

Physics 260: Turbulence in Magnetized Plasmas.

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Meeting Time: 11:00am. → 12:30pm. Tuesdays & Thursdays.

Room: New Physics and Astronomy Building 4-330 (large seminar room.)

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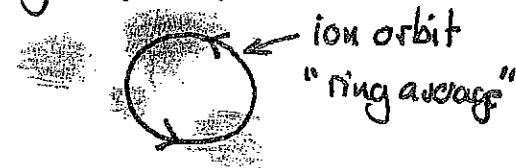
BASIC OUTLINE: Small subset of plasma turbulence - important subset.

(i) INTRODUCTION: Observations and experiments. Turbulence scales and timescales. What is small/big? Mathematics of multiple scales.

(ii) GYRO-KINETICS: Modern approach to turbulence when $\frac{e}{m_i} \ll \frac{\omega_{ci}}{\Omega_i}$.
Ions are rigorously treated as rings of radius $r = \frac{v_{ci}}{\omega_{ci}}$ = LARMOR RADIUS.
simple derivation of the equations in a straight field.
Drifts and orbits.

(iii) PHYSICAL LIMITS OF GYRO KINETICS:

Various simplifying limits, electron behaviour,
2D behaviour, Frozen-in theorems. Relation to M.H.D.



(iv) LINEAR WAVES - ALFVÉN WAVES:

The linear dispersion relation: SOUND WAVES / SLOW WAVES,
ALFVÉN WAVES.

Limits and damping. Relation to M.H.D.

(V) ALFVEN WAVE TURBULENCE:

MHD Theory - AW cascade, Goldreich Sridhar theory.
Collisionless scales and heating.
Observations.

(Vi) INHOMOGENEOUS GYRO-KINETICS:

Extend GK derivation to inhomogeneous plasmas.
• Particle Drifts.
• ω^* effects (temperature and pressure gradient terms).

(Vii) DRIFT INSTABILITIES - ITG, ETG etc.

Instabilities driven by ∇T , ∇n . Physical pictures.
Simple mixing length estimates of transport.
Fusion turbulence - issues.



260 Lecture #1. Introduction & Scales.

(i) Almost all magnetized plasmas are turbulent - the turbulent fluctuations are often low frequency ($\omega \ll \omega_{ci}$) and small in amplitude. Today we will look at some data and the important scales. I am hoping to motivate a simplification of the equations called the GYRO-KINETIC approximation - this is a relatively modern approach that will form the basis of this course.

- (ii) 3 illustrative plasmas.
- (1) LAPD - local experiment
 - (2) Inter Stellar Medium - ISM - astrophysics
 - (3) International Thermonuclear Experimental Reactor - ITER.

BASIC NUMBERS.

	LAPD	ITER	ISM
Particle Density, $n (\text{cm}^{-3})$	10^{12}	$(1-2) \times 10^{14}$	$1 - 10^{-2}$
Temperature, $T (\text{eV})$	1-3	10^4	$10^{-1} - 10$
Magnetic Field, $B (\text{Gauss})$	10^3	5×10^4	$1-3 \times 10^{-6}$
Length Scale \perp to B $L_\perp (\text{cm})$	100	200 (a)	$10^7 - 10^8$
Length Scale \parallel to B $L_\parallel (\text{cm})$	2×10^3	$(2\pi R) 3600$	
Electron thermal velocity $v_{the} = \sqrt{\frac{2T_e}{m_e}}$	$5 \times 10^7 \text{ cm s}^{-1}$	$4 \times 10^9 \text{ cm s}^{-1}$	$\sim 5 \times 10^7 \text{ cm s}^{-1}$
Ion thermal velocity $v_{thi} = \sqrt{\frac{2T_i}{m_i}}$	$5 \times 10^5 \text{ cm s}^{-1}$	$6 \times 10^7 \text{ cm s}^{-1}$	$\sim 10^6 \text{ cm s}^{-1}$
Alfvén velocity $V_A = \sqrt{\frac{B^2}{4\pi \rho m_i}}$	10^8 cm s^{-1}	$4 \times 10^8 \text{ cm s}^{-1}$	$\sim 10^6 \text{ cm s}^{-1}$

FREQUENCIES. (s⁻¹) [Angular]

	LAPD	ITER	ISM.
Plasma frequency: $\omega_{pe} = \left(\frac{4\pi ne^2}{m_e}\right)^{1/2}$	$5 \times 10^{10} \text{ s}^{-1}$	$7 \times 10^{11} \text{ s}^{-1}$	$5 \times 10^3 - 10^4$
Electron gyro-frequency: $\Omega_{ce} = \frac{eB}{m_e c}$	$1.1 \times 10^{10} \text{ s}^{-1}$	$5 \times 10^{11} \text{ s}^{-1}$	50 s^{-1}
Ion gyro-frequency: $\Omega_{ci} = \frac{eB}{m_i c}$	$\frac{1}{2} \times 10^7 \text{ s}^{-1}$	$3 \times 10^8 \text{ s}^{-1}$	0.02 s^{-1}
e-collision rate: γ_e	$3 \times 10^7 \text{ s}^{-1}$	$9 \times 10^3 \text{ s}^{-1}$	$4.5 \times 10^{-6} \text{ s}^{-1}$
i-collision rate: γ_i	$2 \times 10^6 \text{ s}^{-1}$	$1.6 \times 10^2 \text{ s}^{-1}$	$7.5 \times 10^{-8} \text{ s}^{-1}$
ion transit frequency: $\frac{V_{thi}}{L_{ }}$	$2.5 \times 10^2 \text{ s}^{-1}$	$3 \times 10^4 \text{ s}^{-1}$	$10^{-1} - 10^{-12} \text{ s}^{-1}$
Arlén frequency: $V_A / L_{ }$	$5 \times 10^4 \text{ s}^{-1}$	$3 \times 10^5 \text{ s}^{-1}$	$10^{-1} - 10^{-12} \text{ s}^{-1}$
Evolution time/rate:	$10 - 10^2 \text{ s}^{-1}$?	0.3 s^{-1} "confinement time $\sim 3 \text{ s}$ "	?

LENGTHS

Debye Length $\lambda_d = V_{the} / \omega_{pe}$	$7 \times 10^{-4} \text{ cm}$	$7 \times 10^{-3} \text{ cm}$	$7 \times 10^2 \text{ cm}$
Collisionless skin depth $= c / \omega_{pe}$	0.5 cm	0.05 cm	$5 \times 10^5 \text{ cm}$
Ion gyro-radius $r_i = V_{thi} / \Omega_{ci}$	0.1 cm	0.2 cm	$3 \times 10^7 \text{ cm}$
Electron gyro-radius $r_e = V_{thc} / \Omega_{ce}$	$2.5 \times 10^{-3} \text{ cm}$	$0.5 \times 10^{-2} \text{ cm}$	$\sim 10^6 \text{ cm}$
$L_2 / L_{ }$	100/2000	200/3600	$10^2 - 10^3$
Electron mean-free-path $\lambda_{m.f.p.}$	2-10 cm	$3 \times 10^5 \text{ cm}$	10^{13} cm.

DIMENSIONLESS VARIABLES.

	$\sqrt{\frac{m_e}{m_p}} \sim \frac{1}{40}$	LAPD	ITER	ISM
PLASMA PARAMETER $g = \frac{1}{n\lambda_D^3}$	4×10^{-4}		10^{-8}	10^{-9}
PLASMA $\beta = \frac{8\pi n k T}{B^2} \approx \frac{2V_{thi}^2}{V_A^2}$	10^{-4}		$4 \times 10^{-2} \sim \frac{1}{25}$	1
$\frac{R_i}{L_z} = \frac{\text{ion Larmor radius}}{\text{1 scale length}}$	10^{-3}		$< 10^{-3}$	$1 - 10^{-11}$
$\frac{\lambda_{mfp}}{L_z} = \text{"collisionality"}$	10^{-3}		200	$10^{-4} - 10^{-7}$
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I have obviously overdone this list of parameters but
2 clear approximations are critical:

APPROX 1.

$g \ll 1$ "weak coupling."

In 222 a-c we emphasized this approximation - it allows us to write the Fokker-Planck Equations for the plasma species (ions & electrons)

In this course we take this as our starting point - without any questions about its validity.

Fokker-Planck Equation + Maxwell's Equations

Label species: f_α = distribution function of species α $f_\alpha = f_\alpha(r, v, t)$, we will usually deal with $\alpha = \text{electrons or ions}$.

FOKKER-PLANCK EQN.

$$(1) \frac{\partial f_\alpha}{\partial t} + \underline{v} \cdot \frac{\partial f_\alpha}{\partial \underline{r}} + \frac{q_\alpha}{m_\alpha} \left\{ \underline{E} + \underline{v} \times \underline{B} \right\} \cdot \frac{\partial f_\alpha}{\partial \underline{v}} = \sum_{\beta} C_{\alpha\beta} (f_\alpha, f_\beta)$$

COLLISION TERMS
SUM OVER ALL SPECIES β .

where the Landau form of $C_{\alpha\beta}$ is

$$(1.2) C_{\alpha\beta} (f_\alpha, f_\beta) = - \frac{1}{2} \cdot J_{\alpha\beta}$$

$$\text{and } (1.3) J_{\alpha\beta} = 2\pi m \Lambda_{\alpha\beta} \frac{q_\alpha^2 q_\beta^2}{m_\alpha} \int \frac{d^3v'}{u^3} (u^i I_i - u_u) \left\{ \frac{1}{m_\beta} f_\alpha(v) \frac{\partial f_\beta(v')}{\partial v'} \right. \\ \left. - \frac{1}{m_\alpha} f_\beta(v') \frac{\partial f_\alpha(v)}{\partial v'} \right\}$$

MAXWELL'S EQUATIONS

$$(1.4) \nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0} = \frac{1}{\epsilon_0} \sum_{\alpha} \int d^3v q_\alpha f_\alpha(r, v, t)$$

$$(1.5) \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \nabla \times \underline{B} - \mu_0 \underline{J} = \nabla \times \underline{B} - \mu_0 \sum_{\alpha} \int d^3v q_\alpha v f_\alpha(r, v, t)$$

$$(1.6) \nabla \cdot \underline{B} = 0$$

$$(1.7) \frac{\partial \underline{B}}{\partial t} = - \nabla \times \underline{E}$$

THESE EQNS (1.1) → (1.7) ARE
A CLOSED SET. THEY ARE OUR
STARTING POINT

APPROXIMATION 2.— STRONGLY MAGNETIZED.

The examples I give satisfy the condition:

$$\frac{P_i}{L_i} \ll 1 \quad \text{and} \quad \omega = \frac{d}{dt} \ll \Omega_{ci}$$

AT LEAST FOR
THE "EQUILIBRIUM".

NOTE. $P_e \sim \sqrt{\mu_0/m_i} P_i$ and $\Omega_{ce} = \frac{m_i}{m_e} \Omega_{ci}$ so even more true for electrons.

BUT THE TURBULENCE SATISFIES:

$$k_{\perp} p_i \sim D(1)$$

 $k_{\perp} \sim$ typical wavenumber
of turbulence \perp to B .

BUT $\frac{D(1)}{n} \sim 1\% \ll 1$

SEE DATA
FROM DIII-D.

$$\frac{\omega}{\Omega_{ci}} \ll 1, \quad \frac{k_{\parallel}}{k_{\perp}} \ll 1$$

 $k_{\parallel} \sim$ typical wavenumber
along B .
FINE SCALE, SMALL, SLOW FLUCTUATIONS ABOUT A SLOWLY VARYING BACKGROUND.

|| ————— ||

THE GYRO-KINETIC APPROACHtreats this limit with a rigorous development.
WE WILL DEVELOP THIS THEORY.

History Linear Theory 1967. Rutherford & Frieman.

1967. Taylor & Hartlieb.

1978-80 Catto, Tang & Baldwin. & Antoniou & Lane.

Nonlinear Theory 1979. Frieman & Chen

Simulations 1982. Lee

1990... 2004. Dorland, Kotschwarther, Candy, Waleck, Lin, Parker, Leboeuf, Decker.

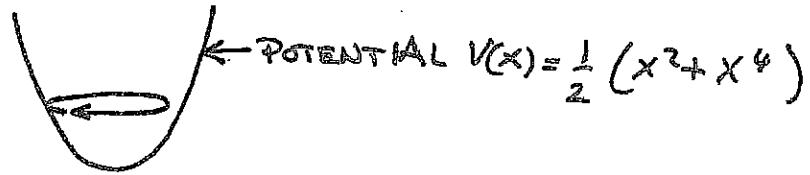
260 Lecture #2. Multiple-scale Analysis.

(i) In this idea we introduce some simple ideas from asymptotic analysis that facilitate the ~~eq~~ solution of multi-scale problems. This is a mathematics lecture - but you will need it. [more details can be found in Benders ~~eq~~ "Advanced Mathematical Methods for Scientists and Engineers".]

(ii) We start with a simple equation:

$$\frac{d^2x}{dt^2} = -(x + 2x^3) = -\frac{\partial}{\partial x} \left(\frac{x^2}{2} + \frac{x^4}{2} \right)$$

NONLINEAR OSCILLATOR.



(iii) This equation can be solved by quadratures to find $t = t(x)$ but this solution isn't very illuminating. For small oscillations, $x \ll 1$, we can try to find a solution in powers of the amplitude (squared).

$$x(t) = x_0(t) + x_1(t) + x_2(t) \dots$$

$$x_1 \sim \mathcal{O}(x_0^3)$$

$$x_2 \sim \mathcal{O}(x_0 x_1^2) \sim \mathcal{O}(x_0^5)$$

SYMBOL $\mathcal{O}(\dots)$ MEANS "OF ORDER" PRECISE

DEFINITION ISN'T WARRANTED HERE.

For small x_0 , successive terms in the series are smaller & smaller.

(iv) The process of expanding is simply bookkeeping in powers of x_0 - it looks much harder than it is:-

(V) **LOWEST ORDER.** $\mathcal{J}(x_0)$
"zeroth order"

$$\frac{d^2x_0}{dt^2} = -x_0 \quad \Rightarrow \quad x_0 = \bar{x}_0 \cos(t + \phi_0)$$

where \bar{x}_0 and ϕ_0 are constants, determined by initial conditions.

(vi) **FIRST ORDER.** $\mathcal{J}(x_1)$

$$\frac{d^2x_1}{dt^2} = -x_1 - 2x_0^3 = -x_1 - 2\bar{x}_0^3 \cos^3(t + \phi_0)$$

we use the identity $\cos^3 \theta = \frac{1}{4} \cos 3\theta + \frac{3}{4} \cos \theta$.

$$\Rightarrow \frac{d^2x_1}{dt^2} + x_1 = -\frac{\bar{x}_0^3}{2} \left\{ \cos\{3(t + \phi_0)\} + 3\cos\{(t + \phi_0)\} \right\}$$

this is an inhomogeneous O.D.E. with one complication, this term is a solution of the L.H.S. and therefore "resonant". It is like pushing a baby on a swing - if we always push in phase the baby tends to swing higher & higher. Nonetheless a solution is easy to write down.

$$(vii) x_1(t) = \frac{\bar{x}_0^3}{16} \cos\{3(t + \phi_0)\} - \frac{3\bar{x}_0^3}{4} t \sin\{(t + \phi_0)\} \\ + \bar{x}_1 \cos\{(t + \phi_0)\}$$

"Homogeneous solution" can be absorbed into zeroth order term without loss of generality.
 $x_0(t)$

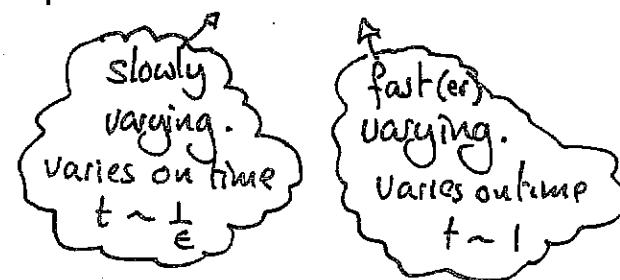
(viii) Unfortunately our formulation has the seeds of its own destruction. The term:

$$-\frac{3\bar{x}_0^3}{4}t \sin\{(t + \phi_0)\}$$

increases "secularly" in time so that for $t > \frac{1}{\epsilon} X_0$ (which was assumed smaller) is bigger than X_0 . Thus \bar{x}_0^2 our perturbation scheme breaks down for longtimes. It's the "baby on the swing" problem - each oscillation the \bar{x}_0^3 term makes a phase coherent nudge to the true system - lots of small nudges over a long time cause a finite change in x .

(ix) To get a perturbation scheme that works for a long time we must deal with functions that change slowly but finitely over a long time. Such a method is the "Method of Multiple Scales" we will now introduce this method.

(x) 2 SCALE FUNCTION e.g. $f(t) = e^{-\epsilon t} \sin t$



$$\frac{df}{dt} = \left(\frac{de^{-\epsilon t}}{dt} \right) \sin t + e^{-\epsilon t} \frac{d}{dt} \sin t$$

$$= -\epsilon e^{-\epsilon t} \sin t + e^{-\epsilon t} \cos t$$

small term
due to slowly
varying part

large term
due to finitely
varying terms.

(xi) So for example \dot{x}_0 could vary slowly on long timescale and our lowest order solution would still be correct.

(xii) THE FORMAL TRICK Introduce two timescales

$$t \quad \text{and} \quad \tau = \epsilon t$$

"FAST" timescale. "SLOW" Timescale.

CHOOSE ϵ IN
A MOMENT

we treat them as independent variables and write:

$$x = x(t, \tau)$$

⇒ $\frac{dx}{dt} = \frac{\partial x}{\partial t} + \frac{dt}{dt} \frac{\partial x}{\partial \tau} = \frac{\partial x}{\partial t} + \epsilon \frac{\partial x}{\partial \tau}$

and: $\frac{d^2x}{dt^2} = \frac{\partial^2 x}{\partial t^2} + 2\epsilon \frac{\partial^2 x}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2 x}{\partial \tau^2}$

(xiii) Now the equation to solve is

$$\frac{\partial^2 x}{\partial t^2} + 2\epsilon \frac{\partial^2 x}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2 x}{\partial \tau^2} = - (x + 2x^3)$$

AND WE SEEK AN EXPANSION IN WHICH

$$x(t, \tau) = x_0(t, \tau) + x_1(t, \tau) + x_2(t, \tau)$$

where even for long times $x_1 \sim \mathcal{O}(x_0^3)$ $x_2 \sim \mathcal{O}(x_0^5)$.

(xiv) CHOOSING ϵ We choose ϵ so that the ^{first} new term

$2\epsilon \frac{\partial^2 x_0}{\partial t \partial \tau}$ appears at the same order as the secular forcing - $\mathcal{O}(x_0^3)$.
(the in-phase nudges)

Thus we require $\epsilon X_0 \sim \mathcal{O}(x_0^3) \Rightarrow \epsilon \approx \mathcal{O}(x_0^2)$
 Now we understand the order of everything we can go back
 and expand again.

(XV) **LOWEST ZEROTH ORDER** : $\mathcal{O}(x_0)$

$$\frac{d^2 X_0}{dt^2} + X_0 = 0$$

looks almost the same as before - but there is a difference!

$$X_0(t, \tau) = \bar{X}_0(\tau) \cos(t + \phi_0(\tau))$$

Now \bar{X}_0 and ϕ_0 are not constant (necessarily) they are
 slowly varying functions of time. In higher order we will
 determine their form.

(XVI) **FIRST ORDER** : $\mathcal{O}(x_0^3)$

$$\frac{d^2 X_1}{dt^2} + X_1 + 2\epsilon \frac{d^2 X_0}{dt d\tau} = -2X_0^3 = -\frac{\bar{X}_0^3}{2} \left[\cos\{3(t + \phi_0)\} + 3\cos(t + \phi_0) \right]$$

calculating the extra term and rearranging we get

$$\frac{d^2 X_1}{dt^2} + X_1 = \left\{ \begin{array}{l} \left[2\epsilon \bar{X}_0 \frac{\partial \phi_0}{\partial \tau} - \frac{3\bar{X}_0^3}{4} \right] \cos(t + \phi_0) \\ + 2\epsilon \frac{\partial \bar{X}_0}{\partial \tau} \sin(t + \phi_0) \\ - \frac{\bar{X}_0^3}{4} \cos\{3(t + \phi_0)\} \end{array} \right.$$

(xvii) If we are to avoid terms that are increasing secularly then the coefficients of $\cos(t+\phi_0)$ and $\sin(t+\phi_0)$ must both vanish.

$$\blacktriangleright 2\epsilon \bar{X}_0 \frac{\partial \phi_0}{\partial t} = \frac{3\bar{X}_0^{-3}}{4} \quad \text{coefficient of } \cos(t+\phi_0)$$

AND

$$\blacktriangleright 2\epsilon \frac{\partial \bar{X}_0}{\partial t} = 0 \quad \text{coefficient of } \sin(t+\phi_0)$$

THESE ARE EQUATIONS FOR THE SLOWLY VARYING VARIABLES.

SOLUTION

$$\bar{X}_0 = \text{constant.}$$

and

$$\phi_0 = \frac{3}{8} \bar{X}_0^2 \frac{t}{\epsilon} + \bar{\phi}_0 \quad = \quad \frac{3}{8} \bar{X}_0^2 t + \bar{\phi}_0$$

↑ CONSTANT

Then we can write the solution to this order.

$$X_0 = \bar{X}_0 \cos \left\{ t \left(1 + \frac{3}{8} \bar{X}_0^2 \right) + \bar{\phi}_0 \right\}$$

$$X_1 = \frac{\bar{X}_0^3}{32} \cos \left\{ t \left(1 + \frac{3}{8} \bar{X}_0^2 \right) + \bar{\phi}_0 \right\}$$

$\bar{\phi}_0$ (does) may depend on an even slower timescale and we may be forced at even higher order to introduce a further timescale.

THIS METHOD WILL BE USED TO DERIVE THE GYRO-KINETIC EQUATION.

Physics 260: Lecture #3. The Gyro-kinetic Ordering.

FIGURE / DIAGRAM ALSO HANDED OUT

(i) Today we will lay out the formal ordering, in full generality. This will prepare us for the grind through the equations next week for the slab case. Talk only about ions.

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(ii) TWO LENGTH SCALES

(a) "slow" $L \equiv \text{MACROSCOPIC LENGTH}$ (size of plasma, $\frac{n}{|\nabla n|}$)

(b) "fast" $\rho_i = \text{ION LARMOR RADIUS}$

(iii) \rightarrow SMALL DIMENSIONAL PARAMETER:

$$\epsilon = \frac{\rho_i}{L} \ll 1$$

(iv) THREE TIMESCALES

FAST :

a) $\Omega_{ci} = \text{ION CYCLOTRON FREQUENCY.}$

MEDIUM :

b) $\frac{V_{thi}}{L} \sim \epsilon \Omega_{ci} = \text{FLUCTUATION/TURBULENCE FREQUENCY.} = \omega$

SLOW :

c) $\frac{V_{thi} \cdot \epsilon^2}{L} \sim \epsilon^3 \Omega_{ci} = \text{TRANSPORT RATE.}$

(v) SMALL FLUCTUATIONS

split quantities into equilibrium and fluctuating parts.

a) Distribution function

$$f(r, v, t) = F_0 + \delta f_1 + \delta f_2 \dots$$

\nearrow
equilibrium.

$$\boxed{\frac{\delta f_1}{F_0} \sim \mathcal{O}(\epsilon)}, \quad \frac{\delta f_2}{F_0} \sim \mathcal{O}(\epsilon^2) \quad \text{etc.}$$

b) Fields: \downarrow equilibrium

$$\underline{B} = \underline{B}_0 + \delta \underline{B}, \quad \frac{|\delta \underline{B}|}{|\underline{B}_0|} \sim \mathcal{O}(\epsilon)$$

$$\underline{E} = \delta \underline{E}, \quad \frac{|\delta \underline{E}|}{B_0 V_{thi}} \sim \mathcal{O}(\epsilon)$$

(VI) SLOWLY VARYING EQUILIBRIUM IN SPACE AND TIME

a) SPACE: Macroscopic length: $\nabla F_0 \sim \mathcal{O}\left(\frac{F_0}{\lambda}\right)$

$$\nabla B_0 \sim \mathcal{O}\left(\frac{B_0}{\lambda}\right)$$

b) TIME: "SLOW" TRANSPORT TIME: $\frac{\partial F_0}{\partial t} \sim \mathcal{O}\left(\frac{V_{thi} \epsilon^2 F_0}{\lambda}\right) \sim \mathcal{O}(\epsilon^3 \Omega_i F_0)$

$$\frac{\partial \underline{B}_0}{\partial t} \sim \mathcal{O}(\epsilon^3 \underline{B}_0)$$

(V) FAST VARYING FLUCTUATIONS ACROSS \underline{B}_0

SLOW VARYING FLUCTUATIONS ALONG \underline{B}_0

a) $\underline{b}_0 = \frac{\underline{B}_0}{B_0}$

a) $|\underline{b}_0 \times \nabla \delta f| \sim \mathcal{O}(k_\perp \delta f) \sim \mathcal{O}\left(\frac{\delta f}{\rho_i}\right)$



$$\boxed{k_\perp \rho_i \sim 1}$$

$$k_\perp \sim \frac{2\pi}{L_\perp}$$

$$b) \quad \underline{B}_0 \cdot \nabla \delta f \sim \mathcal{J}(k_{\parallel} \delta f) \sim \mathcal{J}\left(\frac{\delta f}{L_{\parallel}}\right) \Rightarrow L_{\parallel} \sim L$$

$$k_{\parallel} L \sim \mathcal{J}(1)$$

➡ $\frac{k_{\parallel}}{k_{\perp}} \sim \mathcal{J}(E)$

(VII) MEDIUM VARIATION IN TIME OF FLUCTUATIONS

$$\frac{d \delta f_i}{dt} \simeq \mathcal{J}\left(\frac{v_{tui} \delta f_i}{L}\right) = \mathcal{J}(E \tau_{ci} \delta f) \quad \frac{\omega}{\Omega} \sim \mathcal{J}(E)$$

same ordering on $\frac{d \delta E}{dt}, \frac{d \delta B}{dt}$

(VIII) COLLISIONS ACT ON MEDIUM TIMESCALE

$$\gamma_i \sim \text{ion collision rate} \simeq \mathcal{J}\left(\frac{v_{tui}}{L}\right)$$

(IX) some simple consequences. [refer to diagram]

a) Perpendicular Flow velocity: $v_{\perp} \sim \frac{\delta E \times \underline{B}_0}{B_0^2} \sim \mathcal{J}(E v_{tui})$

b) Fluid Displacement: $\xi_{\text{particle}} \sim \frac{v_{\perp}}{\omega} \sim \mathcal{J}\left(\frac{E v_{tui}}{E \tau_i}\right) \sim \mathcal{J}(\varphi_i)$

AS EXPECTED

c) Field Line Displacement:

Equation for a field line

$$\frac{dr}{dl} = \frac{\underline{B}}{B_0} \Rightarrow \frac{d\xi_B}{dl} \approx \frac{\delta \underline{B}}{B_0}$$



⇒ $\xi_B \sim \int dl \frac{\delta \underline{B}}{B_0} \sim \mathcal{J}\left(L_{\parallel} \frac{\delta \underline{B}}{B_0}\right) \sim \mathcal{J}(E L_{\parallel}) \sim \mathcal{J}(\varphi_i)$

AS EXPECTED

(x) Energy and the transport timescale.

