ll 1 \tag{1}$$

In ITER $\epsilon \leq 10^{-3}$ – a good expansion parameter.

Time Scales. There are *three* basic frequency scales:

- The *fast* cyclotron frequency – Ω. On ITER $\Omega_{ci} \sim 2 \times 10^8 rad/s$.
- The *medium* frequency – $\omega = v_{th}/L \sim \epsilon\Omega$. This is roughly the frequency of the turbulent fluctuations and the rate at which particles sense the inhomogeneity. On ITER $\omega_{ci} \sim \times 10^5 rad/s$
- The *slow* transport rate – $1/\tau = (v_{th}/L)\epsilon^2 \sim \epsilon^3\Omega$. On ITER $\tau \sim 3s$.

The fluctuating density and electric field in current fusion devices is small – $\delta n/n_0 < 0.01$. Therefore we split the distribution functions and fields into slowly varying (in time and space) equilibrium parts and medium time-scale fluctuating parts that vary fast in space. I will suppress any species label and deal for simplicity until we need to discuss electrons and ions separately. We define for the distribution functions:

$$f(\mathbf{r}, \mathbf{v}, t) = F_0(x, \mathbf{v}, t) + \delta f_1(\mathbf{r}, \mathbf{v}, t) + +\delta f_2(\mathbf{r}, \mathbf{v}, t) \;\; .... \tag{2}$$

and for the fields

$$\mathbf{B}(\mathbf{r}, t) = B_0(x, t)\mathbf{z}, \quad \mathbf{E}(\mathbf{r}, t) = \delta\mathbf{E}(\mathbf{r}, t) = -\nabla\phi(\mathbf{r}, t) \tag{3}$$

Now we outline the ordering of all the quantities and their variations in time and space.

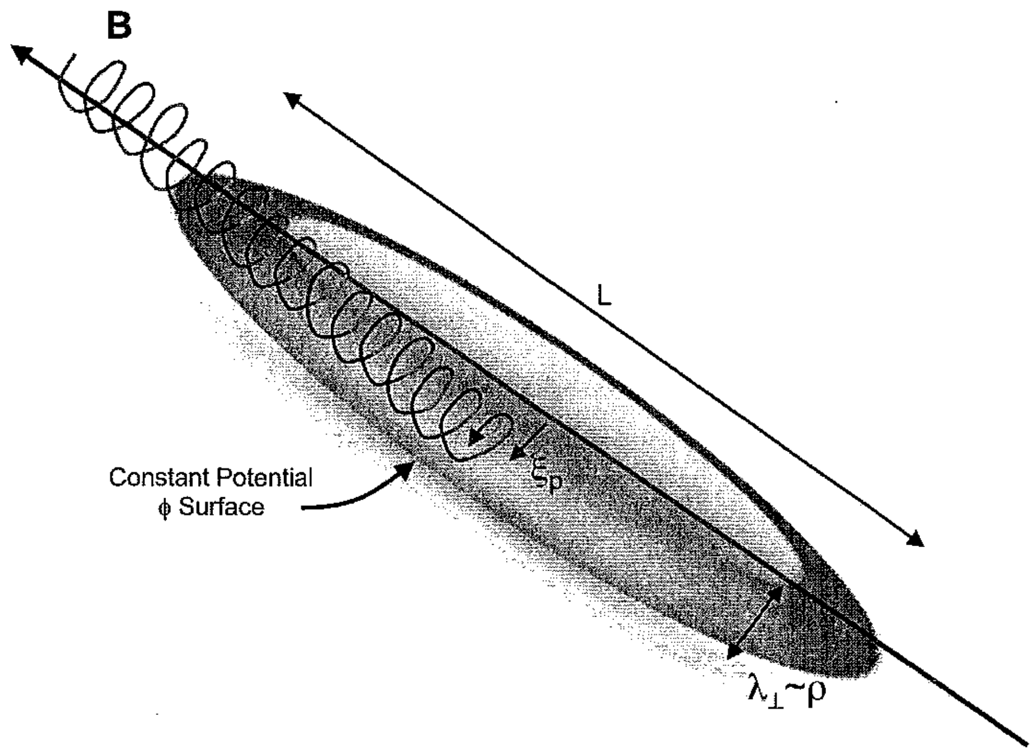

Figure 2: Gyro-kinetic fluctuations, space-scales. Typical electrostatic fluctuation makes cigar shaped potential surface with $L \gg \lambda_\perp \sim \rho_i$. Particle drift off field line gives a step of order the larmor radius, $\xi_p \sim \rho$
.