Energy in the fluctuations $\sim \frac{1}{2} \rho \delta v^2$, $\delta B^2 \sim \epsilon^2 n T$
so if we assume this energy is "damped" at the medium timescale we get

$$\frac{dE}{dt} = \frac{\text{RATE OF CHANGE}}{\text{OF ENERGY}} = \frac{\epsilon^2 n T_0 V_{thi}}{L} = \frac{n T}{\tau_s} \leftarrow \begin{matrix} \text{SLOW} \\ \text{TIME.} \end{matrix}$$

so the plasma is heated by the turbulence on the slow timescale.

(xi) MAXWELL'S EQUATIONS: SIMPLE CONSEQUENCES OF THE ORDERING.

a) FARADAY'S LAW.

$$\frac{\partial \delta \underline{B}}{\partial t} = -\nabla \times \delta \underline{E}$$

?

$$\mathcal{J}(\epsilon^2 \underline{R}_c \underline{B}_0) \quad \quad \quad \mathcal{J}(k_1 \epsilon V_{th} \underline{B}_0) \sim \mathcal{J}(k_{1p} \epsilon R_c \underline{B}_0)$$

SO TO LOWEST ORDER IN EPSILON,

$$\nabla \times \delta \underline{E}'' = 0 \quad \rightarrow \quad \delta \underline{E}'' = -\nabla \phi$$

SO TO LOWEST ORDER THE ELECTRIC FIELD IS "ELECTROSTATIC". TO MAKE IT EASY TO APPLY THIS WE INTRODUCE THE POTENTIALS.

$\delta \underline{B} = \nabla \times \underline{A}$	<u>and</u>	$\delta \underline{E} = -\nabla \phi - \frac{\partial \underline{A}}{\partial t}$
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where $\frac{\partial \underline{A}}{\partial t} \sim \mathcal{J}(\epsilon \nabla_{\perp} \phi)$

But because $k_{11} \sim \mathcal{J}(\epsilon)$
 k_{\perp}

$\frac{\delta E_{11}}{\delta E_{\perp}} \sim \mathcal{J}(\epsilon)$

$$\delta E_{11} = B_0 \cdot \left\{ -\nabla \phi - \frac{\partial \underline{A}}{\partial t} \right\} = -\nabla_{11} \phi - \frac{\partial A_{11}}{\partial t} \sim \mathcal{J}(\epsilon \delta E_{\perp})$$

BOTH INDUCTIVE AND ELECTROSTATIC FIELD IS IMPORTANT IN PARALLEL ELECTRIC FIELD.

11. ————— //

NOTE: The work done by δE_{\perp} is $\sim \mathcal{O}(\delta E_{\perp} \delta p_{\text{particle}}) \sim \mathcal{J}(\delta E_{\perp} p_i)$
 which is the same order as the work done by $\delta E_{11} \sim \mathcal{J}(\delta E_{11} L)$.

b) AMPERE / MAXWELL LAW. (For the fluctuations)

$$\nabla \times \delta \underline{B} = \mu_0 \delta \underline{J} + \frac{1}{c^2} \frac{\partial \delta \underline{E}}{\partial t}$$

DISPLACEMENT CURRENT

DOMINANT ORDER

$$J\left(\frac{\delta B}{P_i}\right) \quad J\left(\frac{V_{thi}^2 B_0 \epsilon}{c^2} \right)$$

SO THE RATIO OF THESE TERMS IS:

$$\frac{\frac{1}{c^2} \frac{\partial \delta \underline{E}}{\partial t}}{\nabla \times \delta \underline{B}} \approx J\left(\epsilon \frac{V_{thi}^2}{c^2}\right)$$

↑ ↑
SMALL SMALL (NON RELATIVISTIC IONS)

Thus we drop displacement current and we get:

$$\nabla \times \delta \underline{B} = \mu_0 \delta \underline{J}$$

AMPERE'S LAW

We need to be a bit careful as there is some divergence of the current - in a sense only two components of AMPERE'S LAW ARE LARGE. But even this is small in V_{thi}/c . see below.

c) POISSON'S EQUATION - QUASI-NEUTRALITY

$$\nabla \cdot \delta \underline{E} = \frac{\rho}{\epsilon_0}$$

$$J\left(\frac{\delta E}{P_i}\right) \sim J\left(\frac{V_{thi} B_0 \epsilon}{P_i}\right) \sim J\left(\frac{q n_0 \epsilon}{\epsilon_0}\right)$$

$$\frac{\nabla \cdot \delta \underline{E}}{\rho/\epsilon_0} \sim J\left(\frac{V_{thi}^2}{c^2} \frac{1}{P_i}\right) \ll 1$$

DROP $\nabla \cdot \delta \underline{E}$ TO GET

QUASI-NEUTRALITY

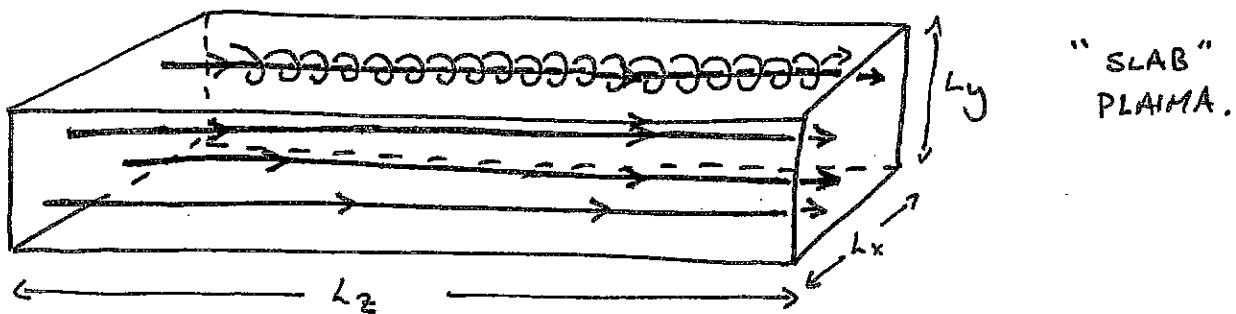
$\rho = \text{CHARGE DENSITY} = 0$

Physics 260: Lecture #4: Gyrokinetics in a Homogeneous Plasma Equilibrium I

(i) We are not going to develop gyrokinetics in full generality where $F_0 \& B_0$ are spatially varying - it can be done but it is too complex to extract the physics.

$$\therefore \boxed{\text{WE ASSUME } \nabla F_0 = 0 \quad B_0 = B_0 \hat{z} \quad \nabla B_0 = 0}$$

(ii) We take a rectangular box with $L_x = L_y$ and periodic boundary conditions on all perturbed quantities. e.g. $\delta E_x(x+L_x) = \delta E_x(x)$ etc



(iii) UNPERTURBED PARTICLE MOTION

Revision for you all + some notation.

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m} (\mathbf{v} \times \mathbf{B}_0) = \Omega_c (\mathbf{v} \times \hat{z})$$

$$\Omega_c = \frac{q B_0}{m}$$

it is trivial to show that.

$$\mathbf{v} = v_{||} \hat{z} + v_{\perp} (\cos\theta \hat{x} + \sin\theta \hat{y})$$

$\Theta = \text{GYROPHASE/GYROANGLE}$

$v_{||}$ and v_{\perp} are constant.

$$\tan\theta = \frac{\mathbf{v} \cdot \hat{y}}{\mathbf{v} \cdot \hat{x}}$$

differentiating $\frac{d\theta}{dt} = -\Omega_c$

$\Theta = -\Omega_c t + \Theta_0$

Integrating the velocity we get the particle position.

$$\boxed{\mathbf{r}(t) = \mathbf{r}_0(t) + v_{||} t \hat{z} - \frac{v_{\perp}}{\Omega_c} (\sin\theta \hat{x} - \cos\theta \hat{y})}$$

Familiar Spiral motion

(iv) **VELOCITY COORDINATES**

Instead of v_x , v_y and v_z we represent the velocity by

$$E = \frac{v^2}{2}, \quad \mu = \frac{v_\perp^2}{2B_0}, \quad \theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$

[note: normalizing v_\perp^2 with B_0 to make μ is no necessary for testlab "but" is very helpful when B varies]

In terms of these coordinates:

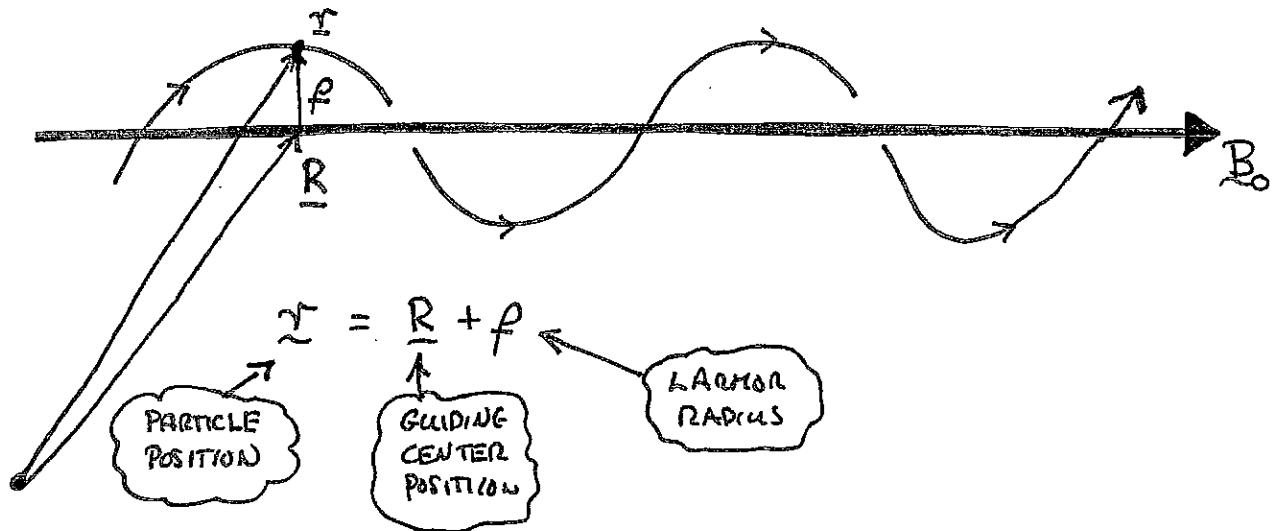
$$\begin{aligned} \frac{\partial}{\partial v} &= \frac{\partial E}{\partial v} \frac{\partial}{\partial E} + \frac{\partial \mu}{\partial v} \frac{\partial}{\partial \mu} + \frac{\partial \theta}{\partial v} \frac{\partial}{\partial \theta} \\ \frac{\partial}{\partial v_\perp} &= \frac{v}{2} \frac{\partial}{\partial E} + \frac{v_\perp}{B_0} \frac{\partial}{\partial \mu} - \frac{\hat{v} \times \hat{z}}{v_\perp^2} \frac{\partial}{\partial \theta} \end{aligned}$$

$$d^3v = \frac{dE \, d\mu \, d\theta}{\left| \frac{\partial E}{\partial v} \times \frac{\partial \mu}{\partial v} \cdot \frac{\partial \theta}{\partial v} \right|} = \frac{B_0 \, dE \, d\mu \, d\theta}{|v_\parallel|}$$

JACOBIAN OF
THE CHANGE
OF VELOCITY
COORDINATES

- (v) In the unperturbed motion E and μ don't change but θ evolves quickly (on the ω_c timescale).

(V) GUIDING CENTER



In the unperturbed motion the guiding center is well defined. The center of the spiral is unambiguous and

$$f = \frac{\hat{z} \times v}{\omega}$$

For the perturbed motion the position of the guiding center is not easily defined. We will DEFINE IT SIMPLY AS THE RELATION

$$R = \underline{z} + \frac{v \times \hat{z}}{\omega}$$

(vi) MOTION OF GUIDING CENTER

$$\frac{d\underline{R}}{dt} = \frac{dr}{dt} + \frac{dv}{dt} \times \frac{\hat{z}}{\omega} = v + \frac{q}{m} (\delta \underline{z} + (v \times \hat{z}) B_0 + v \times \delta \underline{B}) \times \frac{\hat{z}}{\omega}$$

$$\frac{d\underline{R}}{dt} = v_{||} \hat{z} + \frac{\delta E}{B_0} \times \hat{z} + v_{||} \frac{\delta B_{\perp}}{B_0} - v_{\perp} \frac{\delta B_{||}}{B_0}$$

(vii) The perpendicular motion of the guiding center is small compared to the perpendicular motion of the particle. There are two physical terms.

$$\text{PERTURBED } \underline{\underline{E}} \times \hat{\underline{\underline{B}}} \text{ VELOCITY} = \frac{\delta \underline{\underline{E}} \times \hat{\underline{\underline{z}}}}{B_0} = -\nabla \phi \times \hat{\underline{\underline{z}}} \approx \mathcal{J}(\epsilon v_{th})$$

$$\text{MOTION ALONG TILTED FIELD LINE} = V_{||} \frac{\delta \underline{\underline{B}}_{\perp}}{B_0} = \nabla (V_{||} A_{||}) \times \hat{\underline{\underline{z}}} \approx \mathcal{J}(\epsilon v_{th})$$

$$\text{PERTURBED GYROMOTION} = -V_{\perp} \frac{\delta \underline{\underline{B}}_{||}}{B_0} = -V_{\perp} \frac{\underline{\underline{b}} \cdot \nabla \times \underline{\underline{A}}_{||}}{B_0} \approx \mathcal{J}(\epsilon v_{th})$$

 NOT QUITE CIRCULAR MOTION.

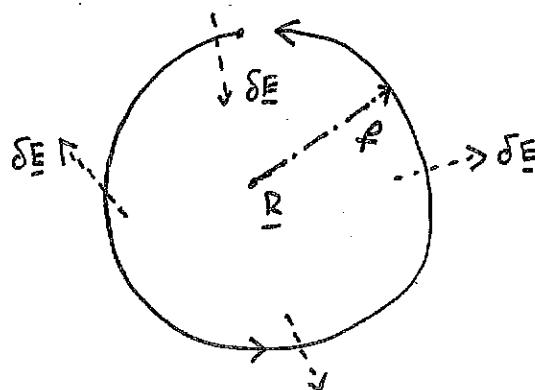
WHERE WE HAVE USED $\delta \underline{\underline{B}}_{\perp} = \nabla A_{||} \times \hat{\underline{\underline{z}}}$ & $\delta \underline{\underline{B}}_{||} = \underline{\underline{b}} \cdot \nabla \times \underline{\underline{A}}_{||}$.

(viii) Now each one of these terms has a fast varying part and a medium timescale varying part. For instance,

$$\delta \underline{\underline{E}}(r(t), t) = \delta \underline{\underline{E}}(R + \rho, t) \quad \begin{matrix} \leftarrow \\ \text{medium scale variation.} \end{matrix}$$

Because $\delta \underline{\underline{E}}$ varies rapidly in the perpendicular direction

$\delta \underline{\underline{E}}$ varies on timescale Ω_c^{-1} due to ρ varying.



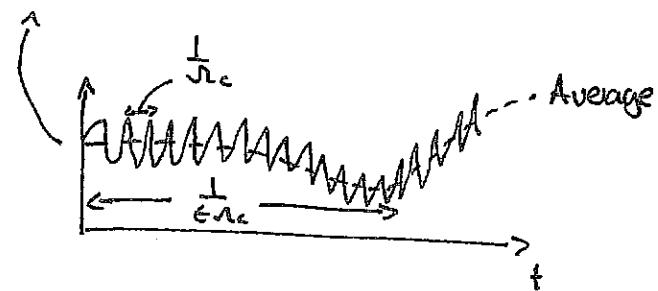
BUT THE CHANGE IN R PERPENDICULAR TO B_0 DURING TIME Ω_c^{-1} IS SMALL

$$\delta R_{\perp} \approx \frac{1}{\Omega_c} \cdot \frac{\delta \underline{\underline{E}} \times \hat{\underline{\underline{z}}}}{B_0} \approx \epsilon p_i$$

(ix) But: we want to compute the changes in \underline{B}_\perp due to $\delta \underline{E}$ over the medium timescale $\frac{1}{\epsilon \Omega_{ci}}$. Then $\Delta \underline{B}_\perp \sim \rho_i$. Formally.

$$\Delta \underline{B}_\perp = \int_{-\infty}^t dt' \frac{\delta \underline{E}(\underline{R} + \underline{r}', t') \times \hat{\underline{z}}}{B_0} + \text{other terms.}$$

The "wiggly" bit (i.e. the fast varying bit) cancels and it is only the average bit that contributes to the integral over times long compared to Ω_c^{-1} .



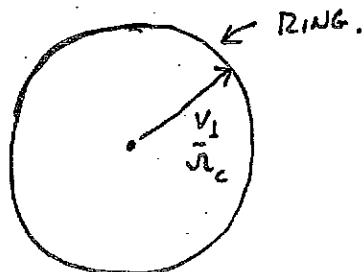
(x) WE DEFINE THE GYRO/RING AVERAGE OF "ANY FUNCTION $H(\underline{r}, t)$ " BY:

$$\langle H(\underline{r}, t) \rangle = \Omega_c \int_0^{\frac{1}{\Omega_c}} dt' H(\underline{R} + \underline{r}(t'), t')$$

WITH \underline{R}
KEPT CONSTANT

$$= \langle H \rangle (\underline{R}, \underline{v}_\perp, t) \quad \text{i.e. is a function of } \underline{R}, \underline{v}_\perp \text{ and } t.$$

NOTICE THAT THIS IS A AVERAGE OVER A RING OF RADIUS v_\perp / Ω_c AND CENTER \underline{R}



(xi) SINCE WE DON'T CARE ABOUT THE SHIFTS OF ORDER ϵp_i TO COMPUTE THE MOTION OF \underline{R} WE TAKE THE RING AVERAGE OF $\frac{d\underline{R}}{dt}$

(xii) [GYRO AVERAGED / RING AVERAGED GUIDING CENTER MOTION.]

$$\left\langle \frac{dR}{dt} \right\rangle = \left\langle V_{||} \hat{z} \right\rangle + \left\langle \frac{\delta E \times \hat{z}}{B_0} + V_{||} \frac{\delta B_{\perp}}{B_0} - V_{\perp} \frac{\delta B_{||}}{B_0} \right\rangle$$

After some algebra:

$$= V_{||} \hat{z} - \frac{\partial}{\partial R} \langle \chi \rangle \times \frac{\hat{z}}{B_0}$$

where $\chi = \phi - v_{||} A_{||} - \underline{v} \cdot \underline{A}_{\perp} = \phi - \underline{v} \cdot \underline{A}$

"POTENTIAL FOR THE PERPENDICULAR MOTION.

(xiv) [DRIFT-KINETIC LIMIT] If the variation of $\frac{\delta E}{\lambda}$ over R and B

Larmor radius is small we can expand in the ring/gyro average

- $\langle \phi(R+\rho, t) \rangle \approx \langle \phi(R, t) \rangle + \langle \rho \cdot \frac{\partial \phi}{\partial R} \rangle \dots \mathcal{O}(\rho^2 k^2)$
 $\approx \langle \phi(R, t) \rangle$

so $E \times B$ drift becomes the usual guiding center expression.

- $\langle V_{\perp} \frac{\delta B_{||}}{B_0} \rangle \approx \langle \chi_f \rho \cdot \frac{\partial (\delta B_{||})}{\partial R} \rangle = \frac{V_{\perp}^2}{2} \frac{1}{\rho} \hat{z} \times \nabla \frac{\delta B_{||}}{B_0}$

"GRAD-B DRIFT"

Physics 260: Lecture #5. Gyro-kinetics in a Straight Field. II

(i) Today, finally, the expansion. We start with expanding the kinetic equation - the Fokker Planck equation. We do Maxwell's Equations later.

(ii) We write: $f = F_0(v, \tau) + \delta f(\tau, v, t, \varepsilon)$

where $\tau = \varepsilon^2 t$ is the long transport timescale (for the evolution of F_0)

We will expand $\delta f = \frac{\delta f^{(0)}}{\partial(\varepsilon)} + \frac{\delta f^{(1)}}{\partial(\varepsilon^2)} \dots \dots \dots$ as we go.

(iii) As before $\delta E = -\nabla\phi - \frac{\partial A}{\partial t}$, $\delta B = \nabla \times A$.

$$\begin{aligned}
 & \text{(5.1)} \quad \varepsilon^2 \frac{\partial F_0}{\partial \tau} + \frac{\partial \delta f}{\partial \tau} + \varepsilon^2 \frac{\partial \delta f}{\partial t} + \underline{v} \cdot \nabla \delta f + V_0 \hat{z} \cdot \nabla \delta f + \frac{q}{m} \left\{ -\nabla\phi - \frac{\partial A}{\partial t} + \underline{v} \times \delta \underline{B} + \underline{v} \times \underline{B}_0 \right\} \frac{\partial \underline{v}}{\partial \varepsilon} \\
 & \text{ORDER} \quad \varepsilon^2 \quad \varepsilon \quad \varepsilon^3 \quad 1 \quad \varepsilon \quad 1 \quad \varepsilon \quad 1 \quad \frac{1}{\varepsilon} \\
 & + \frac{q}{m} \left\{ -\nabla\phi - \frac{\partial A}{\partial t} + \underline{v} \times \delta \underline{B} + \underline{v} \times \underline{B}_0 \right\} \cdot \frac{\partial \delta f}{\partial \underline{v}} = C(F_0, F_0) + C(\delta f, F_0) \\
 & \text{ORDER} \quad \varepsilon \quad \varepsilon^2 \quad \varepsilon \quad \frac{1}{\varepsilon} \quad 1 \quad \frac{1}{\varepsilon} \\
 & + C(F_0, \delta f) + C(\delta f, \delta f) \\
 & \text{ORDER} \quad \frac{1}{\varepsilon} \quad \frac{1}{\varepsilon^2}
 \end{aligned}$$

WE LABEL ORDER RELATIVE TO $\frac{V_0 \hat{z} \cdot \nabla F_0}{E}$

22

22

(iv) Now we take this equation order by order and see what it tells us.

(v) LOWEST ORDER $\mathcal{O}(\frac{1}{\epsilon})$

$$(5.2) \quad \frac{q}{m} \underline{v} \times \underline{B}_0 \cdot \frac{\partial \underline{F}_0}{\partial \underline{v}} = - \Omega_c \frac{\partial \underline{F}_0}{\partial \theta} = 0$$

USING THE COORDINATES
 E, M, θ for \underline{v}

$$(5.3) \quad \rightarrow \underline{F}_0 = \underline{F}_0(E, M, \theta)$$

i.e. NO GYROPHASE DEPENDENCE OF \underline{F}_0 - ON Ω_c TIMESCALE ANY DEPENDENCE IS WIPE OUT BY GYRATION.

At this order \underline{F}_0 is still undetermined in form (and size)

(vi) FIRST ORDER $\mathcal{O}(1)$

$$(5.4) \quad \underline{v}_1 \nabla_{\perp} \delta f^{(1)} + \frac{q}{m} \left\{ -\nabla \phi + \underline{v} \times \delta \underline{B} \right\} \cdot \frac{\partial \underline{F}_0}{\partial \underline{v}} - \Omega_c \frac{\partial \delta f^{(1)}}{\partial \theta} = C(F_0, F_0)$$

This looks like a horrendous equation to solve - both δf & F_0 are unknown. However we can use a trick to annihilate / remove δf and solve for F_0 - this involves Boltzmann's H theorem.

(vi)
 BOLTZMANN'S H THEOREM. $S = \text{ENTROPY} = - \int f \ln f d^3r d^3v$
 from F-equation we can show
 $\frac{dS}{dt} = - \int d^3r \int d^3v \ln f C(f, f) \geq 0$ "2nd law of thermodynamics"

AND if $\frac{dS}{dt} = 0$ then $f = \text{MAXWELLIAN} = \frac{n}{\pi^{3/2}} \frac{1}{V_{th}^3} e^{-\frac{W^2}{V_{th}^2}}$

$V_{th} = \sqrt{\frac{2T}{m}}$ $W = \underline{v} - \underline{v}^{\text{flow}}$ velocity

(viii) To use this result we multiply Eq. (5.4) by $1 + \ln F_0$ and integrate over all space and velocity space. We obtain

$$(5.5) \int dt \int d^3v \ln F_0 C(F_0, F_0) d^3v = 0$$

we use
 $\int d^3v C(F_0, F_0) = 0$

From BOLTZMANN'S H THEOREM this implies:

$$F_0 = \frac{n_0(t)}{\pi^{3/2}} \frac{1}{V_{th}^3} e^{-\frac{V^2}{V_{th}^2}} \quad V_{th}^2 = \sqrt{\frac{2T(t)}{m}}$$

where we have taken the lowest order plasma flow to be zero for simplicity. It is nice to note that the plasma is close to thermal equilibrium.

(ix) So we have determined the form of F_0 - the evolution of F_0 does not enter until $\mathcal{O}(e^2)$. Thus we return to solve for δf from Eq. (5.4). Thus substituting for F_0 we get

$$(5.6) \underline{V}_1 \cdot \nabla \delta f^{(0)} - S_c \frac{\partial \delta f^{(0)}}{\partial \theta} = - \underline{V} \cdot \nabla \left(\frac{q\phi}{T} \right) F_0$$

This is an inhomogeneous equation for δf . It is easy to spot the

PARTICULAR SOLUTION.

$$\delta f_p = - \frac{q\phi}{T} F_0$$

(x) This part of the solution has a simple physical origin. On the fast timescale ϕ is stationary so the total energy is

$$E_{TOTAL} = \frac{1}{2} m v^2 + q\phi$$

So the Boltzmann distribution would be

$$f = \frac{n}{\pi^{3/2} (2T)^{3/2}} e^{-\frac{E_{TOTAL}}{T}}$$

Expanding for
 $\frac{q\phi}{T} \ll 1$

$$\approx F_0 - \frac{q\phi}{T} F_0 + \mathcal{O}\left(\left(\frac{q\phi}{T}\right)^2\right) \dots \dots$$

So our particular solution is just the first term in the expansion of the Boltzmann factor. Over the medium timescale changes can accumulate and these are reflected in the homogeneous solution which we label g

$$(5.7) \quad \delta f^{(1)} = -\frac{q\phi}{T} F_0 + g$$

\nwarrow I will call this the "Boltzmann" part others call it the "adiabatic" part.

\nearrow homogeneous part

(xi) Thus combining (5.7) and (5.6) we have

$$(5.8) \quad \underline{V}_1 \cdot \frac{\partial g}{\partial \underline{r}} - \Omega_c \frac{\partial g}{\partial \theta} = 0$$

this equation does not involve δE or δB so it just involves motion along unperturbed trajectories. Thus it is useful to use the BUILDING CENTER VARIABLES i.e. Express g as

$$g = g(R, \epsilon, \mu, \theta, t, \tau)$$

where $\underline{R} = \underline{r} + \frac{\underline{v} \times \hat{\underline{z}}}{\Omega_c}$

(xii) To convert to these variables we note: $\frac{\partial \underline{R}}{\partial \theta} = \frac{\partial \underline{v}}{\partial \theta} \times \hat{\underline{z}} = \frac{\underline{v}_\perp}{\underline{r}}$

$$\Rightarrow \left(\frac{\partial}{\partial \theta} \right)_r = \left(\frac{\partial}{\partial \theta} \right)_{\underline{R}} + \left(\frac{\partial \underline{R}}{\partial \theta} \right) \cdot \frac{\partial}{\partial \underline{R}} = \frac{\partial}{\partial \theta} + \frac{\underline{v}_\perp}{\underline{r}} \cdot \frac{\partial}{\partial \underline{R}}$$

$$\left(\frac{\partial}{\partial \theta} \right)_r = \left(\frac{\partial}{\partial \underline{R}} \right)_{\underline{v}}$$

Thus Eq. (5.8) becomes:

$$(5.9) \quad \boxed{\left(\frac{\partial g}{\partial \theta} \right)_{\underline{R}} = 0} \Rightarrow g = g(\underline{R}, \epsilon, \mu, t, \tau)$$

INDEPENDANT OF GYROPHASE AT CONSTANT \underline{R} NOT $\underline{\Sigma}$ BUT WE
MUST GO TO NEXT ORDER TO FIND g EVOLVING ON THE MEDIUM TIMESCALE.

(xiii) OK WHERE ARE WE?

After 2 orders of the expansion we have learnt that

$$f = F_0(\epsilon, t) e^{\frac{-q\phi(\epsilon, t)}{T}} + g(\underline{R}, \epsilon, \mu, t, \tau) + \frac{\delta f^{(2)}}{\delta(\epsilon)} \dots \frac{\delta f^{(2)}}{\delta(\epsilon^2)} \dots$$

where we have included some of the $\delta f^{(2)}$ in the Boltzmann factor - i.e. we have written $1 - \frac{q\phi}{T} \Rightarrow e^{-\frac{q\phi}{T}}$ and absorbed the difference into $\delta f^{(2)}$.

NOTE: $\phi(\underline{\Sigma}, t)$ varies around the gyro-orbit by $g(\underline{R}, \epsilon, \mu, t, \tau)$ doesn't.

Physics 260: Lecture #6. Gyro-kinetics in a straight field III.

(i) LAST LECTURE WE LEARNT THAT

F_0 = MAXWELLIAN. and to 2nd order.

(6.1)

$$f = \frac{n_0(t)}{\pi^{3/2}} \left(\frac{m}{2T(t)} \right)^{3/2} e^{-\frac{1}{2} \frac{mv^2}{T} - \frac{q\phi(t, \underline{r})}{T}} + g(\underline{R}, \epsilon, \mu, t, \tau) + \delta f_2 \dots$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$
 $\partial(\epsilon) \quad \partial(\epsilon) \quad \partial(\epsilon) \quad \partial(\epsilon^2)$

where \underline{R} is the "GUIDING CENTER" position.

$$\underline{R} = \underline{z} + \frac{\underline{v} \times \hat{\underline{z}}}{\omega_c}$$

this came from the
 $\partial(\epsilon) \& \partial(\epsilon^2)$ equations.

(ii) Our goal today is to obtain equations for $g(\underline{R}, \epsilon, \mu, t, \tau)$, ϕ , $A_{||}$ & $\delta B_{||}$.

i.e. all the fluctuating quantities. Substituting the above form of f into the Fokker-Planck equation we obtain $\partial \partial(\epsilon)$ [ignoring $\partial(\epsilon^2)$ and higher]

(6.2)

$$\frac{\partial g}{\partial t} + \frac{d\underline{R}}{dt} \cdot \frac{\partial g}{\partial \underline{R}} + \frac{q}{m} \left\{ -\nabla_{\perp} \phi + \underline{v} \times \delta \underline{B} \right\} \cdot \left\{ \underline{v} \frac{\partial g}{\partial \epsilon} + \frac{\underline{v}_{\perp}}{B_0} \frac{\partial g}{\partial \mu} \right\}$$

$$- C_L(g, F_0) = S \left(\frac{\partial \delta f}{\partial \theta} \right)_{B, \epsilon, \mu} + \frac{q}{T} \left(\frac{\partial \phi}{\partial t} - \frac{\underline{v} \cdot \partial \underline{A}}{\partial t} \right) F_0 + \partial(\epsilon^2) \dots$$

LINEARIZE COLLISION
OPERATOR.

↑
DERIVATIVE AT
CONSTANT $\underline{R}, \epsilon, \mu$

$$\text{where: } \frac{d\underline{R}}{dt} = \underline{v}_{||} \hat{\underline{z}} + \left[-\frac{\nabla \phi \times \hat{\underline{z}}}{B_0} + \underline{v}_{||} \frac{\delta \underline{B}}{B_0} - \underline{v}_{\perp} \frac{\delta \underline{B}_{||}}{B_0} \right] \quad (6.4)$$

velocity of guiding center.

$$v_{||} = \sqrt{2(\epsilon - \mu B_0)} \quad \sigma = \pm \text{ sign of } v_{||}$$

(iii) Now to get an equation for g we need to eliminate δf_2 from this equation. We can do this by integrating over θ at fixed R and using that δf_2 must be periodic in θ . Thus

$$\int_0^{2\pi} d\theta \left(\frac{\partial \delta f_2}{\partial \theta} \right)_{R, E, \mu} = 0$$

But $\frac{1}{2\pi} \int_0^{2\pi} d\theta H(R + \rho(\theta), t) = \frac{1}{\pi} \int dt H(R + \rho(t), t)$ is just the "RING AVERAGE" of Lecture #3.

$$= \langle H \rangle_R^E(R, V_1, t)$$

(iv) so RING AVERAGING Eq. (6.2) WE OBTAIN \downarrow reminds us to keep R constant.

$$(6.3) \quad \boxed{\frac{dg}{dt} + \left\langle \frac{dR}{dt} \right\rangle_R \cdot \frac{\partial g}{\partial R} - \left\langle C_L(g, F_0) \right\rangle_R = \frac{q}{T} \frac{\partial}{\partial t} \langle x \rangle_R^E F_0}$$

where $X = \phi - \underline{v} \cdot \hat{\underline{z}}$,

AND. (see Lecture #3.)

$$\begin{aligned} \left\langle \frac{dR}{dt} \right\rangle &= V_{||} \hat{\underline{z}} + \left\langle -\nabla \phi \cdot \hat{\underline{z}} + V_{||} \frac{\delta R}{B_0} - V_1 \frac{\delta R_{||}}{B_0} \right\rangle_R \\ &= V_{||} \hat{\underline{z}} - \frac{\partial \langle x \rangle_R^E}{\partial R} \times \frac{\hat{\underline{z}}}{B_0} \end{aligned}$$

we write $[\langle x \rangle_R^E, g] = \left(\frac{\partial \langle x \rangle_R^E}{\partial R} \times \frac{\partial g}{\partial R} \right) \cdot \hat{\underline{z}} = \frac{\partial \langle x \rangle_R^E}{\partial X} \frac{\partial g}{\partial Y} - \frac{\partial \langle x \rangle_R^E}{\partial Y} \frac{\partial g}{\partial X}$
"POISON BRACKET"

NOTE Deriving Eq. (6.3) from averaging (6.2) requires a couple of pages of algebra. One identity to note is:

$$\left\langle V_i \cdot \nabla \phi(x, t) \right\rangle_R^E = 0 \quad (\text{for any } \phi)$$

(v) We have derived the GYRO-KINETIC EQUATION.

$$6.5 \quad \frac{\partial g}{\partial t} + V_{||} \hat{z} \cdot \frac{\partial g}{\partial \underline{R}} + \frac{1}{B_0} [\langle x \rangle, g] - \langle C_L(g, F_0) \rangle = \frac{q}{T} \frac{\partial}{\partial E} \langle x \rangle F_0$$


 RING MOTION
ALONG \underline{B}_0 RING MOTION
ACROSS \underline{B}_0 COLLISIONS ENERGY IN/OUT

I THINK OF THIS LOOSELY AS THE EQUATION FOR THE MOTION OF RINGS MOVING ALONG AND ACROSS β . IN THIS EQUATION F_0 IS FORMALLY FIXED (INDEPENDANT OF t).

Solving Eq. (6.5) is hard(it is nonlinear through the "poisson bracket").

But in principle it determines g and completes the solution for "f" to $\partial\psi$.

// ————— //

(vi) To complete the $O(\epsilon)$ solution we need to return to Maxwell's Equations at this order. The equations we need are:

$$(VLL) \quad \text{with} \quad \delta \underline{\underline{E}} = -\nabla \phi - \frac{\partial \underline{\underline{A}}}{\partial t} \quad \delta \underline{\underline{B}} = \nabla \times \underline{\underline{A}}$$

and we take the coulomb gauge $\nabla \cdot \underline{A} = 0$, which to this order gives

$$D_i A_j = 0 \quad \Rightarrow \quad A_{\perp} = \nabla \lambda \times \hat{z} \quad \text{and} \quad \delta B_{ii} = - \nabla_i^2 \lambda.$$

$$\nabla \times \delta B = \nabla^2 A_{\parallel} + \nabla \delta B_{\parallel} \times \hat{z}$$

Now for Ions & Electrons

(viii) QUASI-NEUTRALITY. To lowest order $n_{oi} q = n_{oe} e$ $\partial(\zeta)$

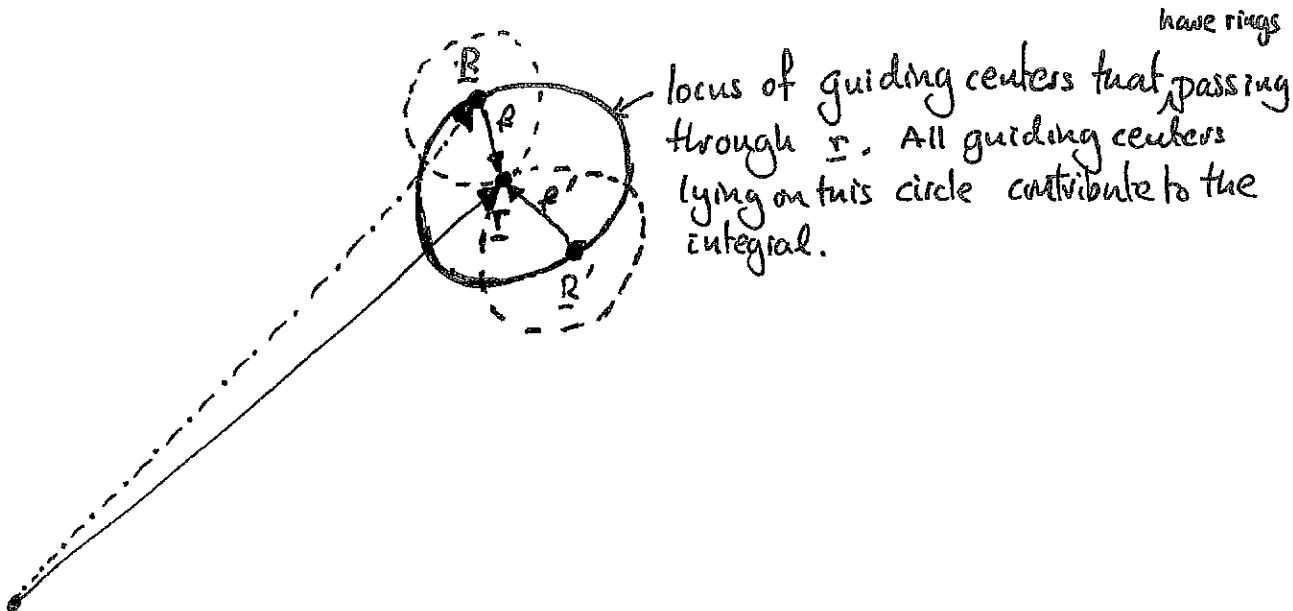
$\partial(\epsilon)$

$$-\frac{q^2 \phi}{T_{oi}} n_{oi} + \sum_i \int g_i \left(\zeta + \frac{\epsilon \times \hat{z}}{R_i}, \epsilon, \mu, t \right) B_0 \frac{d\epsilon d\mu d\theta}{|V_{ii}|}$$

$$= + \frac{e^2 \phi}{T_{oe}} n_{oe} + \sum_e \int g_e \left(\zeta + \frac{\epsilon \times \hat{z}}{R_e}, \epsilon, \mu, t \right) B_0 \frac{d\epsilon d\mu d\theta}{|V_{ii}|}$$

Note we have to take the charge at fixed ζ (not of course R) so the integral over θ is done at fixed ζ .

It can be seen pictorially that the charge at ζ comes from guiding centers a distance $-\rho$ away on a circle about ζ



THE θ INTEGRATION FOR THE CHARGE PRODUCES ANOTHER RING AVERAGE THIS TIME AN AVERAGE OVER A RING ABOUT A CONSTANT ζ POINT. THIS IS SUBSTLY DIFFERENT, WE DENOTE:

$$\frac{1}{2\pi} \int_0^{2\pi} d\theta g(\zeta - \rho(\theta), \epsilon, \mu, t, \tau) = \langle g \rangle_{\zeta} (\epsilon, \mu, t, \tau)$$

RING AVERAGE AT FIXED ζ

Thus we write:

QUASI-NEUTRALITY

(6.6)

$$-\frac{q^2 n_{oi}}{T_i} \phi + \mu q B_0 \iint \frac{d\mu d\varepsilon}{|V_{||}|} \sum_i \langle g_i \rangle_r = \frac{e^2 n_{oe}}{T_e} \phi + e B_0 \iint \frac{d\mu d\varepsilon}{|V_{||}|} \sum_e \langle g_e \rangle_r$$

$$\text{with } |V_{||}| = \sqrt{2(\varepsilon - \mu B_0)}$$

similarly after a little algebra

PARALLEL AMPERE'S LAW

(6.7)

$$\nabla_{\perp}^2 A_{||} = \mu_0 \delta J_{||} = \mu_0 B_0 \iint \frac{d\mu d\varepsilon}{|V_{||}|} \sum_i V_{||} \langle g_i \rangle_r - e B_0 \iint \frac{d\mu d\varepsilon}{|V_{||}|} \sum_e V_{||} \langle g_e \rangle_r$$

PERPENDICULAR AMPERE'S LAW

(6.8)

$$\nabla_{\perp} \delta B_{||} = \mu_0 \hat{\underline{z}} \times \delta \underline{J} = \mu_0 \left\{ q_i \iint \frac{d\mu d\varepsilon}{|V_{||}|} \hat{\underline{z}} \times V_{\perp} \langle g_i \rangle_r - e \iint \frac{d\mu d\varepsilon}{|V_{||}|} \hat{\underline{z}} \times V_{\perp} \langle g_e \rangle_r \right\}$$

This equation can also be written using some algebra in a more physical way.

$$\nabla_{\perp} [\delta B_{||} B_{||} \hat{\underline{z}} + \delta P_{\perp}] = 0$$

Perpendicular Pressure Balance.

where the ^{perurbed} pressure tensor δP_{\perp} is

$$\delta P_{\perp} = \sum_i \iint \frac{d\mu d\varepsilon}{|V_{||}|} \left[\langle m_i V_{\perp} V_{\perp} g_i \rangle_r + \langle m_e V_{\perp} V_{\perp} g_e \rangle_r \right]$$

THE GK EQUATION (6.5) PLUS THE THREE FIELD EQUATIONS (6.6) - (6.8) DETERMINE ALL THE $O(\epsilon)$ FLUCTUATING QUANTITIES. THIS IS THE SYSTEM COMPUTER SOLVE.

Lecture # 7.

Transport Timescale: Heating Terms in Gyro Kinetics and Energy Conservation.

Warning, needs checking for minor errors and the notation needs cleaning up.

1 Summary of Gyro-kinetic Equations.

Here I am just tidying up Lecture # 6's notes. We write:

$$f = F_{s0}(\mathcal{E}, t) \exp(-q\frac{\phi(\mathbf{r}, t)}{T_0}) + g(\mathbf{R}, \mu, \mathcal{E}, t) + \delta f_{s2}(\mathbf{r}, \mathbf{v}). \dots$$

Where F_{s0} is a Maxwellian and s labels species. We take a periodic slab (box) with $\mathbf{B} = B_0 \hat{\mathbf{z}}$ and volume V . We denote the order in the gyro-kinetic expansion by the subscript (i.e. $\delta f_1 \sim \epsilon F_0$ and $\delta f_2 \sim \epsilon \delta f_1$) - note $g \sim \mathcal{O}$. The velocity is given by $\mathbf{v} = v_{\parallel} \hat{\mathbf{z}} + v_{\perp} (\cos \theta \hat{\mathbf{x}} + \sin \theta \hat{\mathbf{y}})$ where θ is the *gyro-phase*. We use the velocity variables $\theta, \mu = v_{\perp}^2/B_0, \mathcal{E} = (1/2)mv^2$ and σ the sign of v_{\parallel} . The guiding center is $\mathbf{R} = \mathbf{r} + \frac{\mathbf{v} \times \mathbf{z}}{\Omega}$. We define the gyro or ring average at fixed \mathbf{R} as:

$$\langle a(\mathbf{r}, \mathbf{v}, t) \rangle_{\mathbf{R}} = \frac{1}{2\pi} \oint d\theta a(\mathbf{R} - \frac{\mathbf{v} \times \mathbf{z}}{\Omega}, \mathbf{v}, t),$$

where the $\omega = qB/m$ and the θ integration is done keeping \mathbf{R} fixed. Note, these gyro-averages are functions of \mathbf{R}, μ and \mathcal{E} and σ . We also define a ring average at fixed \mathbf{r} as:

$$\langle a(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_{\mathbf{r}} = \frac{1}{2\pi} \oint d\theta a(\mathbf{r} + \frac{\mathbf{v} \times \mathbf{z}}{\Omega}, \mathcal{E}, \mu, \sigma, \theta, t),$$

where the θ integration is done keeping \mathbf{r}, μ and \mathcal{E} and σ fixed. The ring distribution $g(\mathbf{R}, \mu, \mathcal{E}, \sigma, t)$ satisfies the gyro-kinetic equation:

$$\frac{\partial g}{\partial t} + v_{\parallel} \frac{\partial g}{\partial z} + [\langle \chi \rangle_{\mathbf{R}}, g] - C(g) = -q \frac{\partial F_0}{\partial \mathcal{E}} \frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial t} = q \frac{F_0}{T_0} \frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial t} \quad (1)$$

where

$$\chi = [\phi - \frac{v_{\parallel} A_{\parallel}}{c} - \frac{\mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp}}{c}].$$

It is important to note that ϕ, h and χ all have zero spatial average over the box. Maxwell's equations (assuming a plasma with one species of ion and electrons) become:

Quasi-Neutrality.

$$-\frac{n_i q^2 \phi}{T_i} + 2\pi q B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle g_i(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_{\mathbf{r}} = \frac{n_i e^2 \phi}{T_i} + 2\pi e B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle g_e(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_{\mathbf{r}} \quad (2)$$

Parallel Ampere's Law.

$$\nabla^2 A_{\parallel} = \mu_0 J_{\parallel} = \mu_0 2\pi q B_0 \sum_{\sigma} \int \int d\mu d\mathcal{E} \sigma \langle g_i(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_r - \mu_0 2\pi e B_0 \sum_{\sigma} \int \int d\mu d\mathcal{E} \sigma \langle g_e(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_r \quad (3)$$

Perpendicular Ampere's Law.

$$\begin{aligned} \nabla \delta B_{\parallel} = \mu_0 \mathbf{z} \times \mathbf{J} = & \mu_0 2\pi q B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle (\mathbf{z} \times \mathbf{v}) g_i(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_r \\ & - \mu_0 2\pi e B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle (\mathbf{z} \times \mathbf{v}) g_e(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_r \end{aligned} \quad (4)$$

or after a little algebra this becomes:

Perpendicular Pressure Balance.

$$\begin{aligned} \nabla \delta B_{\parallel} B_0 = -\mu_0 \nabla \cdot \delta \mathbf{P}_{\perp} = & \mu_0 2\pi B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle m_i(v_{\perp} v_{\perp}) g_i(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_r \\ & + \mu_0 2\pi B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle m_e(v_{\perp} v_{\perp}) g_e(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_r \end{aligned} \quad (5)$$

where clearly $\delta \mathbf{P}_{\perp}$ is the ring averaged perturbed perpendicular pressure tensor.

2 Transport timescale and Heating Rate

Here we calculate the heating terms in Gyrokinetics. There are two derivations here – the first is more conventional but longer the second uses entropy conservation and is, in some sense, more intuitive. Let us also define a time average over times long compared to the fluctuation time (the medium timescale) but short compared to the heating time (which turns out to be $t_{heat} \sim (\omega \epsilon^2)^{-1}$) where ω is the typical frequency of the fluctuations – i.e. $\tau \sim 1$)

$$\bar{A} = \frac{1}{T} \int_{t-T/2}^{t+T/2} A \quad (6)$$

where $1 \ll \omega T \ll \frac{1}{\epsilon^2}$. During this average we may hold τ constant.