Small Fluctuations The fluctuations are order ϵ in the gyro-kinetic expansion *i.e.*

$$\frac{\delta f_1}{F_0} \sim \mathcal{O}(\epsilon), \qquad \frac{\delta f_2}{F_0} \sim \mathcal{O}(\epsilon^2) \quad \text{.....} \; etc. \qquad \frac{|\delta \mathbf{E}|}{|v_{th}\mathbf{B}_0|} \sim \mathcal{O}(\epsilon), \qquad \frac{e\phi}{T} \sim \mathcal{O}(\epsilon). \tag{4}$$

Slowly varying Equlilibrium The equilibrium varies in space on the macroscopic length scale and in time on the transport time τ, *i.e.*

$$\nabla F_0 \sim \mathcal{O}(\frac{F_0}{L}), \qquad \nabla \mathbf{B_0} \sim \mathcal{O}(\frac{\mathbf{B_0}}{L}), \tag{5}$$

$$\frac{\partial F_0}{\partial t} \sim \mathcal{O}(\frac{F_0}{\tau}) \sim \mathcal{O}(\frac{v_{th}}{L}\epsilon^2 F_0), \qquad \frac{\partial \mathbf{B_0}}{\partial t} \sim \mathcal{O}(\frac{\mathbf{B_0}}{\tau}) \sim \mathcal{O}(\frac{v_{th}}{L}\epsilon^2 \mathbf{B_0}), \tag{6}$$

Fast Spatial Variation of Fluctuations across $\mathbf{B_0}$. The variation of the fluctuating quantities across the magnetic field is on the microscopic length scale, *i.e.*

$$|\mathbf{b_0} \times \nabla \delta f| \sim \mathcal{O}(\frac{\delta f}{\rho}), \qquad |\mathbf{b_0} \times \nabla | \delta \mathbf{E} \sim \mathcal{O}(\frac{\delta \mathbf{E}}{\rho}), \tag{7}$$

where $\mathbf{b_0} \sim \frac{\mathbf{B_0}}{B_0}$ is the unit vector along $\mathbf{B_0}$. We will often loosely write $k_\perp$ to mean the approximate inverse perpendicular scale, thus $k_\perp \rho \sim 1$.

Slow Spatial Variation Along $\mathbf{B_0}$ The variation of the fluctuating quantities along the magnetic field is on the macroscopic length scale, *i.e.*

$$\mathbf{b_0} \cdot \nabla \delta f \sim \mathcal{O}(\frac{\delta f}{L}), \qquad \mathbf{b_0} \cdot \nabla \delta \mathbf{E} \sim \mathcal{O}(\frac{\delta \mathbf{E}}{L}), \tag{8}$$

note that $\delta E_{\|} \sim \mathcal{O}(\epsilon \delta E_{\perp})$ follows directly from applying these orderings to ϕ.

Medium Time Scale Variation of Fluctuations. The fluctuating quantities vary on the mediium time scale, *i.e.*

$$\frac{\partial \delta f}{\partial t} \sim \mathcal{O}(\frac{v_{th} \delta f}{L}), \qquad \frac{\partial \delta \mathbf{E}}{\partial t} \sim \mathcal{O}(\frac{v_{th} \delta \mathbf{E}}{L}) \tag{9}$$

Optional. **Collisions Act on the Medium Time Scale.** We make this specific specific ordering of the collision rate that is consistent with the physical situations we wish to describe.

$$\nu \sim \mathcal{O}(\frac{v_{th}}{L}) \tag{10}$$

These orderings have the simple consequences for the fluctuations illustrated in Figure 2. Specifically: the typical perpendicular flow velocity, roughly the $\mathbf{E} \times \mathbf{B}$ velocity, is of order ϵv_{th}; the typical fluid displacement is roughly $\xi_p \sim \rho$. Note also that $\nabla \delta f \sim \mathcal{O}(\nabla f)$ – *i.e.* the perturbed gradients are comparable with the equilibrium gradients. Thus the fluctuations can locally flatten the gradients driving the turbulence.

2 Field Equations.

Since, $\nabla \cdot \delta \mathbf{E} = -\nabla^2 \phi$, Maxwell's equations reduce to Poisson's equation for the fluctuations.

Poisson's Equation.

$$\nabla^2 \phi = -\frac{1}{\epsilon_0}(q n_i - e n_e) = -\frac{1}{\epsilon_0}(q \int d^3\mathbf{v} \delta f_i - e \int d^3\mathbf{v} \delta f_e) \tag{11}$$

where n_i and n_e are the ion and electron densities. When $k_\perp^{-1}$ is long compared to the Debye length one can drop the left hand side of Poisson's equation and obtain *quasi-neutrality* – *i.e.* $q n_i = e n_e$.

Clearly we must solve for the distribution functions of ions and electrons to obtain the charge. In the slab equilibrium we have equilibrium currents in the y direction and these should be balance the variation of B_0. This just gives the equilibrium relation $p(x) + B_0^2(x)/(2\mu_o) = \textit{constant}$ and it will not be needed here. I have dropped the equilibrium *Electric Field* $\mathbf{E}_0$. since because the variation of B_0 is slow in time does not enter the equations at the order we want to keep.