To obtain the long time dependance of n_0 and T_0 we consider the moment equations of the full un-gyro-averaged kinetic equation for any species ($\frac{df_s}{dt} = C_{sr}(f_s, f_r) + C_{ss}(f_s, f_s)$ where $C_{sr}(f_s, f_r)$ is collisions of species s on species r and $C_{ss}(f_s, f_s)$ is like particle collisions). This avoids having to write out all the $\mathcal{O}(\epsilon^2)$ terms from the expanded equations. Integrating over velocity we obtain, after integration by parts, the well known result:

$$\int \frac{d^3 r}{V} \int d^3 v \frac{\partial f_s}{\partial t} = \epsilon^2 \frac{dn_{0s}}{d\tau} + \frac{d}{dt} \int \frac{d^3 r}{V} \int d^3 v \delta f_{s2} = 0$$

which is of course conservation of particles (we have used that the first order distributions space average to zero). Averaging over the medium timescale removes the δf_{s2} term and we are left with:

$$\frac{dn_{0s}}{d\tau}$$

so for both species n_0 is constant. More interesting is to calculate the energy exchange, multiplying the FP equation by $\frac{1}{2}mv^2$ and integrating we obtain:

$$\begin{aligned}\int \frac{d^3r}{V} \int d^3v \frac{1}{2}mv^2 \frac{\partial f_s}{\partial t} &= \frac{3}{2}n_0 \epsilon^2 \frac{dT_{0s}}{d\tau} + \frac{d}{dt} \int \frac{d^3r}{V} \int d^3v \frac{1}{2}mv^2 \delta f_{s2} \\ &= \int \frac{d^3r}{V} \int d^3v q(\mathbf{E} \cdot \mathbf{v}) f_s + \int \frac{d^3r}{V} \int d^3v \frac{1}{2}mv^2 C_{sr}(f_s, f_r)\end{aligned}$$

where the collisional energy exchange terms between species are now included – note like particle collisions do not produce a loss of energy and thus do not appear (this is easy to prove). The collisional energy exchange is standard since to this order it is between Maxwellian species. Specifically:

$$\int \frac{d^3r}{V} \int d^3v \frac{1}{2}mv^2 C_{sr}(f_s, f_r) = n_0 s \nu_E^{sr} (T_r - T_s).$$

Where ν_E^{sr} is given on page 33 of the plasma formulary – it is $\sqrt{m_e/m_i}$ smaller than the ion collision rate which is itself $\sqrt{m_e/m_i}$ smaller than the electron collision rate. Note we are taking (as I say in the first paragraph) the lowest order the distribution function F_0 to be a Maxwellian. In the heating term the scalar potential is the largest but as we see it mostly cancels.

$$\begin{aligned}\int \frac{d^3r}{V} \int d^3v [-q\mathbf{v} \cdot \nabla \phi f] &= \int \frac{d^3r}{V} \int d^3v [q \frac{\partial \phi}{\partial t} f] - \int \frac{d^3r}{V} \int d^3v [q(\frac{\partial \phi}{\partial t} + \mathbf{v} \cdot \nabla \phi) f] \\ &= \int \frac{d^3r}{V} \int d^3v [q \frac{\partial \phi}{\partial t} \delta f] + \int \frac{d^3r}{V} \int d^3v [q(\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f) \phi] - \int d^3v [q \frac{\partial(\phi f)}{\partial t}] \\ &= \int \frac{d^3r}{V} \int d^3v [q \frac{\partial \phi}{\partial t} \delta f] - \frac{q^2}{m} \int \frac{d^3r}{V} \int d^3v \frac{\partial}{\partial v} \cdot [(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \phi) f] - \int \frac{d^3r}{V} \int d^3v [q(C(f, f)) \phi] - \int d^3v [q \frac{\partial(\phi f)}{\partial t}] \\ &= \int \frac{d^3r}{V} \int d^3v [q \frac{\partial \phi}{\partial t} \delta f] - \int d^3v [q \frac{\partial(\phi f)}{\partial t}]\end{aligned}$$

where: we have integrated by parts in time and space between lines one and two, between lines two and three we use $\frac{df}{dt} = C(f, f)$ and, finally, we use gauss's law in velocity space and that all collisions conserve particles to obtain line four. Therefore gathering terms we get,

$$\begin{aligned}\frac{3}{2}n_0 \epsilon \frac{T_0}{d\tau} + \frac{d}{dt} \int \frac{d^3r}{V} \int d^3v \frac{1}{2}mv^2 \delta f_{s2} &= \int \frac{d^3r}{V} \int d^3v \{q \frac{\partial}{\partial t} [\phi - \frac{v_{||} A_{||}}{c} - \frac{\mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp}}{c}] \delta f\} \\ &\quad + \int \frac{d^3r}{V} \int d^3v \frac{1}{2}mv^2 C_{sr}(f_s, f_r) - \int d^3v [q \frac{\partial(\phi \delta f)}{\partial t}]\end{aligned}$$

a very familiar form. Note putting in the scalings for the fluctuating quatities now gives the order

$$\frac{3}{2}n_0 \epsilon^2 \frac{T_0}{d\tau} \sim \omega \epsilon^2 T_0.$$

This shows that the equilibrium variations are, as expected, one order higher than is kept in gyro-kinetic calculations of the fluctuating quantities i.e. $\frac{\partial F_0}{\partial t} \sim \epsilon^2 \frac{\partial \delta f}{\partial t}$. It is not hard to show that this ordering is consistent with all the cascading energy in the Alfvén wave cascade becoming heat. The adiabatic/Boltzmann term gives.

$$\int \frac{d^3 r}{V} \int d^3 v \left\{ q \frac{\partial}{\partial t} [\phi - \frac{v_{\parallel} A_{\parallel}}{c} - \frac{v_{\perp} \cdot A_{\perp}}{c}] (-q \frac{\phi}{T_0} F_0) \right\} = -\frac{d}{dt} \int \frac{d^3 r}{V} \left[\frac{n_0}{T_0} \left(\frac{\phi^2}{2} \right) \right]$$

since: the A_{\parallel} term is odd in v_{\parallel} and the A_{\perp} gyro-averages to zero (or equivalently is odd in v_{\perp}). Doing the gyro averages we obtain,

$$\begin{aligned} \frac{3}{2} n_0 s \epsilon^2 \frac{dT_{0s}}{d\tau} + \frac{d}{dt} \left[\int \frac{d^3 r}{V} \int d^3 v \left(\frac{1}{2} m_s v^2 \delta f_{s2} + q_s \phi h_s \right) - \int \frac{d^3 r}{V} \left[\frac{q_s^2 n_0 s}{T_{0s}} \left(\frac{\phi^2}{2} \right) \right] \right] \\ = \int \frac{d^3 R}{V} \int d^3 v \left(q \frac{\partial \langle \chi \rangle_{h_s}}{\partial t} \right) + n_0 s \nu_{\mathcal{E}}^{sr} (T_r - T_s) \end{aligned} \quad (7)$$

Note the fast time average (see Equation (6)) of the second term on the left hand side is zero and therefore doesn't contribute to the average heating. The first term on the right hand side is the desired heating term – we have written it in a slightly confusing way as an integral over v at constant gyro-center variable R then an integral over R this is to connect with the gyro-kinetic equation.

Multiplying the gyro-kinetic equation, Eq. (5) by $(Th)/F_0$ and integrating over space and velocity we get:

$$\frac{d}{dt} \int \frac{d^3 r}{V} \int d^3 v \frac{T}{2F_0} h_s^2 - \int \frac{d^3 r}{V} \int d^3 v \frac{T_{0s}}{F_{0s}} h_s C(h_s) = \int \frac{d^3 R}{V} \int d^3 v \left(q \frac{\partial \langle \chi \rangle_{h_s}}{\partial t} \right) \quad (8)$$

Clearly the time average of the heating term arises from the collisional term Combining Eqs. (8) and (7) we obtain:

$$\begin{aligned} \frac{3}{2} n_0 s \epsilon^2 \frac{dT_{0s}}{d\tau} + \frac{d}{dt} \left[\int \frac{d^3 r}{V} \int d^3 v \left(\frac{1}{2} m_s v^2 \delta f_{s2} + q_s \phi h_s - \frac{T_{0s}}{2F_{0s}} h_s^2 \right) - \int \frac{d^3 r}{V} \left[\frac{q_s^2 n_0 s}{T_{0s}} \left(\frac{\phi^2}{2} \right) \right] \right] \\ = - \int \frac{d^3 r}{V} \int d^3 v \frac{T_{0s}}{F_{0s}} h_s C(h_s) + n_0 s \nu_{\mathcal{E}}^{sr} (T_r - T_s) \end{aligned} \quad (9)$$

The second term on the left hand side of Eq. (9) is the sloshing energy and averages to zero over the long time-scale. The collisional term (with the negative sign) is positive definite for like particle and pitch angle collision operators. Thus it yields the long time-scale heating of species s:

$$\frac{3}{2} n_0 s \epsilon^2 \frac{dT_{0s}}{d\tau} = - \int \frac{d^3 r}{V} \int d^3 v \frac{T}{F_0} \overline{[h_s C_s(h_s)]} + n_0 s \nu_{\mathcal{E}}^{sr} (T_r - T_s) \quad (10)$$

This form is much easier to evaluate numerically than the fast time average of Eq. (7) since it is positive definite and therefore there cannot be cancelation between positive and negative parts making the average hard to calculate. It is also clear that heating is collisional —when collisions are small h will develop small scales in velocity space (typically $\Delta v \sim v^{1/2}$) so that the

heating to be is essentially independent of ν . This is roughly the argument made by Landau to justify the collisionless rate of landau damping when it is known that entropy can increase only due to collisions. We believe that it is essential for any kinetic code to have some collisions to smooth the distributions at small velocity scales. We now rederive the result of Eq. (10) using an entropy argument.

3 Entropy Argument

For ions the Boltzman H theorem is (ignoring the small electron ion collisions):

$$\frac{dS_i}{dt} = -\frac{d}{dt} \int \frac{d^3r}{V} \int d^3v [f_i \ln f_i] = -\int \frac{d^3r}{V} \int d^3v \ln f_i C_{ii}(f_i, f_i). \quad (11)$$

The standard tricks (for example see Lifshitz and Pitaevski *Physical Kinetics*) show that the right hand side is positive and therefore that entropy increases. We expand entropy about the Maxwellian as in Eq. (1). We obtain to second order:

$$\begin{aligned} \frac{dS_i}{dt} &\sim -\frac{d}{dt} \int \frac{d^3r}{V} \int d^3v [F_{0i} \ln F_{0i} + (1 + \ln F_{0i})\delta f_{2i} + \frac{\delta f_{1i}^2}{2F_{0i}}] \\ &\sim -\int \frac{d^3r}{V} \int d^3v [\frac{\delta f_{1i}}{F_{0i}}] C_{ii}(\delta f_{0i}) \end{aligned} \quad (12)$$

Where we have used the energy conservation properties of the ion-ion collisions and the fact that δf_{1i} has zero average over space. Using the conservation of particles, the form of the Maxwellian and the definition of h we obtain the slow evolution of temperature.

$$\begin{aligned} \frac{3}{2} n_{0i} \frac{\epsilon^2}{T_i} \frac{dT_i}{dr} + \frac{d}{dt} \left[\int \frac{d^3r}{V} \int d^3v (\frac{1}{2} \frac{m_i v^2}{T_i} \delta f_{1i} + \frac{q_i \phi h_i}{T_i} - \frac{T_i^2}{2F_{0i}} h_i^2) - \int \frac{d^3r}{V} [\frac{q_i^2 n_{0i}}{T_{0i}} (\frac{\phi^2}{2})] \right] \\ = -\int \frac{d^3r}{V} \int d^3v \frac{1}{F_{0i}} h_i C_{ii}(h_i) \end{aligned} \quad (13)$$

this is just the result of Eq. (9) – we see the heating directly as irreversible entropy production. Again the heating is just the average of the entropy production term – i.e. the right hand side of Eq. (13) – and the fast time average of the sloshing term of course vanishes (otherwise the perturbed distributions accumulate).

4 Energy Conservation

Lets now move on to energy conservation: Poyntings theorem is:

$$\frac{\partial}{\partial t} \int d^3r (\frac{\epsilon_0 E^2}{2} + \frac{B^2}{2\mu_0}) + \oint \frac{cE \times B}{4\pi} \cdot dS = - \int d^3r (\mathbf{J} + \mathbf{J}_{antenna}) \cdot \mathbf{E}.$$

Where \mathbf{J} is the plasma current. In a periodic box the surface term is zero. Now lets examine the gyro-kinetic orderings.

$$v_{\perp} \sim \frac{\delta E_{\perp}}{B} \sim V_{diamagnetic} \sim v_{thi}\epsilon$$

$$\delta E_{\perp} \sim v_{thi}\epsilon B \sim \frac{\delta E_{\parallel}}{\epsilon}$$

$$\frac{\delta B_{\parallel}}{B} \sim \frac{\delta B_{\perp}}{B} \sim \epsilon$$

thus

$$\epsilon_0 E^2 \sim \epsilon_0 (v_{thi})^2 \epsilon^2 B^2$$

$$\frac{\delta B^2}{\mu_0} \sim \epsilon_0 (c^2) \epsilon^2 B^2.$$

thus because we are non-relativistic the magnetic energy dominates as it should since we drop the displacement current from which we get the electric field energy term. Thus we have:

$$\frac{\partial}{\partial t} \int d^3 r \left(\frac{\delta B^2}{2\mu_0} \right) = - \int d^3 r (\mathbf{J} + \mathbf{J}_{antenna}) \cdot \mathbf{E}.$$

during any growth phase prior to saturation both sides are of order $\omega \epsilon^2 B^2 \sim \omega \epsilon^2 n_0 T_0$ since we treat $\beta = \frac{\mu_0 n_0 T_0}{\delta B^2} \sim \mathcal{O}(1)$. Averaging over the medium timescale requires the B^2 energy to be constant and we arrive at the steady state balance:

$$\int d^3 r (\overline{\mathbf{J} + \mathbf{J}_{antenna}}) \cdot \overline{\mathbf{E}} = 0.$$

Or summing the heating over species the time average energy conservation is:

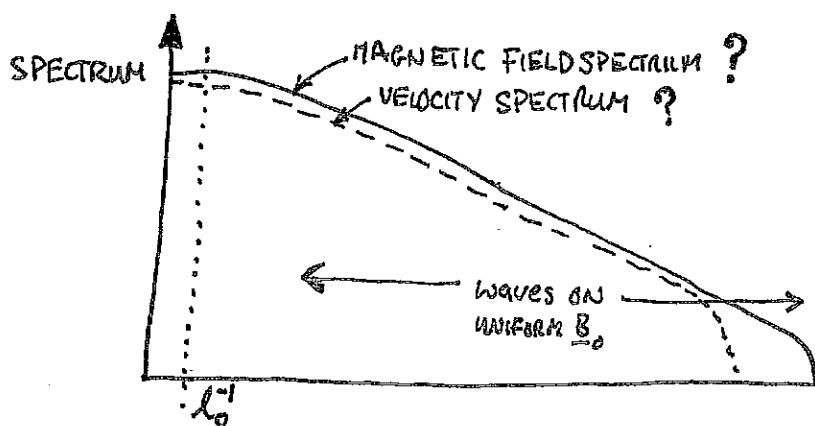
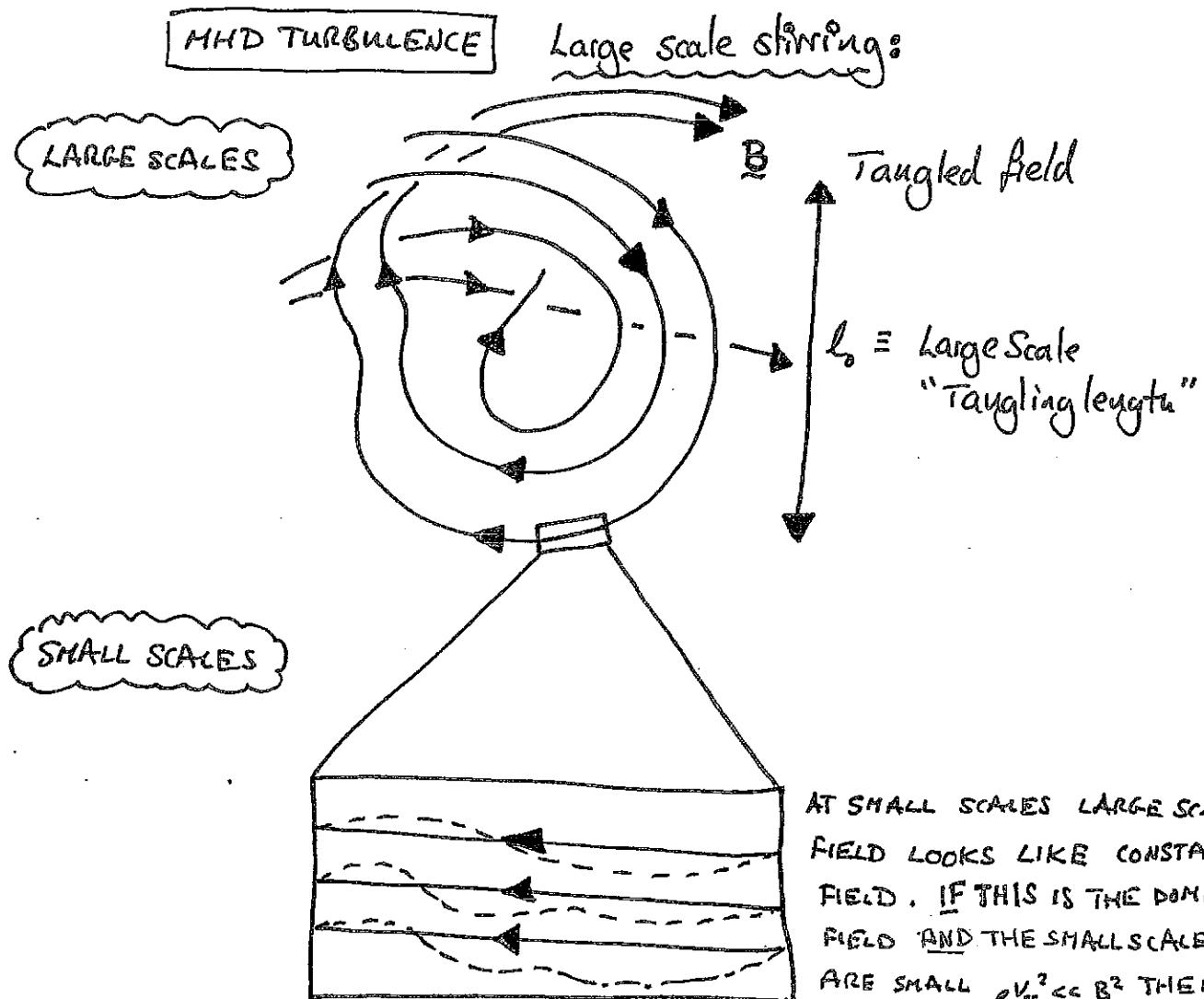
$$\sum_s \frac{3}{2} n_{0s} \epsilon^2 \frac{T_{0s}}{d\tau} = - \int d^3 r \overline{\mathbf{J}_{antenna}} \cdot \overline{\mathbf{E}} \quad (14)$$

Lecture #8: Alfvén-Waves.

first in MHD then in gyro-kinetics.

(i) We shift gears a bit to look at MHD waves, — this will set us up for looking at MHD turbulence and the "Alfvén Wave Cascade".

(ii) As motivation lets consider a standard argument (which might be wrong)

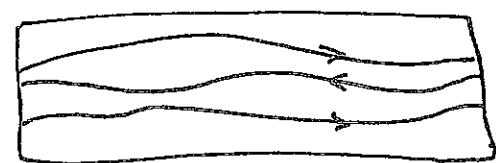


IT IS normally assumed that the dissipation takes place at small/tiny scales and that energy is "cascaded" to these dissipation scales.

(see future lecture)

(iii) Although I won't demonstrate it here the waves in the cascade have $k_{\perp} \gg k_{\parallel}$ - like gyrokinetics. Here k_{\parallel}/k_{\perp} refer to \underline{k} vectors parallel or perpendicular to the local field. Let us therefore study MHD waves in the limit $k_{\parallel} \ll k_{\perp}$ - this must be the fluid limit of M.H.D..

(iv) M.H.D. Waves with $k_{\perp} \gg k_{\parallel}$



$$\underline{B}_0 = B_0 \hat{\underline{z}} \quad P_0 = \text{constant.}$$

$$\rho_0 = \text{constant.}$$

$$V_0 = 0.$$

Perturbation $\xi = \text{DISPLACEMENT} = \sum \xi_i e^{ik_{\perp} \cdot \underline{r} - i\omega t}$ "Plane Waves"

Equations

a) MAGNETIC FIELD } $\frac{\partial \underline{B}}{\partial t} = \nabla \times (\delta \underline{v} \times \underline{B}_0) \Rightarrow \delta \underline{B} = ik_{\parallel} B_0 \xi_{\perp} - iB_0 k_{\parallel} \xi_{\parallel} \hat{\underline{z}}$
 "FROZEN IN LAW" $= \delta \underline{B}_{\perp} + \delta B_{\parallel} \hat{\underline{z}}$

[NOTE: $\delta \underline{E} + \delta \underline{v} \times \underline{B}_0 = 0 \Rightarrow \delta E_z = 0$]

① Field line bending Field line compression.

b) PRESSURE EQUATION } $\frac{\partial P}{\partial t} + \underline{v} \cdot \nabla P = -P_p \nabla \cdot \underline{v} \Rightarrow$

② Adiabatic compression.

c) MOMENTUM EQUATION } $\rho \frac{d \underline{v}}{dt} = -\nabla P + \underline{J} \times \underline{B} = -\nabla(P + \frac{B^2}{2\mu_0}) + \frac{\underline{B} \cdot \nabla P}{\mu_0}$
 $\delta \nabla \times \underline{B} = \mu_0 \underline{J}$

③ - $\rho_0 \omega^2 \xi = -ik \left\{ \delta P + \frac{\delta B_{\parallel} B_0}{\mu_0} \right\} - \frac{k_{\parallel}^2 B_0^2}{\mu_0} \xi_{\perp} + ik_{\parallel} B_0 \frac{\delta B_{\parallel}}{\mu_0} \hat{\underline{z}}$

largest force from
this term in k_{\perp} direction.

(v) We get the three usual waves:

FAST WAVE

If $\frac{\delta p + \delta B_{\parallel} B_0}{M_0} \neq 0$ then the largest force is in the \underline{k}_{\perp} direction
hence $\underline{\xi}_{\perp} = \underline{k}_{\perp} \underline{\xi}_F$ due to this force.

then from (1), (2) and (3) (in the \underline{k}_{\perp} direction)

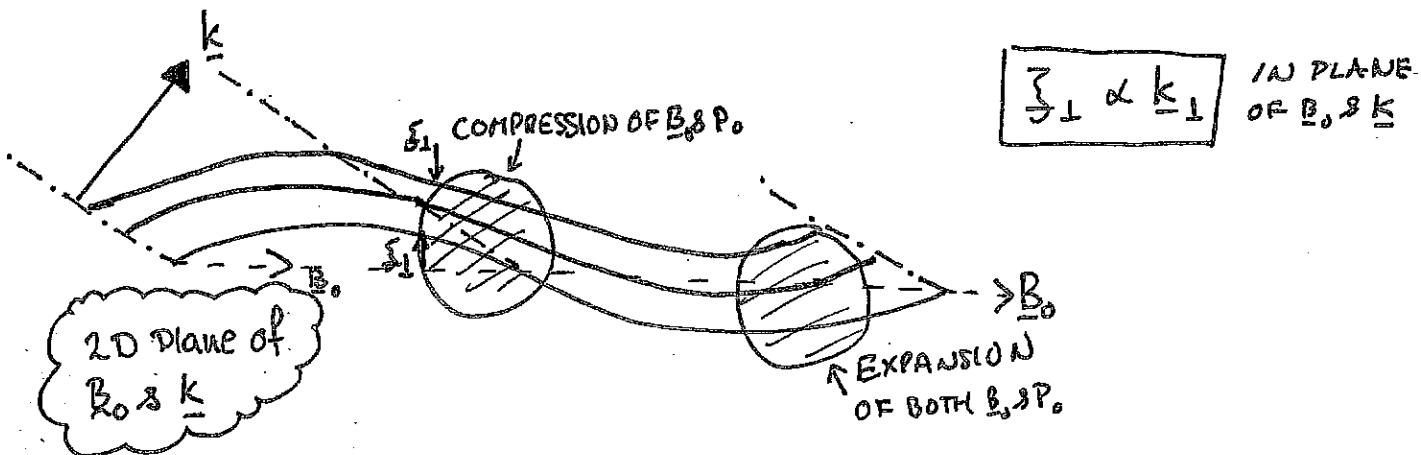
$$-\rho_0 \omega^2 \xi_F = -k_{\perp}^2 \left[\Gamma P_0 + \frac{B_0^2}{\mu_0} \right] \xi_F$$

$$\Rightarrow \omega^2 = k_{\perp}^2 (c_s^2 + V_A^2)$$

$$c_s^2 = \frac{\Gamma P_0}{P_0}$$

$$V_A^2 = \frac{B_0^2}{\mu_0 P_0}$$

$$\frac{\delta B_{\parallel i}}{B_0} = \frac{\delta p}{\Gamma P_0} \neq 0$$



High frequency wave that is ordered out of gyro-kinetics
since with $k_{\perp} \approx \frac{1}{\rho_i}$, $\omega \sim \omega_{ci}$. The fast wave steepens
until it forms shocks.

ALFUEN WAVE

$\delta p + \frac{\delta B_{\parallel i} B_0}{M_0} = 0$ so largest force is
now cancelling itself. In Alfvén wave $\delta B_{\parallel i} = 0 \Rightarrow \underline{k} \cdot \underline{\xi}_{\perp} = 0$

$$\underline{\xi}_{\perp} = (\underline{k} \times \hat{\underline{z}}) \hat{\underline{z}} \underline{\xi}_A$$

Displacement out of
plane of $\underline{k} \times \hat{\underline{z}} (B_0)$

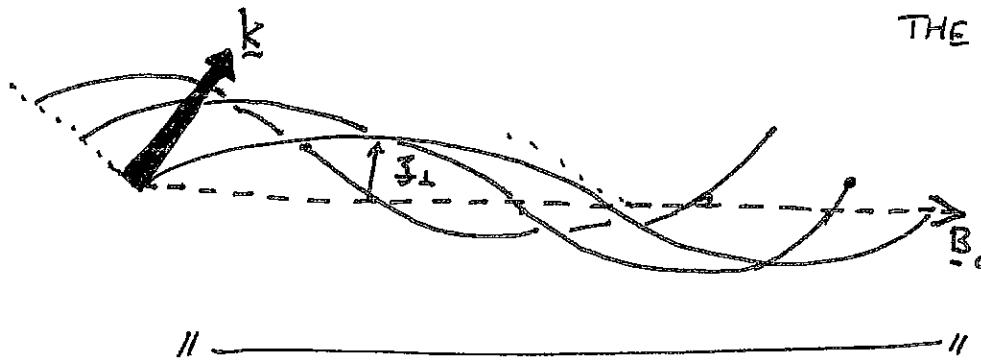


$$\omega^2 = k_{\parallel}^2 V_A^2$$

$$\delta B_{\parallel} = \delta E_{\parallel} = 0$$

SORRY ABOUT THE DRAWING.

DISPLACEMENT OUT OF
THE PLANE OF \underline{k} AND \underline{B}_0



Motion due to
curvature force of
bent field lines
no "pressure forces"

(vii) SLOW WAVE

$$\delta p + \frac{\delta B_{\parallel} B_0}{\mu_0} = 0 \quad \text{Again but } \delta p = - \frac{\delta B_{\parallel} B_0}{\mu_0} \neq 0$$

$$\rightarrow \Gamma_{p_0} i k_{\parallel} \xi_{\parallel} + \Gamma_{p_0} i \underline{k} \cdot \underline{\xi}_{\perp} = - \frac{B_0^2}{\mu_0} i \underline{k} \cdot \underline{\xi}_{\perp}$$

$$i k_{\parallel} \xi_{\parallel} = \left(1 + \frac{1}{\gamma \beta} \right) \frac{\delta B_{\parallel}}{B_0}$$

$$\beta = \frac{\mu_0 P}{B_0^2}$$

To maintain zero total pressure plasma is squeezed
along field line. $\xi_{\parallel} \sim \frac{k_{\perp}}{K_{\parallel}} \xi_{\perp} \gg \xi_{\perp}$.

From ③

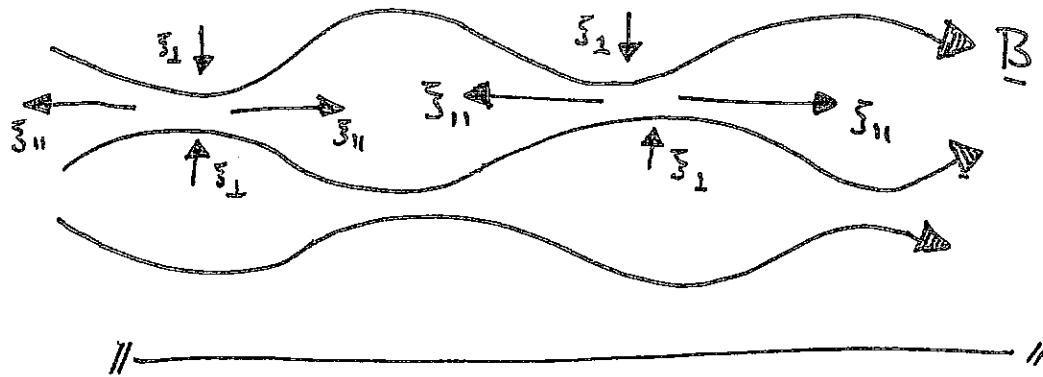
$$-\rho_0 \omega^2 \xi_{\parallel} = -i k_{\parallel} \delta p = i k_{\parallel} \frac{\delta B_{\parallel}}{B_0} \frac{B_0^2}{\mu_0}$$

$$\omega^2 = k_{\parallel}^2 \left[\frac{V_A^2 C_S^2}{V_A^2 + C_S^2} \right]$$

At low β $\omega^2 = k_{\parallel}^2 C_S^2$ sound waves

At high β $\omega^2 = k_{\parallel}^2 V_A^2$ "pseudo alfvén" waves

2.



(viii) The Alfvén wave and slow wave are in our gyro-kinetic description.
It is instructive to look at them in GK variables.

(ix) ALFVEN-WAVE: GK.VARIABLES

$$\delta \underline{E} = -\nabla \phi - \frac{d \underline{A}}{dt} \quad \delta \underline{B} = \nabla \times \underline{A}$$

$$\nabla \cdot \underline{A} = 0 \Rightarrow \nabla_{\perp} \cdot \underline{A}_{\perp} = 0$$

$$\delta B_{\parallel} = \nabla \cdot (\underline{A}_{\parallel} \times \underline{e}) = 0$$

a) $\delta E_{\parallel} = 0 \Rightarrow -ik_{\parallel}\phi + i\omega A_{\parallel} = 0$

b) $\delta B_{\parallel} = 0 \Rightarrow \underline{A}_{\perp} = 0$

c) Quasi-Neutrality $\Rightarrow \nabla \cdot \underline{j} = -\frac{d(\text{CHARGE})}{dt} = 0 \quad -ik_{\parallel} A_{\parallel} = ik_{\perp} \underline{A}_{\perp}$

d) From Perpendicular Force Balance

$$\rho \frac{\partial \underline{v}_{\perp}}{\partial t} = \underline{j}_{\perp} \times \underline{B}_0$$

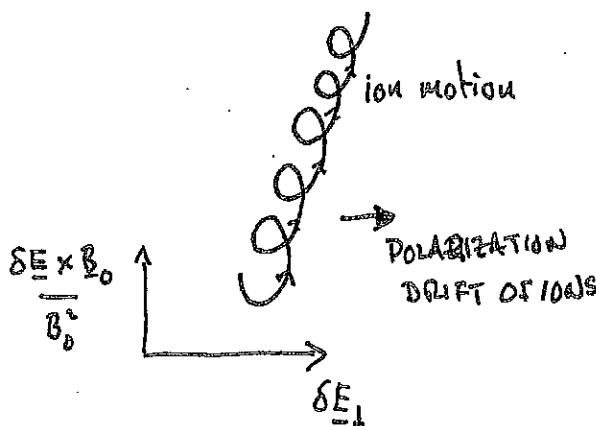
e) From Ohms law

$$\delta \underline{v}_{\perp} = \frac{\delta \underline{E} \times \underline{B}_0}{B_0^2} = -ik_{\perp} \phi \times \hat{z}$$

$$\underline{j}_{\perp} = \frac{\rho}{B_0^2} \frac{\partial \delta \underline{E}_{\perp}}{\partial t}$$

POLARIZATION CURRENT

COMES FROM
ION POLARIZATION
DRIFT, ELECTRON
DRIFT IS TINY.



6.

f) Parallel Ampere's law.

$$\nabla_{\perp}^2 A_{\parallel} = -k_{\perp}^2 A_{\parallel} = -\mu_0 J_{\parallel} = \mu_0 \frac{k_{\perp} J_{\perp}}{k_{\parallel}}$$

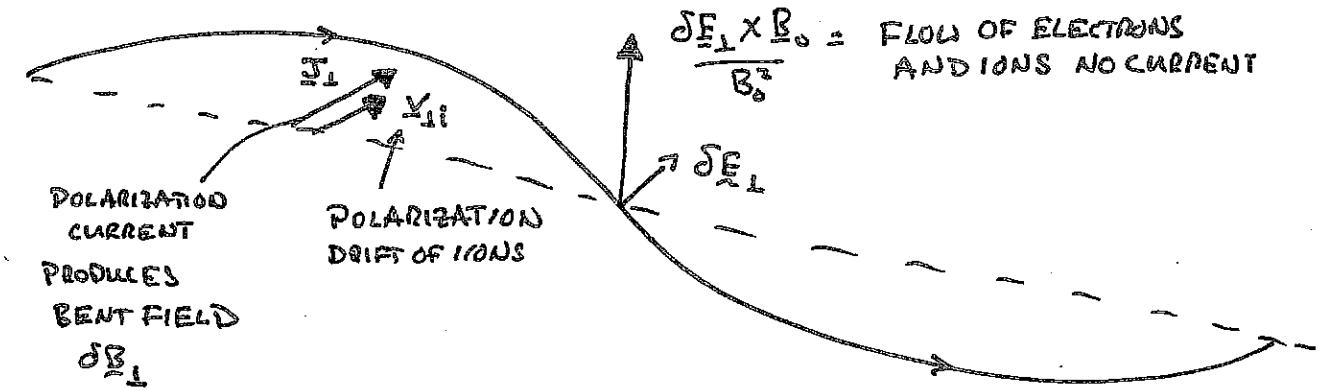
SUBSTITUTING FOR THE POLARIZATION CURRENT WE OBTAIN

$$-k_{\perp}^2 A_{\parallel} = -\frac{\omega}{k_{\parallel}} v_A^2 k_{\perp}^2 \phi$$

AND USING $\delta E_{\parallel} = 0$ (a) WE GET

$$\boxed{\omega^2 = k_{\parallel}^2 v_A^2}$$

PICTURE



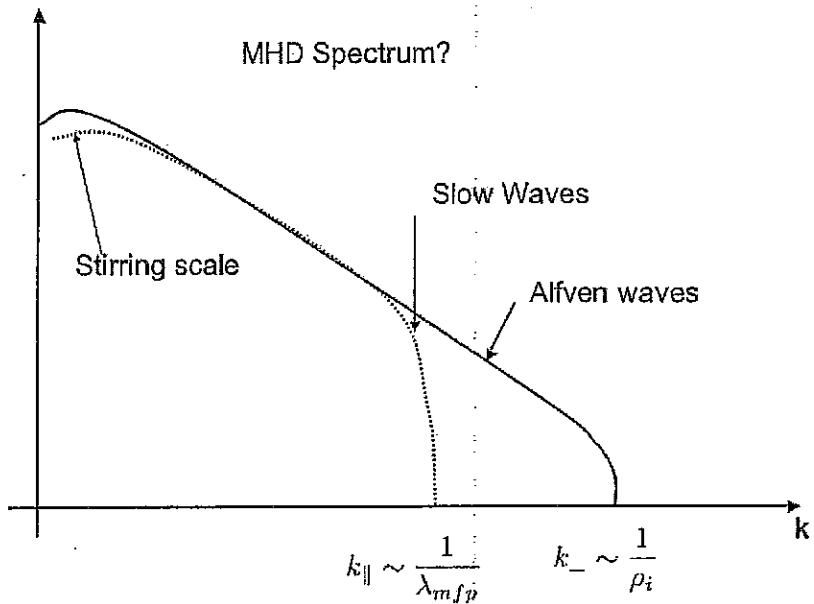
It is a little odd since we never have to drive J_{\parallel} - it is whatever it has to be to make $\nabla \cdot \underline{J} = 0$. Actually a small δE_{\parallel} drives a large J_{\parallel} in electrons ^{but} we never have to solve for it.

*Lecture # 8.***Linear Waves in the Collisionless Gyrokinetic Slab – Alfvén Waves.**

Warning, needs more checking for errors.

1 Collisionless waves.

In many cases of MHD turbulence the small scale fluctuations continue below the mean free path scale i.e. to scales where $k_{\parallel} \lambda_{mfp} \gg 1$. For example in fusion the mean free path is about $1km$ and the fluctuations can be on parallel scales of a few meters and less. In the solar wind the mean free path is about equal to the distance to the sun $1.5 \times 10^{13} cm$, and fluctuations of much smaller parallel scales are observed. Alfvén waves in these systems don't dissipate/damp until $k_{\perp} \rho_i \sim 1$ and this is usually a very collisionless scale. Thus there are reasons to believe that even when the system is stirred at scales well above the mean free path, energy cascades to collisionless scales before dissipating. We would like to understand Alfvén wave turbulence in this regime – for all kinds of applications. In fact, as



we shall see, we expect a spectrum like:

Today we will look at the properties of collisionless linear waves in the gyro-kinetic slab – this is a fairly algebraic topic but I hope to insert as much physics as possible. These waves will be the building blocks of the turbulence. As we show below the slow wave is damped in this regime while the Alfvén wave continues to the larmor radius scales $k_{\perp} \rho_i \sim 1$.

2 Summary of Gyro-kinetic Equations.

Here I am just tidying up Lecture # 7's notes – getting more of the typos. We write:

$$f = F_{s0}(\mathcal{E}, \tau) \exp(-q \frac{\phi(\mathbf{r}, t)}{T_0}) + g(\mathbf{R}, \mu, \mathcal{E}, t) + \delta f_{s2}(\mathbf{r}, \mathbf{v}) \dots$$

Where F_{s0} is a Maxwellian and s labels species. We take a periodic slab (box) with $\mathbf{B} = B_0 \mathbf{z}$ and volume V . We denote the order in the gyro-kinetic expansion by the subscript (i.e. $\delta f_1 \sim \epsilon F_0$ and $\delta f_2 \sim \epsilon \delta f_1$) – note $g \sim \mathcal{O}(\epsilon)$. The velocity is given by $\mathbf{v} = v_{\parallel} \mathbf{z} + \mathbf{v}_{\perp} (\cos \theta \mathbf{x} + \sin \theta \mathbf{y})$ where θ is the *gyro-phase*. We use the velocity variables θ , $\mu = v_{\perp}^2/B_0$, $\mathcal{E} = (1/2)mv^2$ and σ the sign of v_{\parallel} . The guiding center is $\mathbf{R} = \mathbf{r} + \frac{\mathbf{v} \times \mathbf{z}}{\Omega}$. We define the gyro or ring average at fixed \mathbf{R} as:

$$\langle a(\mathbf{r}, \mathbf{v}, t) \rangle_{\mathbf{R}} = \frac{1}{2\pi} \oint d\theta a(\mathbf{R} - \frac{\mathbf{v} \times \mathbf{z}}{\Omega}, \mathbf{v}, t),$$

where the $\omega = qB/m$ and the θ integration is done keeping \mathbf{R} fixed. Note, these gyro-averages are functions of \mathbf{R} , μ and \mathcal{E} and σ . We also define a ring average at fixed \mathbf{r} as:

$$\langle a(\mathbf{R}, \mathcal{E}, \mu, \sigma, \theta, t) \rangle_{\mathbf{r}} = \frac{1}{2\pi} \oint d\theta a(\mathbf{r} + \frac{\mathbf{v} \times \mathbf{z}}{\Omega}, \mathcal{E}, \mu, \sigma, \theta, t),$$

where the θ integration is done keeping \mathbf{r} , μ and \mathcal{E} and σ fixed. The ring distribution $g(\mathbf{R}, \mu, \mathcal{E}, \sigma, t)$ satisfies the gyro-kinetic equation:

$$\frac{\partial g}{\partial t} + v_{\parallel} \frac{\partial g}{\partial z} + [\langle \chi \rangle_{\mathbf{R}}, g] - \langle C(g) \rangle_{\mathbf{R}} = -q \frac{\partial F_0}{\partial \mathcal{E}} \frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial t} = q \frac{F_0}{T_0} \frac{\partial \langle \chi \rangle_{\mathbf{R}}}{\partial t} \quad (1)$$

where

$$\chi = [\phi - \frac{v_{\parallel} A_{\parallel}}{c} - \frac{\mathbf{v}_{\perp} \cdot \mathbf{A}_{\perp}}{c}].$$

It is important to note that ϕ , h and χ all have zero spatial average over the box. Maxwell's equations (assuming a plasma with one species of ion and electrons) become:

Quasi-Neutrality.

$$-\frac{n_i q^2 \phi}{T_i} + 2\pi q B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle g_i(\mathbf{R}, \mathcal{E}, \mu, \sigma, t) \rangle_{\mathbf{r}} = \frac{n_i e^2 \phi}{T_i} + 2\pi e B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle g_e(\mathbf{R}, \mathcal{E}, \mu, \sigma, t) \rangle_{\mathbf{r}} \quad (2)$$

Parallel Ampere's Law.

$$\nabla^2 A_{\parallel} = \mu_0 J_{\parallel} = \mu_0 2\pi q B_0 \sum_{\sigma} \int \int d\mu d\mathcal{E} \sigma \langle g_i(\mathbf{R}, \mathcal{E}, \mu, \sigma, t) \rangle_{\mathbf{r}} - \mu_0 2\pi e B_0 \sum_{\sigma} \int \int d\mu d\mathcal{E} \sigma \langle g_e(\mathbf{R}, \mathcal{E}, \mu, \sigma, t) \rangle_{\mathbf{r}} \quad (3)$$

Perpendicular Ampere's Law.

$$\nabla \delta B_{\parallel} = \mu_0 z \times J = \mu_0 2\pi q B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle (z \times v) g_i(R, \mathcal{E}, \mu, \sigma, t) \rangle_r - \mu_0 2\pi e B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle (z \times v) g_e(R, \mathcal{E}, \mu, \sigma, t) \rangle_r \quad (4)$$

or after a little algebra this becomes:
Perpendicular Pressure Balance.

$$\nabla \delta B_{\parallel} B_0 = -\mu_0 \nabla \cdot \delta P_{\perp} = \mu_0 2\pi B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle m_i (v_{\perp} v_{\perp}) g_i(R, \mathcal{E}, \mu, \sigma, t) \rangle_r + \mu_0 2\pi B_0 \sum_{\sigma} \int \int \frac{d\mu d\mathcal{E}}{|v_{\parallel}|} \langle m_e (v_{\perp} v_{\perp}) g_e(R, \mathcal{E}, \mu, \sigma, t) \rangle_r \quad (5)$$

where clearly δP_{\perp} is the ring averaged perturbed perpendicular pressure tensor.

3 Linear Collisionless Gyro-kinetics.

To find the collisionless linear waves we linearize the GK equation and drop the collisions:

$$\frac{\partial g}{\partial t} + v_{\parallel} \frac{\partial g}{\partial z} = -q \frac{\partial F_0}{\partial \mathcal{E}} \frac{\partial \langle \chi \rangle_R}{\partial t} = q \frac{F_0}{T_0} \frac{\partial \langle \chi \rangle_R}{\partial t} \quad (6)$$

In treating the MHD Alfvén wave we found it convenient to use $\nabla \cdot J = 0$. The gyro-kinetic version of this relation can be derived by multiplying Eq. (6) for both species (ions and electrons) by the charge, integrating over velocity and summing. Using Eqs. (2) we obtain:

$$\frac{\partial J_{\parallel}}{\partial z} = -\frac{\partial}{\partial t} \left(\frac{q^2 n_i}{T_i} [\phi - \langle \langle \phi \rangle \rangle_i] + \frac{e^2 n_e}{T_e} [\phi - \langle \langle \phi \rangle \rangle_e] + \frac{q^2 n_i}{T_i} \langle \langle v_{\perp} \cdot A_{\perp} \rangle \rangle_i + \frac{e^2 n_e}{T_e} \langle \langle v_{\perp} \cdot A_{\perp} \rangle \rangle_e \right) \quad (7)$$

where the double bracket is for species α is defined by:

$$\langle \langle a(r, v, t) \rangle \rangle_{\alpha} = \int_0^{\infty} dv_{\perp} v_{\perp} \left(\frac{m_{\alpha}}{T_{\alpha}} \right) \exp \left(-\frac{m_{\alpha} v_{\perp}^2}{2 T_{\alpha}} \right) \langle \langle a(r, v, t) \rangle \rangle_{R_{\alpha}} \quad (8)$$

and the two ring averages are defined in the previous section – note they depend on the species since $R_{\alpha} = r + \frac{v \times z}{\Omega_{\alpha}} = r + \rho_{\alpha}$. Clearly the right hand side of Eq. (7) is minus the divergence of the perpendicular current. It is instructive to take the drift kinetic limit where we can expand in small larmor radius (for example $\phi(r, t) \sim \phi(R, t) + \rho_{\alpha} \cdot \nabla_R \phi(R, t) + \frac{1}{2} \rho_{\alpha} \cdot \nabla_R (\rho_{\alpha} \cdot \nabla_R \phi(R, t)) \dots$). In this limit we get

$$\frac{\partial J_{\parallel}}{\partial z} = -\frac{\partial}{\partial t} \left(\frac{q^2 n_i}{2 T_i} [\rho_i^2 \nabla^2 \phi] \right) + \mathcal{O}(\rho_i^2 \nabla^2 \frac{\delta B_{\parallel}}{B_0}) \quad (9)$$

the first term on the right hand side is the divergence of the ion polarization current. The second term is negligible in this limit as we shall see. Note there is no divergence of the lowest order $E \times B$ velocity and since it is independent of the mass and charge it carries no current anyway.

For plane waves the ring averages are conveniently written in terms of Bessel functions. For example let,

$$\phi(r, t) = \hat{\phi} e^{ik \cdot r}$$

since the right hand side of the linearized GK equation is proportional to $e^{ik \cdot R}$ it is clear that we can also write

$$g(R, \mathcal{E}, \mu_0, t) = \hat{g}(\mathcal{E}, \mu_0, t) e^{ik \cdot R} \quad (11)$$

and the ring averages at constant r of g are also expressed in terms of Bessel functions. For example:

$$\langle g_\alpha(R_\alpha, \mathcal{E}, \mu_0, t) \rangle_r = J_0\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) \hat{g}_\alpha(\mathcal{E}, \mu_0, t) e^{ik \cdot r}.$$

Substituting Eqs. (10) and (11) into Eq. (6) and taking the time dependence of all perturbed quantities to be $e^{-i\omega t}$ we obtain:

$$\begin{aligned} \hat{g}_\alpha &= \frac{q_\alpha F_{0\alpha}}{T_\alpha} \frac{\omega}{\omega - k_\parallel v_\parallel} \left[J_0\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) (\hat{\phi} - v_\parallel \hat{A}_\parallel) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)}{\frac{k_\perp v_\perp}{\Omega_\alpha}} \frac{m_\alpha v_\perp^2}{q_\alpha} \frac{\delta \hat{B}_\parallel}{B_0} \right] \\ &= \frac{q_\alpha F_{0\alpha}}{T_\alpha} J_0\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) \left(\frac{\omega}{k_\parallel} \hat{A}_\parallel \right) + \frac{q_\alpha F_{0\alpha}}{T_\alpha} \frac{\omega}{\omega - k_\parallel v_\parallel} \left[J_0\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) (\hat{\phi} - \frac{\omega}{k_\parallel} \hat{A}_\parallel) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)}{\frac{k_\perp v_\perp}{\Omega_\alpha}} \frac{m_\alpha v_\perp^2}{q_\alpha} \frac{\delta \hat{B}_\parallel}{B_0} \right] \end{aligned} \quad (12)$$

You will notice that \hat{g}_α is the product of functions of v_\parallel and functions of v_\perp . This property makes it possible to write all the velocity integrals (which we do as integrals over v_\parallel , v_\perp and θ) in Eqs. (2), (3) and (5) as products of modified Bessel functions and plasma dispersion functions. For the parallel velocity integrals we use the plasma dispersion function defined as:

$$Z(\xi_\alpha) = \frac{1}{\sqrt{\pi}} \int_L dx \frac{e^{-\omega^2}}{x - \xi_\alpha} \text{ for argument } \xi_\alpha = \frac{\omega}{k_\parallel \sqrt{2T_\alpha/m_\alpha}} \quad (13)$$

where the integral is taken along the Landau contour i.e. from $-\infty$ to ∞ below ξ_α in the complex x plane. For the perpendicular integrals we use the Bessel function identity (see *Bessel Functions* by Watson):

$$\int_0^\infty e^{-a^2 x^2} J_n(px) J_n(qx) x dx = \frac{1}{2a^2} I_n\left(\frac{pq}{2a^2}\right) e^{-\left(\frac{p^2+q^2}{4a^2}\right)} \quad (14)$$

where I_n is the n 'th order modified Bessel function. We define two functions:

$$\begin{aligned} \Gamma_0(\beta_\alpha) &= e^{-\beta_\alpha} I_0(\beta_\alpha) \\ &\sim 1 - \beta_\alpha \dots \quad \text{for } \beta_\alpha \ll 1 \end{aligned} \quad (15)$$

$$\begin{aligned} \Gamma_1(\beta_\alpha) &= e^{-\beta_\alpha} [I_0(\beta_\alpha) - I_1(\beta_\alpha)] \\ &\sim 1 - \frac{3}{2}\beta_\alpha \dots \quad \text{for } \beta_\alpha \ll 1 \end{aligned} \quad (16)$$

where $\beta_\alpha = \frac{k^2 \rho_\alpha^2}{2}$ and $\rho_\alpha = \frac{\sqrt{2T_\alpha/m_\alpha}}{\Omega_\alpha}$ is the thermal larmor radius of species α .