3 Gyro-Kinetic Particle Motion

Before we plough through the derivation of the gyro-kinetic equation and sweat over the algebra we can gain a little physical insight by looking at the single particle motion in the gyro-kinetic ordering. I will start this in a general slowly varying field and then give the non-uniform slab result. First we define the gyro-center position by a vector version of the simple uniform field (slab) result.

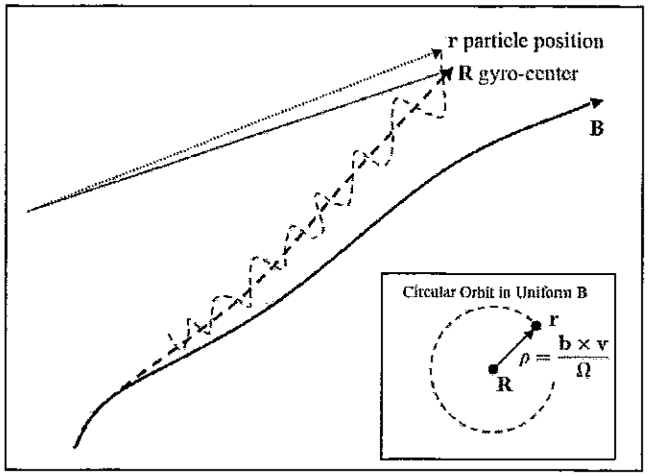

Figure 3: Defining the Gyro-center.

The exact gyro-center position is not actually a well defined quantity. However to lowest and first order our ordering shows that the particle orbit looks locally to be like the orbit in a uniform field. Thus we define (see Figure 2.) the gyro-center position $\mathbf{R}$ in terms of the particle position r and particle velocity $\mathbf{v}$:

$$\mathbf{R} = \mathbf{r} + \frac{\mathbf{v} \times \mathbf{b}_0}{\Omega_0} \tag{12}$$

where (as before) $\mathbf{b}_0 = \mathbf{b}_0(\mathbf{r}) = \mathbf{B}_0/B_0$ is the unit vector along the local equilibrium field and $\Omega_0 = \Omega_0(\mathbf{r}) = qB_0/m$ is the local equilibrium gyro-frequency. The transformation to gyro-center position is sometimes called the *Catto Transformation* after its inventor. We define the perpendicular, $v_\perp$, and parallel, $v_\|$, and gyro-angle, θ with respect to the equilibrium field from the expression:

$$\mathbf{v} = v_\| \mathbf{b}_0 + v_\perp(\cos\dot{\theta}\, \mathbf{e}_1 + \sin\theta\, \mathbf{e}_2). \tag{13}$$

The unit vectors $\mathbf{b}_0$, $\mathbf{e}_1$ and $\mathbf{e}_2$ form a local right handed coordinate basis *i.e.* $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{b}_0$, and they vary on the macroscopic, L, spacial scale and the slow, τ, time scale. Now we specialize to the straight field (electrostatic) case so that $\mathbf{b}_0 = \mathbf{z}$, $\mathbf{e}_1 = \mathbf{x}$ and $\mathbf{e}_2 = \mathbf{y}$. The fastest motion is the gyro-motion and indeed;

$$\frac{d\theta}{dt} = -\Omega_0 + \mathcal{O}(\epsilon\Omega). \tag{14}$$

We will show shortly that both $v_\perp$ and $v_\|$ are slowly varying and therefore can be considered constant on the fast (Ω) time scale. Now consider the evolution of $\mathbf{R}$. We differentiate Eq. (12) with respect to time:

$$\frac{d\mathbf{R}}{dt} = \mathbf{v} + \frac{d\mathbf{v}}{dt} \times \frac{\mathbf{b}_0}{\Omega_0} + \mathbf{v} \times \frac{d}{dt}(\frac{\mathbf{b}_0}{\Omega_0}) \tag{15}$$

now using the equation of motion:

$$m\frac{d\mathbf{v}}{dt} = q(\delta\mathbf{E} + \mathbf{v} \times \mathbf{B}_0), \tag{16}$$

we obtain,

$$\frac{d\mathbf{R}}{dt} = v_{\parallel}\mathbf{b}_0 + (\delta\mathbf{E} \times \frac{\mathbf{b}_0}{B_0}) - (\mathbf{v} \times \frac{\mathbf{b}_0}{\Omega_0})\frac{\mathbf{v} \cdot \nabla B_0}{B_0}. \tag{17}$$

Note that the dominant gyro-center motion is along the field lines and the cross field motion comes from the perturbed **E** cross **B** drift and, as we shall see, the grad B drift.