With these algebraic tricks we write Ampere's law Eq. (3) using Eq. (7) as :

$$-k_\perp^2 A_\parallel = -\mu_0 J_\parallel = -\mu_0 \frac{\omega}{k_\parallel} \left[\frac{q^2 n_i}{T_i} (1 - \Gamma_0(\beta_i)) \hat{\phi} + \frac{e^2 n_e}{T_e} (1 - \Gamma_0(\beta_e)) \hat{\phi} + (q n_i \Gamma_1(\beta_i) - e n_e \Gamma_1(\beta_e)) \frac{\delta \hat{B}_\parallel}{B_0} \right] \quad (17)$$

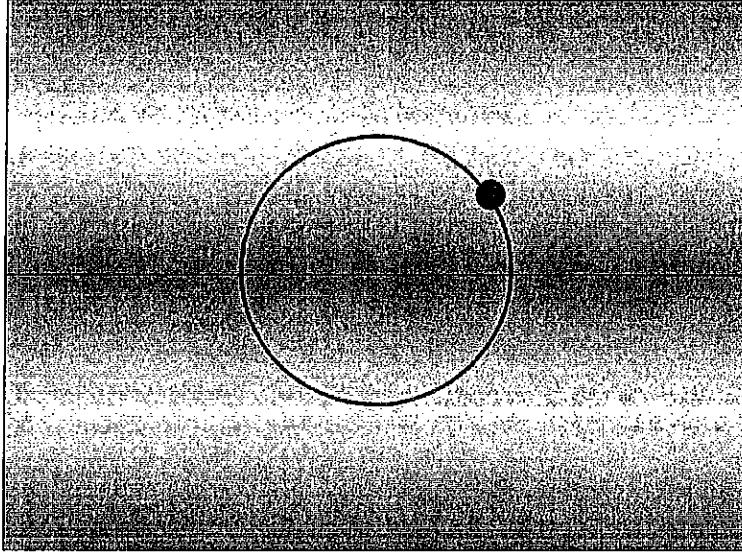
or since from now on we are interested in $\beta_e = \frac{k^2 \rho_e^2}{2} \ll 1$:

$$-k_\perp^2 A_\parallel = \mu_0 J_\parallel = -\mu_0 \frac{\omega}{k_\parallel} \left[\frac{q^2 n_i}{T_i} (1 - \Gamma_0(\beta_i)) \hat{\phi} + q n_i (\Gamma_1(\beta_i) - 1) \frac{\delta \hat{B}_\parallel}{B_0} \right] \quad (18)$$

then

$$\langle \phi(r, t) \rangle_{R_\alpha} = \frac{1}{2\pi} \oint d\theta \hat{\phi} e^{ik \cdot R + ik \cdot \rho_\alpha} = \frac{1}{2\pi} \hat{\phi} e^{ik \cdot R} \oint d\theta e^{ik \cdot \rho_\alpha} = J_0\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) \hat{\phi} e^{ik \cdot R}$$

J_0 is the zeroth order Bessel function of the first kind. We have used $ik \cdot \rho_\alpha = i \frac{k_\perp v_\perp}{\Omega_\alpha} \sin \theta'$ where θ' is the angle between ρ_α and k_\perp and identified the ring integral with the integral form of J_0 . This is the ring average of a plane wave:



In the drift kinetic limit when $k_\perp \rho \sim \frac{k_\perp v_\perp}{\Omega_\alpha} \ll 1$ we have the usual small argument expansion of the Bessel functions:

$$J_0\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) \sim 1 - \frac{1}{4}\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)^2,$$

so that to lowest order in drift kinetics the ring average is the same as the value at the guiding center position. The polarization drift correction is the next order term. In the opposite limit when the rings are large compared to the wavelength $k_\perp \rho \sim \frac{k_\perp v_\perp}{\Omega_\alpha} \rightarrow \infty$,

$$J_0\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right) \sim \mathcal{O}\left(\frac{\Omega_\alpha}{k_\perp v_\perp}\right) \rightarrow 0$$

and the gyro motion "averages out" the potential. Taking plane waves for all the potentials we obtain (after a little algebra)

$$\langle \chi \rangle_{R_\alpha} = \left[J_0\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)(\hat{\phi} - v_\parallel \hat{A}_\parallel) + \frac{J_1\left(\frac{k_\perp v_\perp}{\Omega_\alpha}\right)}{\frac{k_\perp v_\perp}{\Omega_\alpha}} \frac{m_\alpha v_\perp^2}{q_\alpha} \frac{\delta \hat{B}_\parallel}{B_0} \right] e^{ik \cdot R} \quad (10)$$

where $\delta \hat{B}_\parallel = i(k_\perp \times \hat{A}_\perp) \cdot z$. It is very helpful that the averages become simple multiplications by Bessel functions in "k space" – integral equations become algebraic equations.

where we have used charge neutrality, $qn_i = en_e$. Quasi-neutrality gives:

$$\left[\frac{q^2 n_i}{T_i} \Gamma_0(\beta_i) (1 + \xi_i Z(\xi_i)) + \frac{e^2 n_e}{T_e} (1 + \xi_e Z(\xi_e)) \right] (\hat{\phi} - \frac{\omega}{k_{\parallel}} \hat{A}_{\parallel}) = \frac{q^2 n_i}{T_i} (1 - \Gamma_0(\beta_i)) \hat{\phi} - [qn_i \Gamma_1(\beta_i) \xi_i Z(\xi_i) - en_e \xi_e Z(\xi_e)] \frac{\delta \hat{B}_{\parallel}}{B_0} \quad (19)$$

and finally the perpendicular pressure balance gives:

$$\frac{\delta \hat{B}_{\parallel}}{B_0} = \frac{\mu_0}{B_0^2} [qn_i (\Gamma_1(\beta_i) - 1) \hat{\phi} - (qn_i \Gamma_1(\beta_i) (1 + \xi_i Z(\xi_i)) - en_e (1 + \xi_e Z(\xi_e))) (\hat{\phi} - \frac{\omega}{k_{\parallel}} \hat{A}_{\parallel}) - 2 [n_i T_i \Gamma_1(\beta_i) \xi_i Z(\xi_i) + n_e T_e \xi_e Z(\xi_e)] \frac{\delta \hat{B}_{\parallel}}{B_0}] \quad (20)$$

Eqs. (18), (19) and (20) are simultaneous algebraic equations for the field quantities. Clearly the condition that a solution exists is that the determinant of the matrix of coefficients of the field quantities is zero – *the dispersion relation*. It is not difficult to write this down but it is not illuminating. Rather we will look at the limits of these equations themselves.

4 The Alfvén Wave at $k_{\perp} \rho_i \ll 1$

When $k_{\perp} \rho_i \ll 1$ we can expand $\Gamma_0(\beta_i)$ and $\Gamma_1(\beta_i)$. It is clear from Eqs. (18), (19) and (20) that a consistent ordering is:

$$\begin{aligned} \frac{\delta \hat{B}_{\parallel}}{B_0} &\sim \mathcal{O}(k_{\perp}^2 \rho_i^2) \frac{q \hat{\phi}}{T_i} \\ (\hat{\phi} - \frac{\omega}{k_{\parallel}} \hat{A}_{\parallel}) &\sim \mathcal{O}(k_{\perp}^2 \rho_i^2) \hat{\phi} \end{aligned} \quad (21)$$

note that at lowest order there are no forces along the field since E_{\parallel} and the mirror force (which is $\propto k_{\parallel} \delta B_{\parallel}$) are both small. Ampere's law and $E_{\parallel} = (\hat{\phi} - \frac{\omega}{k_{\parallel}} \hat{A}_{\parallel}) = 0$ then yield:

$$\begin{aligned} A_{\parallel} &= \frac{\omega}{k_{\parallel} V_A^2} \hat{\phi} \\ \hat{\phi} &= \frac{\omega}{k_{\parallel}} \hat{A}_{\parallel} \end{aligned} \quad (22)$$

and thus

$$\omega = \pm k_{\parallel} V_A \quad (23)$$

Clearly the parallel dynamics are unimportant in this wave and although there is a J_{\parallel} it requires only a small parallel force to drive it.

260 Lecture #10 Collisionless Waves in Gyro-kinetic Slab.

(i) Last time we derived the "dispersion relation" for waves in a gyro-kinetic slab. We obtained.

AMPERE'S LAW

$$\hat{A}_{\parallel} = \frac{\omega}{k_{\parallel} V_A^2} \left\{ \frac{1 - \Gamma_0}{k_{\perp}^2 p_i^2} \hat{\phi} + \frac{T_i}{q} \frac{(\Gamma_i - 1)}{k_{\perp}^2 p_i^2} \frac{\delta \hat{B}_{\parallel}}{B_0} \right\}$$

"ion polarization drift"

"inductive $E \times B$ flow."
ion - electron.

Note: $\frac{E \times B}{B_0^2} = -\frac{\hat{\phi}}{B_0} ik \times \frac{\hat{z}}{2} - \omega \frac{\delta B_{\parallel}}{B_0} \frac{k_{\perp}}{k_{\parallel}^2}$

compression

QUASI-NEUTRALITY

$$\frac{T_i}{T_e} = \gamma \quad \xi_i = \frac{\omega}{k_{\parallel} V_{tear}} \quad \frac{q}{e} = \beta$$

(due to ion and electron contributions not quite canceling)

$$[\beta M_0 (1 + \xi_i \gamma (\xi_i)) + \gamma (1 + \xi_e \gamma (\xi_e))] \frac{e}{T_i} (\hat{\phi} - \frac{\omega}{k_{\parallel}} \hat{A}_{\parallel})$$

Parallel E_{\parallel} drift terms

$$= \beta (1 - \Gamma_0) \frac{e \hat{\phi}}{T_i} - [\Gamma_i \xi_i \gamma (\xi_i) - \xi_e \gamma (\xi_e)] \frac{\delta B_{\parallel}}{B_0}$$

"Parallel Mirror Forces"

Polarization drift.

PERPENDICULAR CURRENT/AMPERE PRESSURE BALANCE

CURRENT IN THE $k \times z$ direction.

$$\frac{\delta B_{\parallel}}{B_0} = -\frac{\mu_0 n_e T_e}{B_0^2} \left\{ \gamma (\Gamma_i - 1) \frac{e \hat{\phi}}{T_i} - [\Gamma_i (1 + \xi_i \gamma (\xi_i)) - (1 + \xi_e \gamma (\xi_e))] \frac{e}{T_i} (\hat{\phi} - \frac{\omega}{k_{\parallel}} \hat{A}_{\parallel}) \right\}$$

↑
ion $E \times B$ ↑
electron $E \times B$
"Slippage"

parallel response to E_{\parallel}

$$- 2 \left[\frac{\gamma}{\beta} \Gamma_i \xi_i \gamma (\xi_i) + \xi_e \gamma (\xi_e) \right] \frac{\delta B_{\parallel}}{B_0} \}$$

"DB drift terms causing current"

(ii) DRIFT-KINETIC LIMIT: $k_{\perp} p \ll 1$. \nexists

$$\Gamma_0 = 1 - \frac{k_{\perp}^2 p_i^2}{2} \quad \Gamma_i = 1 - \frac{3}{2} \frac{k_{\perp}^2 p_i^2}{2}$$

① Alfvén Wave (Again!)

$$\frac{\delta B_{\parallel}}{B_0} \sim \mathcal{O}\left(\frac{k_{\perp}^2 p_i^2}{2}\right) \frac{q\phi}{T} \quad \text{small } \delta B_{\parallel}$$

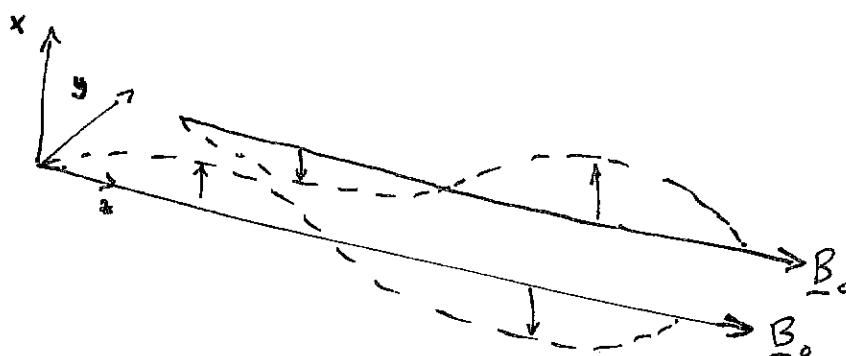
QN \rightarrow $\frac{E_{\parallel}}{ik_{\parallel}} = \hat{\phi} - \frac{\omega}{k_{\parallel}} \hat{A}_{\parallel} \sim \mathcal{O}\left(\frac{k_{\perp}^2 p_i^2}{2}\right) \hat{\phi} \quad \text{small } E_{\parallel}$ [No charge build up due to perpendicular dynamics]

\Rightarrow Parallel forces are small} - parallel dynamics is unimportant.
No minor, No E_{\parallel} Forces

Ampere's law: $\hat{A}_{\parallel} = \frac{\omega}{k_{\parallel} V_A^2} \hat{\phi} + \mathcal{O}(k_{\perp}^2 p_i^2)$

$E_{\parallel} = 0$: $\hat{\phi} = \frac{\omega}{k_{\parallel}} \hat{A}_{\parallel} + \mathcal{O}(k_{\perp}^2 p_i^2)$

$$\Rightarrow \boxed{\omega^2 = k_{\parallel}^2 V_A^2}$$



$E_{\parallel} = 0$
no forces along
 B_0 !
("no susip"
↓
no damp")

Field still "Frozen" into plasma at these scales.

② Slow Wave: (still $k_p \rho \ll 1$)

For this wave we have to deal with the parallel dynamics so we have to deal with the "Z function". This wave is heavily damped unless $T_e \gg T_i$ so we take this limit:

DEFINE: $C_s^2 = \frac{2\pi e}{m_i}$ SOUND SPEED.

$$\frac{C_s^2}{V_A^2} = \beta_e = \text{Electron } \beta. = \frac{\mu_0 n_i T_e}{B_0^2}$$

$$\frac{V_{thi}^2}{V_A^2} = \beta_i \ll \beta_e \quad \beta_i = \text{Ion } \beta. = \frac{\mu_0 n_i T_i}{B_0^2}$$

We will take

\nwarrow PHASE VELOCITY OF WAVE.

LIMITS OF PLASMA DISPERSION FUNCTION.

$V_{thi} \ll \frac{\omega}{k_{\parallel}} \ll V_{the}$

$$\xi_e \ll 1.$$

$$\xi_i \gg 1.$$

$$Z(\xi_e) = i\sqrt{\pi} - 2\xi_e \dots \quad \xi_e \ll 1$$

$$Z(\xi_i) = \underbrace{i\sqrt{\pi} e^{-\xi_i^2}}_{\text{SMALL}} - \frac{1}{\xi_i} \left(1 + \frac{1}{2} \frac{1}{\xi_i^2} \dots \right) \quad \xi_i \ll 1$$

Ions too slow to damp the wave much, electrons too fast.

To start with we ignore damping term.

AMPERE'S LAW

$$A_{||} = \frac{\omega}{k_{||} V_A^2} \left\{ \hat{\phi} + \frac{3T_i}{2q} \frac{\delta B_{||}}{B} \right\}$$

QUASI-NEUTRALITY

$$\frac{\epsilon}{3} \left[1 - \frac{k_{||}^2 c_s^2}{\omega^2} \right] \left(\hat{\phi} - \frac{\omega}{k_{||}} \hat{A}_{||} \right) = \frac{T_i}{q} \frac{\delta B_{||}}{B}$$

PRESSURE BALANCE

$$(1 + \beta_i) \frac{\delta B_{||} T_i}{B q} = -\beta_e \frac{\epsilon}{\omega} \left(\phi - \frac{\omega}{k_{||}} \hat{A}_{||} \right)$$

COMBINING

$$\underbrace{\left(\frac{1 + \beta_i}{\beta_e} \right) \left(1 - \frac{k_{||}^2 c_s^2}{\omega^2} \right)}_{=} = -1$$

Not quite the MHD dispersion relation because parallel dynamics are different.

$$\beta_e \rightarrow 0$$

$$\omega^2 = k_{||}^2 c_s^2 \quad \text{SOUND WAVE.}$$

$$\beta_e \rightarrow \infty$$

$$\omega^2 = \frac{k_{||}^2 c_s^2}{\beta_e} = k_{||}^2 V_A^2 \quad \text{PSEUDO ALFVEN WAVE.}$$

" "

DAMPING If we include the ion damping term $i m e^{-\xi_i^2}$ we get the Landau damping and Barnes or Transit time damping of this wave.

LANDAU DAMPING — Force on ions from $E_{||}$ term.

BARNES DAMPING / TRANSIT TIME DAMPING — Force on ions from $\delta B_{||}$ term.

$\beta_e \rightarrow 0$ Landau Damping Dominates (from E_{\parallel})

$$\frac{\delta B_{||}}{B_0} - \text{small} \sim J(\beta_e) \quad \frac{x}{\omega} \sim -\sqrt{\pi} \left(\frac{T_e}{T_i} \right)^{3/2} e^{-\frac{T_e}{T_i}}$$

$\beta_e \rightarrow \infty$ T.T.D. / Barnes Damping Dominates.

$$\frac{\delta B_{\parallel}}{B} = \underline{\text{Lager.}}$$

$$m\ddot{v} \sim qE_{\text{II}}$$

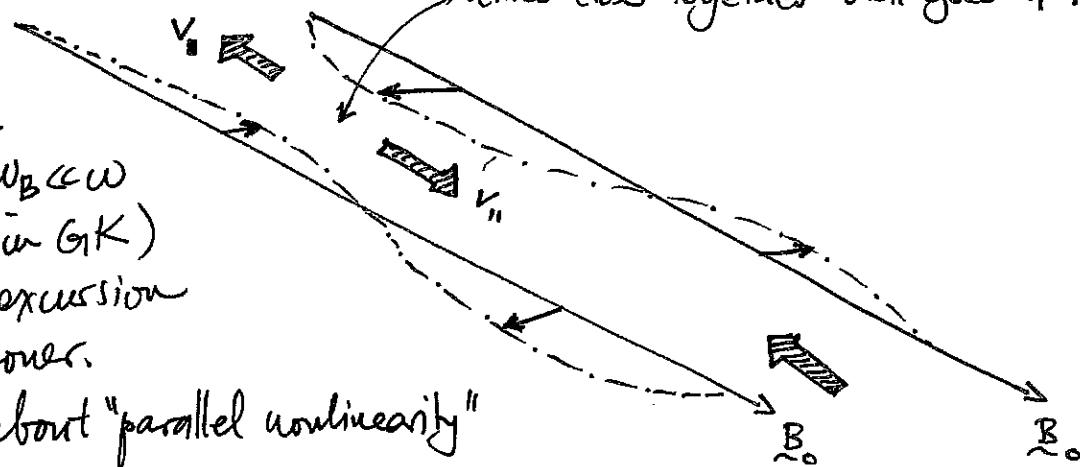
$\omega_B^2 \sim \frac{qE_{||}k_{||}}{m}$ will be on the heating timescale

$$\frac{\gamma}{\omega} \propto e^{-\frac{V_A^2}{V_{th1}^2}} \propto e^{-\frac{1}{\beta_1}}$$

lines closer together δB_n goes up. (δp_1 goes down)

B bounce
frequency $\omega_B \ll \omega$
(ordered out in GIK)

- so perp. excursion happens sooner.
 - debate about "parallel nonlinearity"



To see how the field lines must move imagine a sound wave travelling along a line the pressure perturbation makes a $\nabla_1 \delta p$ force that causes the line to expand or contract in the ~~perpendicular~~ direction.

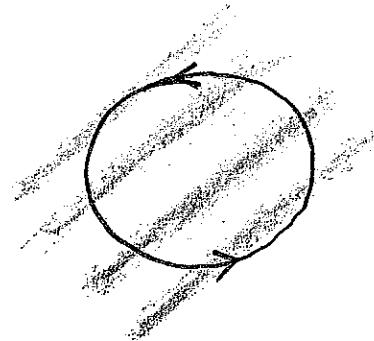
SLOW WAVE HAS DENSITY PERTURBATION - ALFVÉN WAVE DOES
 BUT SLOW WAVE IS USUALLY STRONGLY DAMPED (WHEN $T_e \approx T_i$).

$$(iii) \quad k_{\perp} p_i \gg 1 \quad P_0 \sim \frac{1}{k_{\perp} p_i} \ll 1 \quad P_i \sim \frac{1}{k_{\perp} p_i} \ll 1$$

ALL ION RESPONSE EXCEPT BOLTZMANN TERM BECOMES SMALL

$$\boxed{\delta f_i \sim -\frac{q}{T_i} \phi F_0}$$

SETS UP EQUILIBRIUM ACROSS B



ALL DYNAMICS DUE TO ELECTRONS.

$$\text{AMPERES LAW} \quad P_0 = F_i = 0$$

$$\frac{\omega A_{||}}{F_{||}} = \frac{\omega^2}{K_{||} V_A^2} \frac{2}{k_{\perp}^2 p_i^2} \left(\phi - \frac{T_i}{2} \frac{\delta B_{||}}{B_0} \right)$$

$$\text{QUASI-NEUTRALITY} : V_{the} \gg \frac{\omega}{K_{||}} \quad \leftarrow \text{Assume.}$$

$$\begin{aligned} \text{Electrons have} \\ \text{Boltzmann response} &= e \delta n_e = \frac{e^2 n_e (\phi - \omega A_{||})}{T_e} = q \delta n_i = \frac{q^2 n_i}{T_i} \hat{\phi} \\ \text{along } B_0 \text{ to } E_{||} \end{aligned}$$

PERPENDICULAR-PRESSURE :

$$\frac{\delta B_{||}}{B_0} = -\frac{\mu_0}{B_0^2} \left\{ -q n_i \hat{\phi} - e n_e \left(\hat{\phi} - \frac{\omega}{K_{||}} E_{||} \right) \right\}$$

Ion pressure perturbation Electron pressure perturbation.

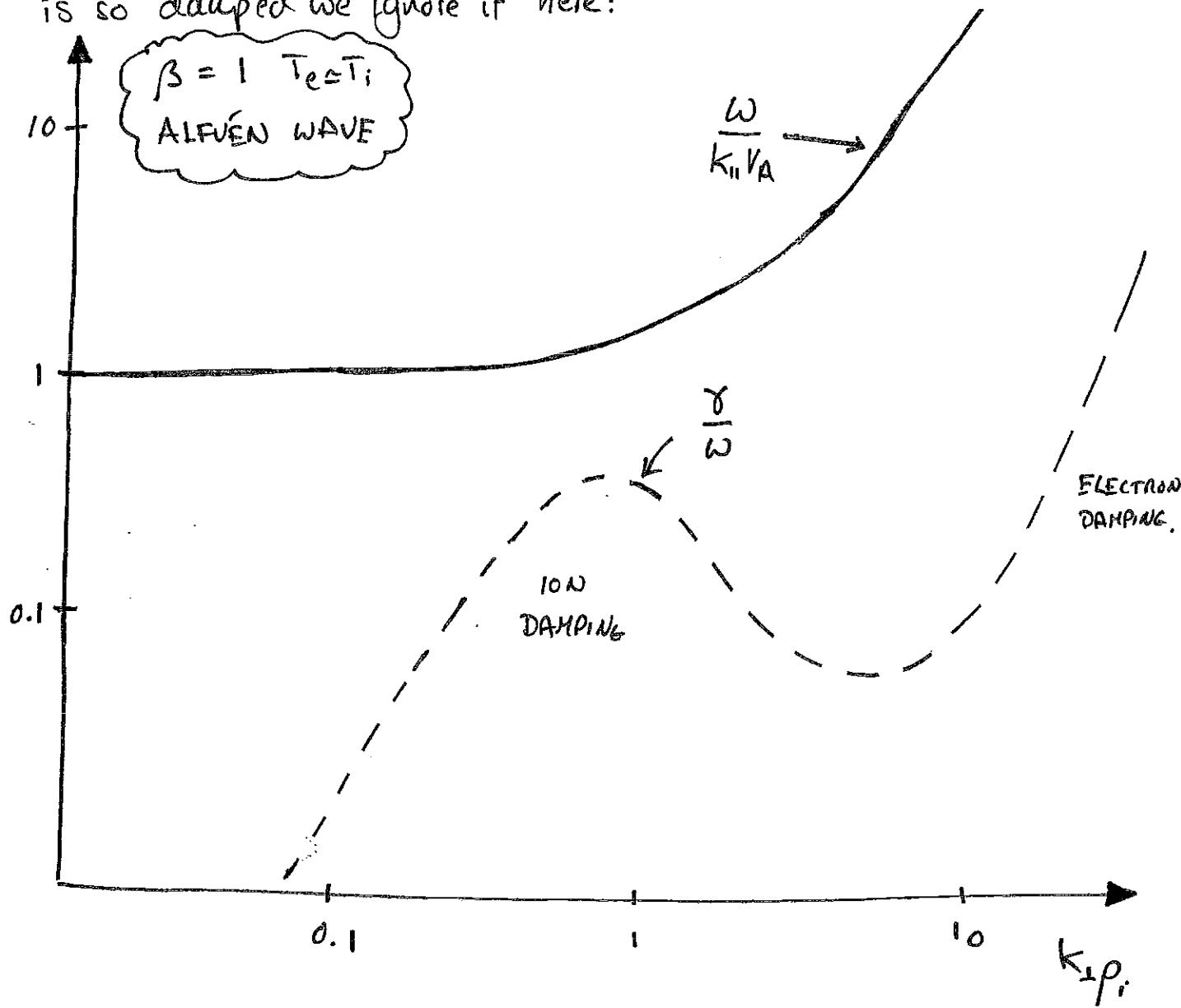
COMBINING

$$\boxed{\omega^2 = \frac{k_{||}^2 V_A^2 k_{\perp}^2 p_i^2}{(1 + \beta)}}$$

"Kinetic Alfvén Wave"
sometimes wrongly called
the "whistler"

All free current is electron $E \times B$ motion
(ions have been averaged out)

(iv) The general $\beta \sim J(1)$ $k_{\perp} p_i \sim J(1)$ $T_e \sim T_i$
 dispersion relation must be solved numerically. The slow wave
 is so damped we ignore it here:



260 Lecture #11: Non-linear Interaction of MHD Waves: Weak-Turbulence.

(i) In this lecture we look at the physics of wave-wave interaction in MHD. — Nonlinear theory is always hard and we must try and simplify the equations as much as possible. No gyro-kinetics today.

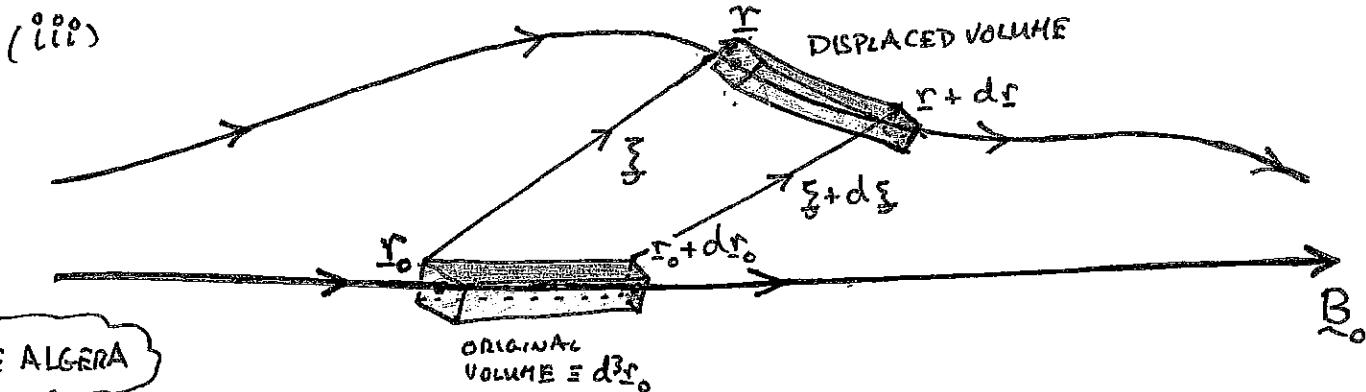
(ii) Lagrangian MHD: Those of you who did 222 with me will recognize this approach — it allows a simple set of equations to be derived for the waves. Let us recall how we formulate the equations in terms of the Lagrangian displacement. [if you haven't seen this before this will be too quick]

$\underline{r} = \text{CURRENT POSITION OF A FLUID ELEMENT}$

$$\underline{r} = \underline{r}_0 + \underline{\xi}(\underline{r}_0, t)$$

$\underline{r}_0 = \text{ORIGINAL POSITION OF FLUID ELEMENT}$

$\underline{\xi}(\underline{r}_0, t) = \text{DISPLACEMENT OF FLUID ELEMENT}$



$J = \text{JACOBIAN OF TRANSFORMATION}$

$$\underline{r}_0 \rightarrow \underline{r}$$

$$= |\nabla_{\underline{r}_0} \underline{r}| = |\underline{I} + \nabla_{\underline{r}_0} \underline{\xi}|$$

$$\nabla_{\underline{r}_0} \underline{\xi} = \left(\begin{array}{c} \frac{\partial \xi_x}{\partial x_0}, \frac{\partial \xi_y}{\partial x_0}, \frac{\partial \xi_z}{\partial x_0} \\ \frac{\partial \xi_x}{\partial y_0}, \frac{\partial \xi_y}{\partial y_0}, \frac{\partial \xi_z}{\partial y_0} \\ \frac{\partial \xi_x}{\partial z_0}, \frac{\partial \xi_y}{\partial z_0}, \frac{\partial \xi_z}{\partial z_0} \end{array} \right)$$

CONSERVATION OF SPECIFIC ENTHALPY

 $\gamma = \text{ratio of specific heats.}$

$$P = P_0 \left| \frac{d^3 r_0}{d \Sigma_0} \right|^{\gamma} = \frac{P_0}{J^{\gamma}}$$

CONSERVATION OF FLUX - FROZEN IN
FIELD LINES

[needs some Algebra]



$$\underline{B} = \frac{\underline{B}_0 \cdot \nabla_0 \Gamma}{J}$$

"Cauchy
solution"

$$\nabla_0 \equiv \frac{\partial}{\partial \Sigma_0}$$

(iv) these relations replace the familiar Eulerian equations (continuity, pressure and "induction equation") - formally we have integrated these equations.

(v) MOMENTUM EQUATION

$$\underline{v} = \frac{d \underline{r}}{dt} = \left(\frac{\partial \underline{r}}{\partial t} \right)_{\Sigma_0} = \left(\frac{\partial \underline{\xi}}{\partial t} \right)_{\Sigma_0}$$

$$\Rightarrow \frac{\rho_0}{J} \frac{\partial^2 \underline{\xi}}{\partial t^2} = -\nabla(P + \frac{B^2}{2}) + \underline{B} \cdot \nabla \underline{B}$$

$$= -\nabla(P_{\text{TOTAL}}) + \frac{\underline{B}_0 \cdot \nabla_0}{J} \left(\frac{\underline{B}_0 \cdot \nabla_0 \underline{\xi}}{J} \right)$$

we use $\nabla_0 \equiv \nabla_0 \Gamma \cdot \nabla$
"Chain Rule"

$$P_{\text{TOTAL}} = \frac{P_0}{J^{\gamma}} + \frac{1}{2J^2} |\underline{B}_0 \cdot \nabla_0 \underline{\xi}|^2$$

$$\frac{\rho_0}{J} \nabla_0 \underline{\xi} \cdot \frac{\partial^2 \underline{\xi}}{\partial t^2} = -\nabla_0 P_{\text{TOTAL}} + \nabla_0 \underline{\xi} \cdot \left[\frac{\underline{B}_0 \cdot \nabla_0}{J} \left(\frac{\underline{B}_0 \cdot \nabla_0 \underline{\xi}}{J} \right) \right]$$

This looks complicated but it has reduced all the 8 MHD equations to 3 for the displacement. Nonetheless it is too complicated for us we need to make further assumptions.

3 2 ASSUMPTIONS to simplify

(vi) High β -incompressible limit.

If we take $\frac{P_0}{B_0^2} \rightarrow \infty$ biggest term is

$$\nabla_0 \left(\frac{P_0}{\beta} \right) \leftarrow \text{force from compression terms.}$$

\therefore If we want to look at the Alfvén & slow waves we must set $\beta = \text{constant}$ to lowest order in $1/\beta$. Since $\beta = 1$ if $t=0$ and $r=r_0$ we have

$$\beta = 1 + \mathcal{O}\left(\frac{1}{\beta}\right) \Rightarrow P = P_0 + \mathcal{O}(B_0^2) \dots$$

(vii) STRAIGHT FIELD-LINES AT $t=0$

$$\underline{B}_0 = B_0 \hat{\underline{z}}$$

$P_0 = \text{constant}$
 $\rho_0 = \text{constant}$.

THEN MOMENTUM EQUATIONS

$$\rho_0 \left(\underline{\dot{r}} + \nabla_0 \xi \right) \cdot \left[\frac{\partial^2 \xi}{\partial t^2} - V_A^2 \frac{\partial^2 \xi}{\partial z_0^2} \right] = - \nabla_0 P_{\text{TOTAL}}$$

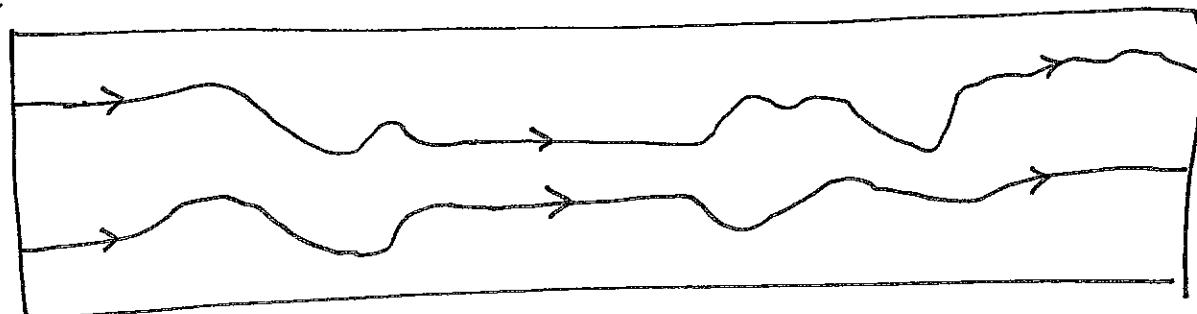
where we do not determine P_{TOTAL} from the pressure equation (because it involves $\frac{1}{\beta}$ terms in β) instead we use P_{TOTAL} to ensure

$$\beta = 1 = |\nabla_0 \xi| = |\underline{\dot{r}} + \nabla_0 \xi|$$

Nonlinear constraint.

INITIALIZE SOME DISTURBANCE
 $\xi(t=0, r_0)$

\underline{B}



(viii) EXACT NON-LINEAR SOLUTIONS We can "spot" exact solutions:-

TAKE $\xi = \xi_+ (z_0 - V_A t, x_0, y_0)$

DISTURBANCE TRAVELLING TO RIGHT AT V_A

Note: $\frac{\partial^2 \xi_+}{\partial t^2} - V_A^2 \frac{\partial^2 \xi_+}{\partial z_0^2} = 0 \rightarrow P_{\text{TOTAL}} = 0$

AND WE MUST HAVE: $\mathcal{J} = 1$ so if we choose.

$$\mathcal{J}(t=0) = |\nabla_0 \xi_+|_{t=0} = 0 \quad \text{i.e. at } t=0 \text{ no compression.}$$

then this will remain true for all time since we can simply transform to the frame moving @ V_A i.e. $z'_0 = z_0 - V_A t$ $x'_0 = x_0$ $y'_0 = y_0$.

$$|\nabla'_0 \xi_+| = 0.$$

OR $\xi = \xi_- (z_0 + V_A t, x_0, y_0) \quad \mathcal{J}(t=0) = |\nabla_0 \xi_-|_{t=0} = 0$

DISTURBANCE TRAVELING TO LEFT AT V_A .

(ix) NO NONLINEAR INTERACTION (EVOLUTION) OF A DISTURBANCE

MOVING IN ONLY ONE DIRECTION. TO GET NONLINEAR EVOLUTION (INTERACTION) WE MUST HAVE WAVES GOING IN BOTH DIRECTIONS.

i.e. COULDING WAVES.

(X) WEAK TURBULENCE LIMIT

$$\nabla_0 \xi \ll I$$

We expand in powers of ξ - the displacement to the order we need:

NOTE.

$$\frac{\partial B}{\partial z} \ll 1$$

$$J = 1 + \nabla_0 \cdot \xi + \left[(\nabla_0 \cdot \xi)^2 - \nabla_0 \cdot \xi : \nabla_0 \cdot \xi \right] \dots$$

$$\text{Expand: } \xi = \xi^{(0)} + \xi^{(1)} + \xi^{(2)} \dots$$

$$\text{where } \xi^{(1)} \approx \mathcal{O}(\xi^{(0)2}) \dots$$

(xi) LOWEST ORDER

①

$$\frac{\partial^2 \xi^{(0)}}{\partial t^2} - V_A^2 \frac{\partial^2 \xi^{(0)}}{\partial z_0^2} = - \nabla_0 P_{\text{TOTAL}}$$

$$J^{(0)} = 1$$

⇒ ②

$$\nabla_0 \cdot \xi^{(0)} = 0$$

Taking the divergence of ① we obtain $\nabla_0^2 P_{\text{TOTAL}} = 0$

If we take $P_{\text{TOTAL}} \rightarrow 0$ as $|z_0| \rightarrow \infty$ this gives $P_{\text{TOTAL}} = 0$.

⇒

$$\left\{ \frac{\partial^2}{\partial t^2} - V_A^2 \frac{\partial^2}{\partial z_0^2} \right\} \cdot \xi^{(0)} = 0$$

③

wave equation

④

$$\xi^{(0)} = \xi_+^{(0)}(z_0 - V_A t, x_0, y_0) + \xi_-^{(0)}(z_0 + V_A t, x_0, y_0)$$

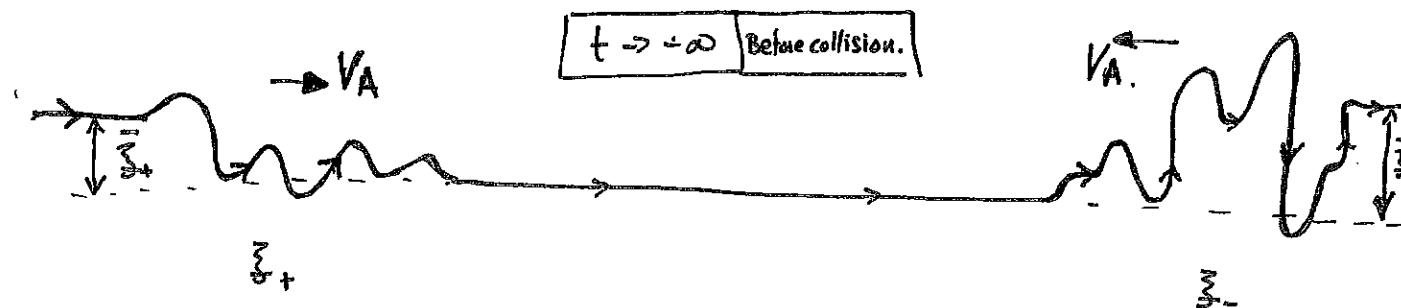
general solution.

6

with $\nabla \cdot \xi^{(0)} = 0$ FOR ALL TIMES we must have

$$\nabla \cdot \xi_+^{(0)} = \nabla \cdot \xi_-^{(0)} = 0 \quad \text{this is as before set in the initial conditions.}$$

(X) WAVES (DISTURBANCE) TRAVELING ALONG B_0 - MIXTURE OF SLOW AND ALFVÉN WAVES. NOW WE HAVE WAVES GOING IN BOTH DIRECTIONS - COLLIDING. SCATTERING.

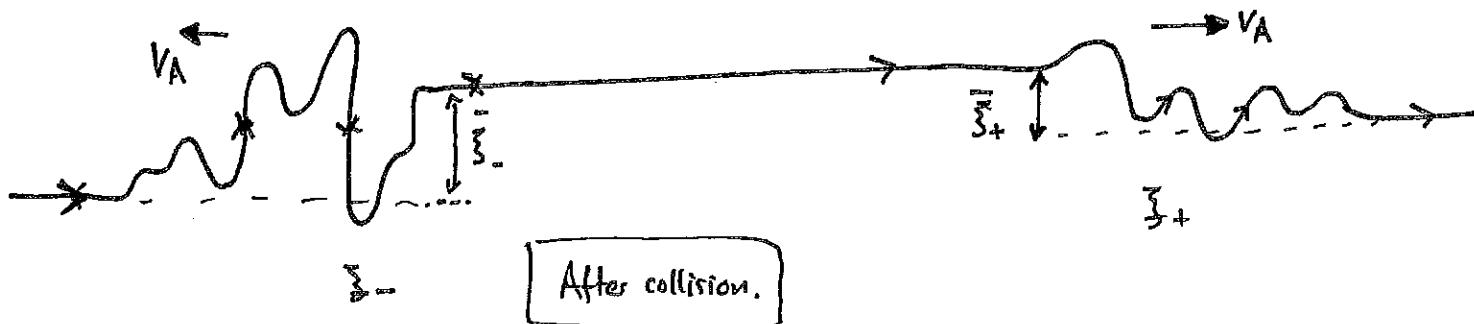


$$z - V_A t \rightarrow +\infty \quad \xi_+ \rightarrow 0, \quad z - V_A t \rightarrow -\infty \quad \xi_+ \rightarrow \bar{\xi}_+(x_0, y_0)$$

$$z + V_A t \rightarrow -\infty \quad \xi_- \rightarrow 0, \quad z + V_A t \rightarrow +\infty \quad \xi_- \rightarrow \bar{\xi}_-(x_0, y_0)$$

(X) Now the waves interact as they pass through each other. We want to calculate what happens - need to go to 2nd order since at lowest order they pass through each other (almost!).

(X) Passing Through as t -> infinity the wave packets above become:



Physics 260. Lecture #12. Alfvén-Wave Turbulence - II.

(LAST TIME)

(i) Lagrangian MHD. equations with incompressibility. $P + \frac{B^2}{2}$

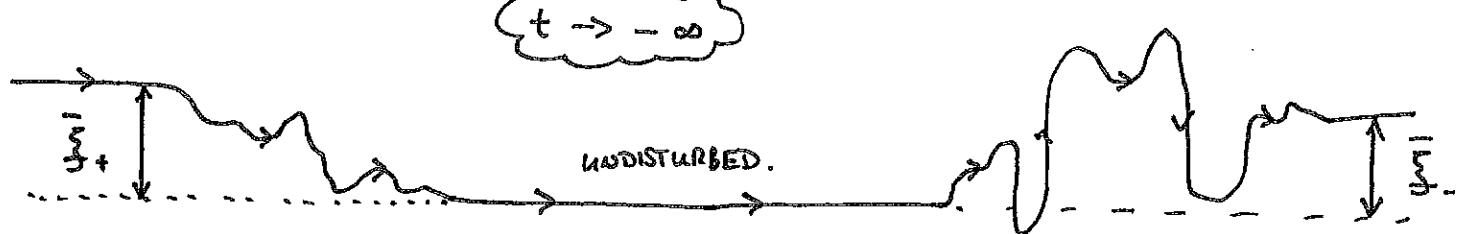
$$\textcircled{1} \quad \rho_0 \left[\dot{\underline{\xi}} + \nabla_0 \cdot \underline{\xi} \right] \cdot \left\{ \frac{\partial^2 \underline{\xi}}{\partial t^2} - V_A^2 \frac{\partial^2 \underline{\xi}}{\partial z^2} \right\} = - \nabla_0 P_{\text{TOTAL}}$$

$$\textcircled{2} \quad \underline{\zeta} = \left| \dot{\underline{\xi}} + \nabla_0 \cdot \underline{\xi} \right| = 1 \approx 1 + \nabla_0 \cdot \underline{\xi} + [(\nabla_0 \cdot \underline{\xi})^2 - \nabla_0 \cdot \underline{\xi} \cdot \nabla_0 \cdot \underline{\xi}]$$

WAVES GOING IN ONE DIRECTION ONLY PROPAGATE UNDISTURBED.

(ii) Waves coming from infinity colliding: Initial conditions.

$t \rightarrow -\infty$



$$\xi_+ (z_0 - V_A t, x_0, y_0)$$

$$\xi_- (z_0 + V_A t, x_0, y_0)$$

$$\text{"STEP"} = \xi_+ (x_0, y_0)$$

$$\text{"STEP"} = \xi_- (x_0, y_0)$$

(iii) Weak Turbulence Theory $|\nabla_0 \cdot \underline{\xi}| \ll 1$ $\underline{\xi} = \underline{\xi}^{(0)} + \underline{\xi}^{(1)}$...

LOWEST ORDER

$$\frac{\partial^2 \underline{\xi}^{(0)}}{\partial t^2} - V_A^2 \frac{\partial^2 \underline{\xi}^{(0)}}{\partial z^2} = - \nabla_0 P_{\text{TOTAL}}^{(0)}$$

③

$$\underline{\zeta}^{(0)} = 1 \iff \nabla_0 \cdot \underline{\xi}^{(0)} = 0 \quad \text{④}$$

Taking the divergence of ③ gives $\nabla_0^2 P_{\text{TOTAL}}^{(0)} = 0 \Rightarrow P_{\text{TOTAL}}^{(0)} = 0$
so from ③

$$\underline{\xi}^{(0)} = \xi_+^{(0)} (z_0 - V_A t, x_0, y_0) + \xi_-^{(0)} (z_0 + V_A t, x_0, y_0)$$

$$\text{and } \nabla_0 \cdot \underline{\xi}_+^{(0)} = \nabla_0 \cdot \underline{\xi}_-^{(0)} = 0.$$

LOWEST ORDER
NO INTERACTION
OF PULSES.

(iv) 1st/next order

$$\frac{\partial^2 \xi^{(1)}}{\partial t^2} - V_A^2 \frac{\partial^2 \xi^{(1)}}{\partial z_0^2} = - \nabla_0 \cdot \frac{P_{\text{TOTAL}}^{(1)}}{\rho_0} \quad (5)$$

$$\mathbf{j}^{(1)} = \mathbf{0} \quad \rightarrow \quad \nabla_0 \cdot \xi^{(1)} - \nabla_0 \xi^{(0)} \cdot \nabla_0 \xi^{(0)} = 0 \quad (6)$$

Taking the rotational and irrotational parts i.e. take $\nabla \times (5)$:

$$\left\{ \frac{\partial^2}{\partial t^2} - V_A^2 \frac{\partial^2}{\partial z_0^2} \right\} \nabla \times \xi^{(1)} = 0$$

since this is just the same equation as the $\nabla_0 \xi^{(0)}$ equation we may set $\nabla \times \xi^{(1)} = 0$ and absorb these terms into $\xi^{(0)}$.

$$(v) \quad \Rightarrow \boxed{\xi^{(1)} = \nabla \chi^{(1)}}$$

and (6) becomes.

$$\begin{aligned} \nabla^2 \chi^{(0)} &= \underbrace{\nabla_0 \xi_+ \cdot \nabla_0 \xi_+}_{\text{Function of } z_0 - V_A t} + \underbrace{\nabla_0 \xi_- \cdot \nabla_0 \xi_-}_{\text{Function of } z_0 + V_A t} \\ &\quad + \underbrace{2 \nabla \xi_+ \cdot \nabla \xi_-}_{\text{Mixed function of } z_0 - V_A t \text{ and } z_0 + V_A t}. \end{aligned}$$

Equation (6) determines the pressure perturbation:

$$\left\{ \frac{\partial^2}{\partial t^2} - V_A^2 \frac{\partial^2}{\partial z_0^2} \right\} \chi = - \frac{P_{\text{TOTAL}}^{(1)}}{\rho_0}$$

where we have peeled off the ∇^2

(vi) We can split up the 1st order displacement into parts driven by one pulse or two. i.e.

$$\nabla^2 \chi_+ = \nabla_0 \xi_+ : \nabla_0 \xi_+, \quad \nabla^2 \chi_- = \nabla_0 \xi_- : \nabla_0 \xi_-,$$

$$\text{and } \nabla^2 \hat{\chi} = 2 \nabla_0 \xi_+ : \nabla_0 \xi_-$$

$\nabla_0 \chi_+$ is just the 1st order part of the wave/pulse going to the right.

$\nabla_0 \chi_-$ is just the - - - - - left.

$\hat{\chi}$ is the interaction term.

Suppose the pulses pass through each other at $t \sim t_I$ then

for $t \ll -t_I$ $\hat{\chi} = 0$. AND FOR

$t \gg t_I$ $\hat{\chi} \neq 0$ - INTERACTION TERMS KEEPING $J = 1$.