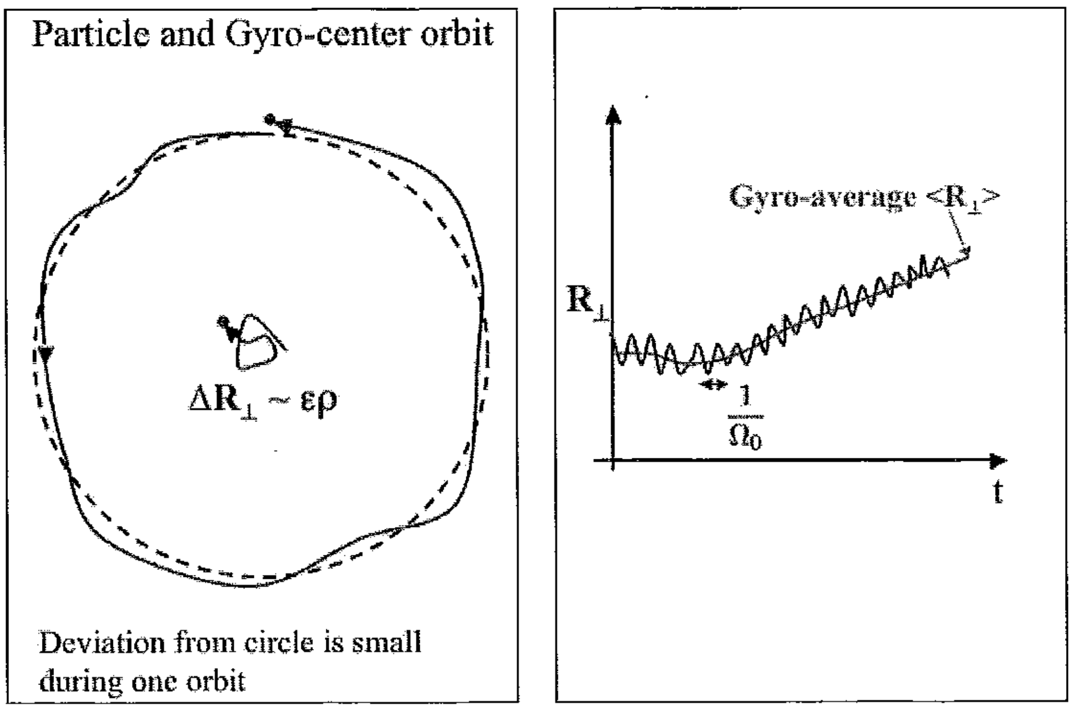

Figure 4: Motion of Gyro-center **R** and its Average. Note that in one orbit the gyro-center moves a small distance of order $\epsilon\rho$. However over ϵ^{-1} gyro-orbits the gyro-center "drifts" a distance $\mathcal{O}(\rho)$.

We wish to know the motion of the gyro-center, $\mathbf{R}$, over the medium time scale (times of order L/v_th). The right hand side of Eq. (38) oscillates on the fast time scale Ω^{-1} but when we integrate the perpendicular motion this averages out *i.e.*:

$$\delta\mathbf{R}_\perp = \int_0^t \left[\delta\mathbf{E} \times \frac{\mathbf{b}_0}{B_0} - (\mathbf{v} \times \frac{\mathbf{b}_0}{\Omega_0})\frac{\mathbf{v}\cdot\nabla B_0}{B_0}\right] dt$$

$$= \int_0^t \left[< \delta\mathbf{E} \times \frac{\mathbf{b}_0}{B_0} >_\mathbf{R} - < (\mathbf{v} \times \frac{\mathbf{b}_0}{\Omega_0})\frac{\mathbf{v}\cdot\nabla B_0}{B_0} >_\mathbf{R}\right] dt + \mathcal{O}(\epsilon\rho). \tag{18}$$

where the gyro-average (ring average) at fixed $\mathbf{R}$ is defined by:

$$< A(\mathbf{r}, \mathbf{v}, t) >_\mathbf{R} = \frac{1}{2\pi}\int_0^{2\pi} A(\mathbf{R} - \frac{\mathbf{v}\times\mathbf{b}_0}{\Omega_0}, \mathbf{v}, t)d\theta. \tag{19}$$

In Eq. (19) the θ integration is done keepin R, $v_\perp$ and $v_\parallel$ fixed. Thus this gyro-average is an **AVERAGE OVER A RING CENTERED ABOUT R OF RADIUS** $v_\perp/\Omega_0$. Thus we think of the gyro-center motion as the motion of this ring obeying the equation:

$$< \frac{d\mathbf{R}}{dt} > = < v_\parallel \mathbf{b}_0 >_\mathbf{R} + < \delta\mathbf{E} \times \frac{\mathbf{b}_0}{B_0} >_\mathbf{R} - < (\mathbf{v} \times \frac{\mathbf{b}_0}{\Omega_0})\frac{\mathbf{v}\cdot\nabla B_0}{B_0} >_\mathbf{R} \tag{20}$$

After some straightforward algebra we obtain:

$$< \frac{d\mathbf{R}}{dt} > = v_\parallel \mathbf{b}_0 \quad - \quad \frac{\partial < \phi >_\mathbf{R}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \quad - \quad \frac{v_\perp^2}{2\Omega_0}\frac{\nabla B_0}{B_0} \times \mathbf{b}_0 \tag{21}$$

and we have dropped the $\mathcal{O}(\epsilon v_{th})$ corrections to the parallel motion as they are small compared to the $v_\parallel \mathbf{b}_0$ term and they are not needed. The expression, Eq. (20) is almost very familiar except that the perpendicular motion is the E cross B drift in the *ring averaged field (potential)* plus the grad B drift in the equilibrium field.

Note that the perpendicular drift is $\mathcal{O}(\epsilon v_{th})$ in the gyro-kinetic ordering. We keep this because perpendicular structures are small scale and this small drift can move the gyro-center across the potential structure on the turbulent time-scale.

To complete our derivation of the particle motion we need the equations for the variation of $v_\perp$ and $v_\parallel$. The variation of energy, $\mathcal{E} = \frac{1}{2}mv^2 + q\phi(\mathbf{r}, t)$, follows in a similar manner to the derivation above, specifically:

$$\frac{d\mathcal{E}}{dt} = q\frac{\partial < \phi >_\mathbf{R}}{\partial t} \tag{22}$$

note that $\mathcal{E}$ varies on the medium time scale whereas the kinetic energy has an $\mathcal{O}(\epsilon)$ variation on the fast time scale due to the variation of ϕ over the gyro-orbit. The net heating of a particle over the medium time scale and longer comes from integrating the right hand side of Eq. (39) over time. Note the same ring averaged perturbed quantity $< \phi >_\mathbf{R}$ enters the energy and gyro-center evolution – you might suspect that this is due to some underlying property of the equations, indeed it is related to the Hamiltonian properties of the collisionless motion. I will not elaborate on this here as it does not illuminate the physical picture. We have kept energy variations up to $\mathcal{O}(\epsilon)$ (they are needed), but we shall only need the $\mathcal{O}(1)$ part of the magnetic moment variation. Thus:

$$\mu = \frac{mv_\perp^2}{2B_0(\mathbf{R}, t)} = constant. \tag{23}$$

To the order that is required Eqs (21), (39) and (23) provide a set of equations to find the particle orbits and energy variation.

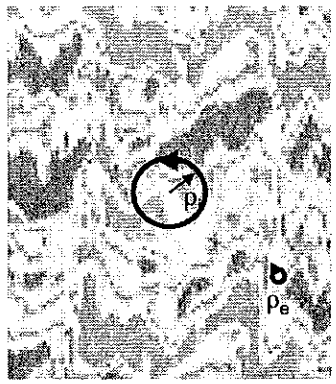

Figure 5: Gyro-average of the fluctuations over a ring of radius the larmor radius $\rho = v_\perp/\Omega$. For small radii the average is almost the same as the value at the center, for large radii the average tends to cancel and is almost zero. Electrons have smaller rings by the factor $\sqrt{m_e/m_i}$

4 Ordered Fokker-Planck Equation.

For convenience we write (see last lecture) $\delta f = \delta f_1 + \delta f_2......etc.$. We use the orderings stated in the last lecture. The FP equation with order (relative to $v_{th}F_0/L$) under each term is:

$$\underbrace{\frac{\partial F_0}{\partial t}}_{\epsilon^2} + \underbrace{\frac{\partial \delta f}{\partial t}}_{\epsilon} + \underbrace{\mathbf{v}\cdot\nabla F_0}_{1} + \underbrace{\mathbf{v}_\perp\cdot\nabla\delta f}_{1} + \underbrace{v_\parallel \mathbf{z}\cdot\nabla\delta f}_{\epsilon} + \frac{q}{m}\left(\underbrace{-\nabla\phi}_{1} + \underbrace{\mathbf{v}\times\mathbf{B}}_{\epsilon^{-1}}\right)\cdot\frac{\partial F_0}{\partial \mathbf{v}}$$

$$+\frac{q}{m}\left(\underbrace{-\nabla\phi}_{\epsilon} + \underbrace{\mathbf{v}\times\mathbf{B}}_{1}\right)\cdot\frac{\partial \delta f}{\partial \mathbf{v}} = \underbrace{C(F_0,F_0)}_{1} + \underbrace{C(\delta f, F_0)}_{\epsilon} + \underbrace{C(F_0,\delta f)}_{\epsilon} + \underbrace{C(\delta f),\delta f))}_{\epsilon^2}. \qquad (24)$$

Now the process of simplifying the equations involves equating orders and solving the resulting equations. In principle we need to go to $\mathcal{O}(\epsilon^2)$ to get the long transport time evolution of F_0 – in fact we will only go to $\mathcal{O}(\epsilon)$ and then use the moment equations to get the evolution of F_0.

4.1 $\mathcal{O}(\epsilon^{-1})$:

At this order from Eq. (24) we have simply:

$$\mathbf{v} \times \mathbf{B} \cdot \frac{\partial F_0}{\partial \mathbf{v}} = -\Omega_0(x) \left(\frac{\partial F_0}{\partial \theta} \right)_{\mathbf{r},v,v_\perp} = 0 \tag{25}$$

from which we deduce that F_0 is independent of gyro-angle θ (any initial dependance would be wiped out by the fast gyration) so that:

$$F_0 = F_0(v, v_\perp, t) \tag{26}$$

and recall that F_0 depends only on the long transport time scale. Now we proceed to $\mathcal{O}(1)$:

4.2 $\mathcal{O}(1)$:

From Eq. (24) we obtain:

$$\mathbf{v} \cdot \nabla F_0 + \mathbf{v}_\perp \cdot \nabla \delta f_1 + \frac{q}{m}(-\nabla\phi) \cdot \frac{\partial F_0}{\partial \mathbf{v}} - \Omega_0 \left(\frac{\partial \delta f_1}{\partial \theta} \right)_{\mathbf{r},v,v_\perp} = C(F_0, F_0) \tag{27}$$

This looks like a horrendous equation to solve – it involves two unknowns F_0 and $\delta f - 1$. In such cases we must isolate one unknown at a time and solve. A trick inspired by Boltzmann's H theorem allows us to solve for F_0. We multiply Eq. (27) by $\ln F_0$ and integrate over all velocities to obtain:

$$\nabla \cdot \int d^3\mathbf{v}(\mathbf{v}\delta f_1 \ln F_0) + \int d^3\mathbf{v}(\ln F_0) C(F_0, F_0) = 0 \tag{28}$$

The first term is a fluctuating term and varies over short distances – the second term is slowly varying. If we average this equation over all z and y and a region in x that is big compared to the small scale but small compared to the big scale we can average out the first term. Then we know from Boltzmann's H theorem that the second term being zero implies that F_0 is a Maxwellian – we will take it to be a stationary Maxwellian, *i.e.*

$$F_0 = \frac{n}{(\sqrt{\pi} v_{th})^3} \exp -(\frac{v^2}{v_{th}^2}) \tag{29}$$

where $v_{th} = \sqrt{(2T/m)}$ and n and T depend on the long time scale and the long x space scale. Essentially this result shows that there is no entropy production at this order. Eq. (27) becomes:

$$\mathbf{v}_\perp \cdot \nabla \delta f_1 - \Omega_0 \left(\frac{\partial \delta f_1}{\partial \theta} \right)_{\mathbf{r},v,v_\perp} = -\mathbf{v} \cdot \nabla(\frac{q\phi}{T})\, F_0 + \mathbf{v} \cdot \nabla F_0 \tag{30}$$

To this order we can drop the factor $v_\parallel \mathbf{z} \cdot \nabla(\frac{q\phi}{T})\, F_0$ from the right hand side. Then this equation has the simple **particular solution:**

$$\delta f_{1p} = -(\frac{q\phi}{T})\, F_0 - \rho \cdot \nabla F_0. \tag{31}$$

Where $\rho = \frac{\mathbf{z} \times \mathbf{v}}{\Omega}$ is the larmor radius. The particluar solution is sum of the perturbed Boltzmann response of the particles to the potential and the expansion of F_0 about $\mathbf{R}$. The potential is static on the gyration time scale and therefore the energy $\mathcal{E} = (1/2)mv^2 + q\phi$ is conserved to this order. Thus this part of the response can be captured by replacing the Maxwellian with the Maxwellian times the Boltzmann factor, *i.e.*

$$F_0 \rightarrow F_0 \exp -(\frac{q\phi}{T}) \tag{32}$$

With the density and temperature functions of $\mathbf{R}$ **not** $\mathbf{r}$. It remains to solve for the **Homogeneous Solution** of Eq. (30). We note that the operator on the right hand side of Eq. (30) is particular part of the collisionless motion in a constant uniform field thus we expect to find the gyro-center variable useful. Indeed we note that:

$$\Omega_0 \left(\frac{\partial}{\partial \theta}\right)_{\mathbf{R}} = \Omega_0 \left(\frac{\partial}{\partial \theta}\right)_{\mathbf{r}} - \mathbf{v}_\perp \cdot \nabla \tag{33}$$

and thus the homogeneous part of δf_1, δf_{1h}, satisfies the equation:

$$\left(\frac{\partial \delta f_{1h},}{\partial \theta}\right)_{\mathbf{R}} = 0. \tag{34}$$

Thus the homogeneous part is independent of gyro-angle at fixed $\mathbf{R}$ (not $\mathbf{r}$) *i.e.*,

$$\delta f_{1h} = h(\mathbf{R}, \mathcal{E}, \mu, t). \tag{35}$$

$h(\mathbf{R}, v, v_\perp, t)$ is sometimes called the **Guiding center distribution**. As we show in next order h satisfies the **gyro-kinetic equation.**

It is convenient to redefine δf_2 slightly to pull out the full Boltzmann response (see Eq. (32)) and to write the maxwellian as a function of $\mathbf{R}$. Thus to $\mathcal{O}(\epsilon^2)$

$$f(\mathbf{r}, \mathbf{v}, t) = n(t, \mathbf{R}) \left(\frac{m}{2\pi T(t, \mathbf{R})}\right)^{3/2} \exp[-(\frac{\mathcal{E}}{T(t, \mathbf{R})})] + h(\mathbf{R}, \mathcal{E}, \mu, t) + \delta f_2 \ldots\ldots\ldots\ldots \tag{36}$$

Where we have written $\mathcal{E} = (1/2)mv^2 + q\phi$. While this is the form of the distribution function we still need to derive equations for $g(\mathbf{R}, v, v_\perp, t)$, $n(t, \mathbf{R})$ and $T(t, \mathbf{R})$. Now we proceed to $\mathcal{O}(\epsilon)$ where we obtain the **gyro-kinetic equation** as a solubility constraint for δf_2:

4.3 $\mathcal{O}(\epsilon)$:

Substituting the form Eq. (36) into Eq. (24) and dropping terms $\mathcal{O}(\epsilon^2)$ and higher we obtain:

$$\frac{\partial h}{\partial t} + \frac{d\mathbf{R}}{dt} \cdot \frac{\partial h}{\partial \mathbf{R}} + \frac{d\mathcal{E}}{dt} \cdot \frac{\partial h}{\partial \mathcal{E}} + \frac{d\mu}{dt} \cdot \frac{\partial h}{\partial \mu} - C(h, F_0) - C(F_0, h) - C(F_0, F_0) = \Omega_0 \left(\frac{\partial \delta f_2}{\partial \theta}\right)_{\mathbf{R}} + \frac{d\mathbf{R}}{dt} \cdot \frac{\partial F_0}{\partial \mathbf{R}} + \frac{d\mathcal{E}}{dt} \cdot \frac{\partial F_0}{\partial \mathcal{E}} \tag{37}$$

where (see Lecture # 1) we have

$$\frac{d\mathbf{R}}{dt} = v_\parallel \mathbf{b}_0 + \delta \mathbf{E} \times \frac{\mathbf{b}_0}{B_0} - (\mathbf{v} \times \frac{\mathbf{b}_0}{\Omega_0}) \frac{\mathbf{v} \cdot \nabla B_0}{B_0}. \tag{38}$$

and

$$\frac{d\mathcal{E}}{dt} = q \frac{\partial \langle \phi \rangle}{\partial t}. \tag{39}$$

We note that because our form of F_0 is not quite a maxwellian we have a collisional term $C(F_0, F_0)$ which to this order is $C(F_0, -\rho \cdot \nabla F_0)$ – this gives classical transport terms which we ignore. To obtain an equation for h we must annihilate δf_2 from Eq. (38) – to do this we average over θ, the gyro-angle, at fixed $\mathbf{R}$, $\mathcal{E}$ and μ. Thus we define the gyro or ring average, (see in Lecture # 1) at fixed $\mathbf{R}$, $\mathcal{E}$ and μ as:

$$\langle a(\mathbf{r}, \mathbf{v}, t)\rangle_{\mathbf{R}} = \frac{1}{2\pi} \oint d\theta a(\mathbf{R} - \frac{\mathbf{v} \times \mathbf{z}}{\Omega}, \mathbf{v}, t),$$

where the $\Omega = qB/m$ and the θ integration is done keeping $\mathbf{R}$, $\mathcal{E}$ and μ fixed. Note, these gyro-averages are functions of $\mathbf{R}$, $\mathcal{E}$ and μ and σ – the sign of $v_\parallel$.

We note that to this order the $\frac{\partial h}{\partial \mathcal{E}}$ and $\frac{\partial g}{\partial \mu}$ terms average out. The ring distribution $h(\mathbf{R}, \mu, \mathcal{E}, \sigma, t)$ satisfies the **Gyro-kinetic equation**:

$$\frac{\partial h}{\partial t} + v_\parallel \frac{\partial h}{\partial z} + \mathbf{v_D} \cdot \frac{\partial h}{\partial \mathbf{R}} - \frac{\partial < \phi >_{\mathbf{R}}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial h}{\partial \mathbf{R}} - \langle C(h)\rangle_{\mathbf{R}} = q\frac{F_0}{T_0}\frac{\partial \langle\phi\rangle_{\mathbf{R}}}{\partial t} - \frac{\partial < \phi >_{\mathbf{R}}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial F_0}{\partial \mathbf{R}} \tag{40}$$

and

$$\mathbf{v_D} = -\frac{v_\perp^2}{2\Omega_0}\frac{\nabla B_0}{B_0} \times \mathbf{b}_0 = \frac{v_\perp^2}{2B_0}(\frac{1}{\Omega_0})\frac{dB_0}{dx}\mathbf{y}$$

is the equilibrium grad B drift. In some loose sense the **Gyro-kinetic equation** is the kinetic equation for rings of charge centered at $\mathbf{R}(t)$ of radius $v_\perp/\Omega$. It is important to note that ϕ and h have zero spatial average over the box. The physical interpretation of the terms in Eq. (40) is straight forward, for example:

- $\mathbf{v_D} \cdot \frac{\partial h}{\partial \mathbf{R}}$ is the convection of the perturbed ring distribution by the equilibrium grad B drift.
- $-\frac{\partial<\phi>_{\mathbf{R}}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial h}{\partial \mathbf{R}}$ is the convection of the perturbed distribution by the ring averaged E cross B drift. This is the only nonlinear term.
- $q\frac{F_0}{T_0}\frac{\partial\langle\phi\rangle_{\mathbf{R}}}{\partial t}$ is the work done on the particles by the field.
- $-\frac{\partial<\phi>_{\mathbf{R}}}{\partial \mathbf{R}} \times (\frac{\mathbf{b}_0}{B_0}) \cdot \frac{\partial F_0}{\partial \mathbf{R}}$ is the convection of the equilibrium distribution by the ring averaged E cross B drift.

We define a second ring average at fixed $\mathbf{r}$ as:

$$\langle a(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t)\rangle_{\mathbf{r}} = \frac{1}{2\pi} \oint d\theta a(\mathbf{r} + \frac{\mathbf{v} \times \mathbf{z}}{\Omega}, \mathcal{E}, \mu, \sigma, \theta, t),$$

This average arises in Maxwell's equations where for example the charge at $\mathbf{r}$ is due to particles with gyro-centers on a circle of radius $v_\perp/\Omega$ about $\mathbf{r}$. Maxwell's equations (assuming a plasma with one species of ion and electrons) become:

Quasi-Neutrality. We ignore the left hand side of Poisson's equation to obtain:

$$-\frac{n_i q^2 \phi}{T_i} + 2\pi q \sum_\sigma \int \int v_\perp dv_\perp dv_\parallel \langle h_i(\mathbf{R}, \mathcal{E}, \mu, \sigma, t)\rangle_{\mathbf{r}} = \frac{n_e e^2 \phi}{T_e} + 2\pi e \sum_\sigma \int \int v_\perp dv_\perp dv_\parallel \langle h_e(\mathbf{R}, \mathcal{E}, \mu, \sigma, t)\rangle_{\mathbf{r}} \tag{41}$$

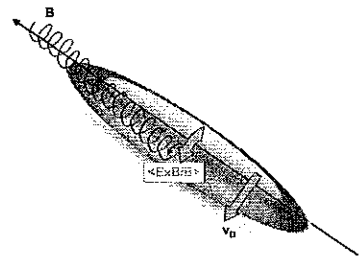

Figure 6: Perpendicular motion of the guiding center is the E cross B drift plus the equilibrium grad B drift.

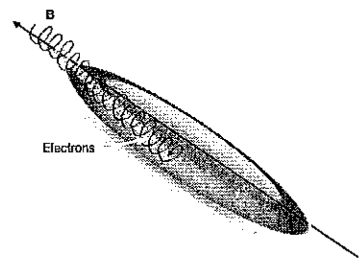

Figure 7: Perpendicular motion of the guiding center is the E cross B drift plus the equilibrium grad B drift.

The two equations, Eq. (40) and Eq. (41) are essentially an autonomous set on the turbulent time-scale. They are the **Electrostatic Gyro-kinetic system.** Of course F_0 must also be known, this requires calculating evolution on the long transport time-scale and is outside the purview of this lecture. However on the turbulent time-scale F_0 must be kept fixed and we can simply take it as known.

5 Gyro-averages and Bessel Functions

. Strictly speaking the Eq. (40) and Eq. (41) are an integro-differential system in space since they involve the gyro-averages. It is common to use a fourier basis in both $\mathbf{r}$ and $\mathbf{R}$ since this "diagonalizes" the gyro-average. Specifically

$$< \exp i\mathbf{k}\cdot\mathbf{r} >_{\mathbf{R}} = J_0(\frac{k_\perp v_\perp}{\Omega}) \exp i\mathbf{k}\cdot\mathbf{R} \tag{42}$$

$$< \exp i\mathbf{k}\cdot\mathbf{R} >_{\mathbf{r}} = J_0(\frac{k_\perp v_\perp}{\Omega}) \exp i\mathbf{k}\cdot\mathbf{r} \tag{43}$$

Where $J_0(x)$ is the zeroth order Bessel function of the first kind. Thus in the fourier space gyro-averaging just becomes multiplication by a Bessel function (that depends on $v_\perp$).