VELOCITY

$t \ll -t_I$

$$v = \underbrace{\frac{\partial \xi_+}{\partial t} + \nabla_0 \frac{\partial \chi_+}{\partial t}}_{V_+(z_0 - V_A t, x_0, y_0)} + \underbrace{\frac{\partial \xi_-}{\partial t} + \nabla_0 \frac{\partial \chi_-}{\partial t}}_{V_-(z_0 + V_A t, x_0, y_0)}$$

$t \gg t_I$

$$(vii) v = V_+(z_0 - V_A t, x_0, y_0) + \delta V_+(z_0 - V_A t, x_0, y_0) + V_-(z_0 + V_A t, x_0, y_0) + \delta V_-(z_0 + V_A t, x_0, y_0)$$

where we take $\delta V_+ = \nabla \tilde{V}_+$ and $\nabla^2 \tilde{V}_+ = 2 \nabla_0 \frac{\partial \xi_+}{\partial t} : \nabla_0 \tilde{\xi}_-$

$\delta V_- = \nabla \tilde{V}_-$ and $\nabla^2 \tilde{V}_- = 2 \nabla_0 \tilde{\xi}_+ : \nabla_0 \frac{\partial \xi_-}{\partial t}$

since for $t \gg t_I$ the pulses have passed through each other.

we have:

$$t \gg t_I \quad \left\{ \begin{array}{l} \nabla^2 \tilde{V}_+ \approx 2 \nabla_0 \frac{\partial \xi_+}{\partial t} : \nabla_0 \tilde{\xi}_- \approx \text{function of } z - v_A t \\ \nabla^2 \tilde{V}_- \approx 2 \nabla_0 \tilde{\xi}_+ : \nabla_0 \frac{\partial \xi_-}{\partial t} \cdots \cdots z + v_A t. \end{array} \right.$$

Thus only the "step" part of $\tilde{\xi}_-$ alters $\tilde{\xi}_+$ and vice versa.

(viii) To 1st order IN EULERIAN VARIABLES.

$$V(\xi, t) = V(x - \xi, t) \approx V(x, t) + \xi \cdot \nabla V(x, t) \dots$$

for $t > t_I$

$$V = V_+(z - v_A t, x, y) + \xi \cdot \nabla V_+(x, t) + \delta V_+(z - v_A t, x, y)$$

$$V_- (z - v_A t, x, y) + \xi \cdot \nabla V_-(x, t) + \delta V_-(z - v_A t, x, y)$$

CHANGE FROM $t < t_I$ ARE

$$\tilde{\xi}_+ \cdot \nabla V_+ + \delta V_+ : \text{propagating to right.}$$

AND

$$\tilde{\xi}_+ \cdot \nabla V_- + \delta V_- : \text{propagating to left.}$$

IMPORTANT

CHANGES TO THE PULSES DUE TO INTERACTION

ARE ENTIRELY DUE TO THE STEPS IN THE DISPLACEMENT

THEY CHANGE THE PERPENDICULAR STRUCTURE BUT NOT THE PARALLEL STRUCTURE OF THE PULSES.

(ix) [THREE WAVE INTERACTION]

One common way to discuss this lack of parallel change in structure is through the 3 wave conditions for resonant interaction: consider 3 waves interacting with frequencies such that: $\omega_1 + \omega_2 = \omega_3$ and \underline{k} 's such that: $\underline{k}_1 + \underline{k}_2 = \underline{k}_3$

if: $\omega_i = \omega_i(\underline{k}_i)$ the linear relation $\omega_1 = \omega_1(\underline{k}_1)$ $\omega_2 = \omega_2(\underline{k}_2)$ $\omega_3 = \omega_3(\underline{k}_3)$ then we get strong interaction. But for Alfvén waves we have two facts.

(1) $\omega = \pm k_{\parallel} V_A$ (2) and only waves going in opposite directions interact.

$$\begin{aligned}\omega_1 + \omega_2 &= \omega_3 \Rightarrow k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3} \Rightarrow k_{\parallel 2} = 0 \\ \underline{k}_1 + \underline{k}_2 &= \underline{k}_3 \Rightarrow k_{\parallel 1} + k_{\parallel 2} = k_{\parallel 3} \quad k_{\parallel 1} = k_{\parallel 3}\end{aligned}$$

Only interaction when one of the waves has $k_{\parallel} = 0$

Note though $k_{\perp 2} \neq 0$ and $k_{\perp 3} \neq k_{\perp 1}$

(x) You will hear people say that in weak turbulence \underline{k}_{\perp} "cascades" and k_{\parallel} doesn't.

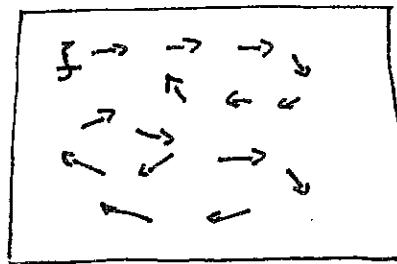
260 Lecture #13: Strong Turbulence : Kolmogorov's Theory.

(i) We looked at the weak interaction of Alfvén wave-packets. This occurs when the displacement is weak enough that

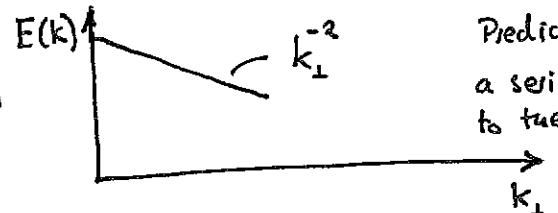
$$\nabla_0 \zeta \ll 1.$$

We deduced that the packets get shredded in the perpendicular direction but not in the parallel direction. \blacktriangleleft CASCADE IN k_{\perp} BUT NOT IN k_{\parallel} .

LOOKING DOWN B .



- SMALL SHEARING OF THE 2D PERPENDICULAR PLANE.
- LARGE v INCOMPRESSIBLE.
- "EDDY" LIKE OBJECTS/MOTIONS



$$(ii) \text{ Since } \zeta_{\perp} \sim \frac{v_{\perp}}{\omega} \quad \nabla_0 \zeta \ll 1 \Rightarrow \frac{k_{\perp} v_{\perp}}{\omega} \ll 1$$

TODAY WE LOOK AT THE STRONG TURBULENCE LIMIT

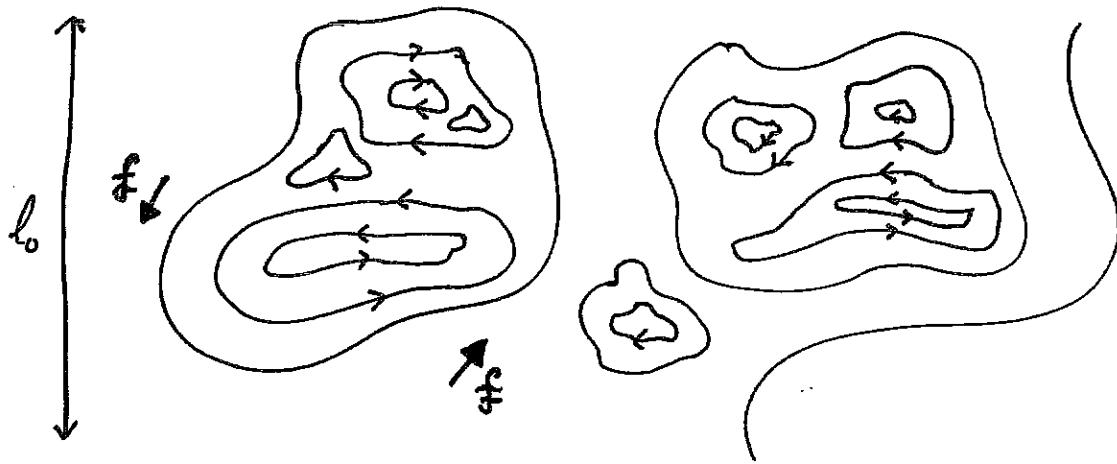
$$\frac{k_{\perp} v_{\perp}}{\omega} \sim 1$$

FINITE DISTORTION OF A WAVE PACKET WHEN IT MOVES ONE WAVELENGTH.

(iii) This limit is hard to treat in any quantitative way - we will use qualitative arguments based on Kolmogorov's famous theory of neutral fluid turbulence. I will start by reviewing his ideas which are now well accepted.

NEUTRAL INCOMPRESSIBLE TURBULENCE.

(iv) The observation of turbulence showed a structure of eddies inside eddies etc. Multiple scales exist coexisting. Usually we talk about a large "stirring" scale : l_0 where the fluid is being moved at velocity v_0 (often randomly). work ($f.v.$) is being done on the fluid so that the kinetic energy of the fluid increases until some turbulent steady state is set up. This steady state of eddies at many scales must dissipate the energy (work done) by the forcing. As we shall see this dissipation is done by the very smallest eddies in the turbulence.



(V) Navier-Stokes Equation.

$$\rho \left\{ \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right\} = - \nabla p + \frac{\nu \rho}{\text{viscosity}} \nabla^2 \underline{v} + f(x, t)$$

forcing. [Large Scale]

we take the incompressible assumption for simplicity.

$$\nabla \cdot \underline{v} = 0$$

We construct the energy equation by dott ing with \underline{v} and

integrating over the volume. (which will take to be 1 here)

$$\frac{d}{dt} \int d^3 r \frac{1}{2} \rho v^2 = -\nu \int d^3 r \rho (\nabla \times \underline{v})^2 + \int d^3 r \underline{v} \cdot \underline{f}$$

CHANGE OF
KINETIC ENERGY

VISCOUS DISSIPATION
(PRODUCES HEAT)

ENERGY INPUT
DUE TO FORCING.

(vi) REYNOLD'S NUMBER. Many of the flows we observe are high Reynolds' number - let us see what this is.

$$Re = \text{REYNOLD'S \#} = \frac{\ell_0 v_0}{\nu}$$

DEFINITION

at the large scale:

$$\frac{|\underline{v}_0 \cdot \nabla \underline{v}_0|}{|\nu \nabla^2 \underline{v}_0|} \approx Re$$

WHEN $Re \gg 1$ viscous forces at the large scale are small compared to inertial forces. Note this means that

$$\underline{f} \sim \rho \underline{v}_0 \cdot \nabla \underline{v}_0 \rightarrow \text{ENERGY INPUT (POWER)} \approx \rho \frac{v_0^3}{\ell_0}$$

IN ORDER FOR VISCOUS DISSIPATION TO BALANCE THE POWER INPUT WE MUST HAVE $\nabla \times \underline{v}$ OF THE SMALL SCALES MUCH LARGER THAN $\frac{v_0}{\ell_0}$ THE $\nabla \underline{v}_0$.

i.e. WE MUST DEVELOP SUFFICIENT SMALL SCALE ACTIVITY TO PROVIDE THE VISCOUS DISSIPATION.

(vii) SCALES: We need some mathematical definition of the idea of "velocity at scale \underline{l} " we heat steady state.

$$\begin{aligned} \langle \underline{V}(\underline{r}) \cdot \underline{V}(\underline{r} + \underline{l}, t) \rangle &= \text{two point correlation.} \\ &\text{one of the 2nd Order "Structure Functions"} \\ \text{average over time or ensemble.} &= F(\underline{r}, \underline{l}) \end{aligned}$$

We will make two further assumptions:

HOMOGENEOUS: - forcing is statistically the same throughout the volume then: $F(\underline{r}, \underline{l}) \rightarrow F(\underline{l})$

ISOTROPIC: - forcing is statistically same in all directions
 $F(\underline{l}) \rightarrow F(l)$

Note $F(0) = \text{TOTAL KINETIC ENERGY}$

(vii) Velocity at scale \underline{l} :

We are interested in the ^{"average"} velocity difference between point \underline{r} and \underline{l} . Define:

$$\sqrt{\langle (\underline{V}(\underline{r} + \underline{l}, t) - \underline{V}(\underline{r}, t))^2 \rangle} = V_l = 2[F(0) - F(l)]$$

(viii) Velocity Spectrum k-space: We pass to a continuous limit

$$\underline{V}(\underline{r}) = \sum_{\underline{k}} V_{\underline{k}} e^{i\underline{k} \cdot \underline{r}} \quad \underline{k} = 2\pi(m, n, p)$$

$$\approx (2\pi)^3 \int d^3k V_{\underline{k}} e^{i\underline{k} \cdot \underline{r}}$$

isotropic $\Rightarrow |V_{\underline{k}}|^2 = \text{FUNCTION OF } k \text{ ONLY.}$

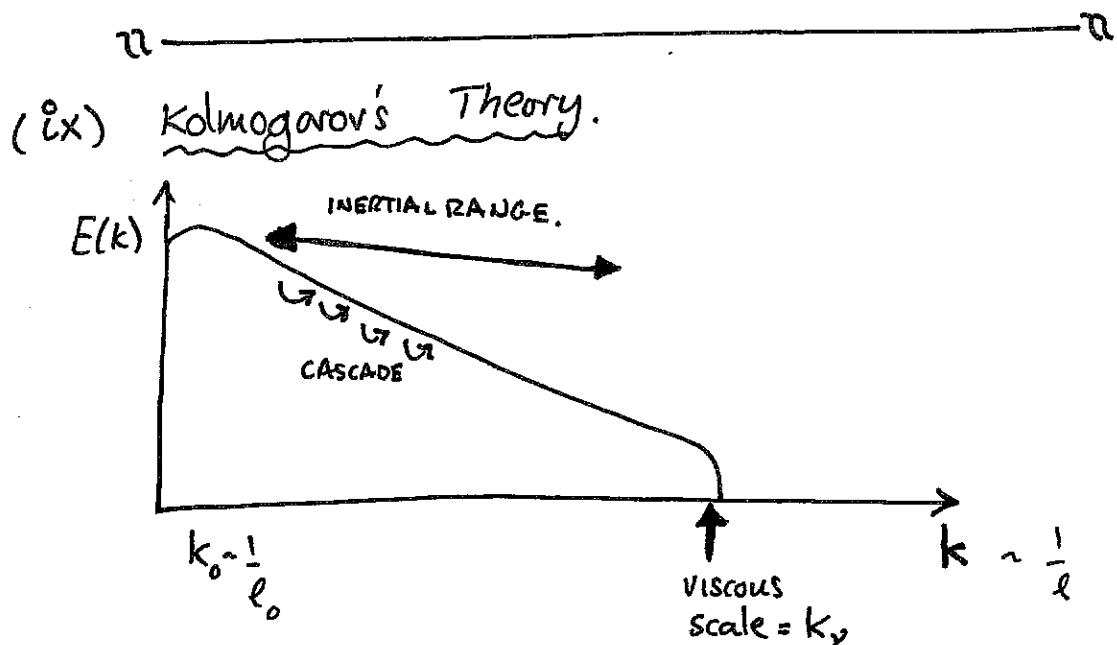
We define the isotropic spectrum

$$E(k) = \frac{1}{2} \rho \frac{k^2 |v_k|^2}{8\pi}$$

Then simple algebra shows that:

$$\text{KINETIC ENERGY} = \int_0^\infty E(k) dk$$

$$F(l) = \int_0^\infty dk E(k) \frac{\sin kl}{kl}$$



2 IDEAS

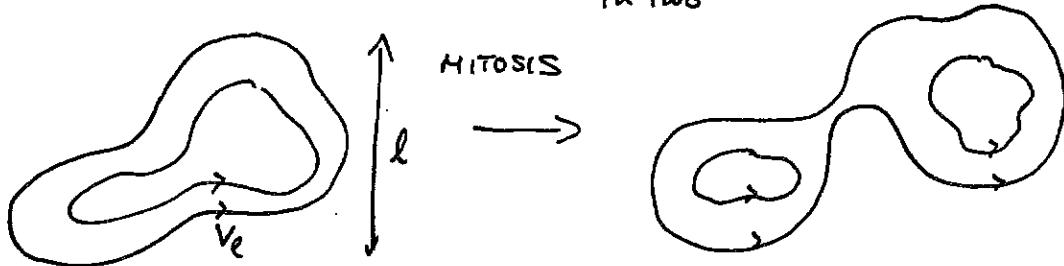
(A) ENERGY IS PASSED FROM SCALE ℓ TO SCALE $\sim 2\ell$ UNTIL IT REACHES $k_v = \frac{1}{l_v}$ WHERE IT IS DISSIPATED.

(B) ENERGY AT SCALE ℓ IS PASSED TO SCALE 2ℓ BY INTERACTION WITH EDDIES OF ROUGHLY THE SAME SCALE - "LOCAL HYPOTHESIS".

(x) ENERGY FLUX FROM SCALE l TO $\sim 2l$

$$\epsilon = \frac{1}{2} \rho \frac{V_e^2}{\tau_e}$$

τ_e = CASCADE TIME, = time to break eddies in two



THIS MUST HAPPEN ON THE ONLY TIMESCALE AT SCALE l (IF WE IGNORE VISCOSITY) i.e.

$$\tau_e = \frac{l}{V_e} \quad \text{"Eddy"}$$

FROM IDEA A $\epsilon = \text{CONSTANT INDEPENDANT OF } l \text{ IN INERTIAL RANGE}$

$$\epsilon = \frac{1}{2} \rho \frac{V_e^3}{l} = \epsilon_0 = \frac{1}{2} \rho \frac{V_0^3}{l_0}$$

$$V_e = V_0 \left(\frac{l}{l_0} \right)^{1/3} \rightarrow \tau_e = \left(\frac{l_0}{V_0} \right) \left(\frac{l}{l_0} \right)^{2/3}$$

NOTE

- BIG EDDIES HAVE MORE KINETIC ENERGY THAN SMALL EDDIES
- SMALL EDDIES TURN OVER FASTER THAN BIG EDDIES.

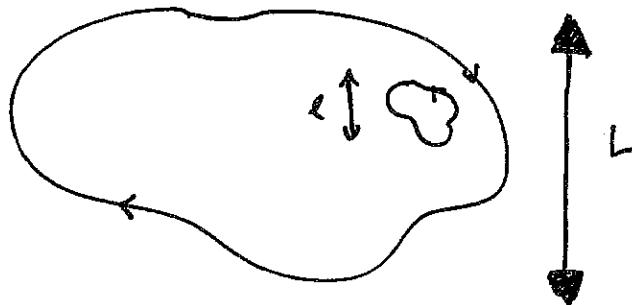
(xi) Dissipation Scale: Viscous dissipation on time τ_e is finite:

$$\text{i.e. } \nu \nabla^2 \sim \frac{1}{\tau_e} \Rightarrow \frac{\nu}{l_e^2} = \left(\frac{V_0}{l_0} \right) \left(\frac{l_0}{l_e} \right)^{2/3} \rightarrow \boxed{\frac{l_e}{l_0} = Re^{3/4}} \gg 1$$

(xii) Let's look at the justification of Kolmogorov's idea (B)

a) Q. CAN BIG EDDIES SHEAR APPART SMALL EDDIES BEFORE THEY CASCADE?

Ans. NO! Shearing rate of Big Eddy $\frac{V_0}{L} \left(\frac{l_0}{L} \right)^{2/3} \ll \tau_e^{-1}$



NOTE THE SMALL EDDY IS CARRIED MANY TIMES ITS OWN SCALE BY THE BIG EDDY DURING THE CASCADE TIME BUT THIS DOES NOT CHANGE THE EDDY.

b) Q. CAN SMALL EDDIES DIFFUSE THE FLUID ACROSS A BIG EDDY IN A BIG EDDY CASCADE TIME?

Ans. No! Diffusion coefficient due to eddies of scale $\ell = \frac{\ell^2}{\tau_e} = D_e$

TIME TO DIFFUSE ACROSS BIG EDDY (SIZE L)

$$= \frac{T_L}{\tau_e} = \frac{L^2}{D_e} \gg \tau_e = \frac{l_0}{V_0} \left(\frac{L}{l_0} \right)^{2/3}$$

$$\underline{\text{NOTE}} \quad \frac{T_L}{\tau_e} = \frac{D_e}{D_e} = \left(\frac{L}{\ell} \right)^{4/3}$$

Big Eddies do most of the diffusion.

Thus: Interaction with eddies of roughly the same scale is stronger than interaction with eddies of different scales.

260 Lecture #14: Strong Turbulence II - M.H.D. Cascade.

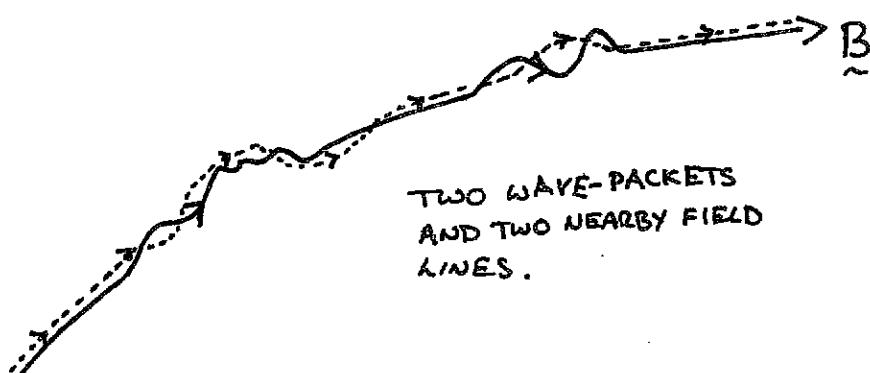
- (i) In the last lecture we discussed Kolmogorov's theory of neutral fluid turbulence. Today we discuss the equivalent theory in M.H.D. - it is due to Goldreich and Sridhar.
- (ii) As before (see Lectures 11 & 12) the "inertial range" of MHD turbulence is expected to be Alfvén waves propagating on the large scale field.

(iii) Although in the INERTIAL RANGE $\frac{\delta B}{B} \ll 1$, $\nabla_0 \xi \approx \mathcal{O}(1)$

DEFINITION OF STRONG TURBULENCE

NOTE: $\frac{\delta B}{B} \sim k_{\parallel} \xi$ and $\nabla_0 \xi \sim k_{\perp} \xi$

therefore we get "strong" turbulence when $\frac{\delta B}{B} \ll 1$
only if $k_{\perp} \gg k_{\parallel}$



- (iv) I will assume - and show later - that we are mostly concerned with Alfvén wave like disturbances and with " $k_{\perp} \gg k_{\parallel}$ ". Thus mostly $|v_{\perp}| \gg v_{\parallel}$ and $\delta B_{\perp} \gg \delta B_{\parallel}$. We will take incompressible motions (this is ok if $\beta \gg 1$ or $v \ll c_s$ which is usually true in the inertial range)

(v) EQUATIONS:

$P + \frac{B^2}{2\mu_0}$ / viscosity

$$\textcircled{1} \quad \rho \left\{ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right\} = - \nabla P + \frac{\underline{B} \cdot \nabla \underline{B}}{\mu_0} + \rho \nu \nabla^2 \underline{V}$$

$$\textcircled{2} \quad \frac{\partial \underline{B}}{\partial t} + \underline{V} \cdot \nabla \underline{B} = \underline{B} \cdot \nabla \underline{V} + \eta \nabla^2 \underline{B}$$

resistivity.

$\nabla \cdot \underline{B} = 0 \quad \text{and} \quad \nabla \cdot \underline{V} = 0 \quad \rho = \text{constant}$

We consider a fixed background field $\underline{B} = B_0 \hat{z}$ and we define "Elsasser Variables" [change notation to coincide with Alfvén & Goldreich]

$$\underline{w}_+ = V_{A0} \hat{z} + \delta \underline{V} - V_{A0} \frac{\delta \underline{B}}{B_0}$$

$$V_{A0} = \sqrt{\frac{B_0^2}{\rho \mu_0}}$$

$$\underline{w}_- = -V_{A0} \hat{z} + \delta \underline{V} + V_{A0} \frac{\delta \underline{B}}{B_0}$$

UNPERTURBED ALFVÉN
VELOCITY.

Adding and subtracting (1) & (2) we obtain:

$$\frac{\partial \underline{w}_+}{\partial t} + V_A \frac{\partial \underline{w}_+}{\partial z} = - \underline{w}_+ \cdot \nabla \underline{w}_+ - \nabla(P/\rho) + \nu \nabla^2 \underline{V} - \eta V_{A0} \nabla^2 \frac{\delta \underline{B}}{B_0}$$

$$\frac{\partial \underline{w}_-}{\partial t} - V_A \frac{\partial \underline{w}_-}{\partial z} = - \underline{w}_- \cdot \nabla \underline{w}_- - \nabla(P/\rho) + \nu \nabla^2 \underline{V} + \eta V_{A0} \nabla^2 \frac{\delta \underline{B}}{B_0}$$

\underline{w}_+ - Alfvénic disturbance moving to "right" towards $z = +\infty$.

\underline{w}_- - Alfvénic disturbance moving to "left" towards $z = -\infty$.

As before only waves going in opposite directions interact.

(vi) Energy Equation

Constructing the energy we get:

$$\frac{d}{dt} \int d^3r \left\{ \frac{\rho v^2}{2} + \frac{B^2}{2\mu_0} \right\} = - \int d^3r \left\{ \rho v (\nabla \cdot v)^2 + \eta \frac{J^2}{\mu_0} \right\}$$

↓ ↓ ↓
 K.E. Magnetic Energy. $J = \frac{\nabla \times B}{\mu_0}$

we can easily add forcing to this picture/equation.

We define 2 Reynolds number things.

$$Re = \text{Reynolds \#} = \frac{v_0 l_0}{\nu}$$

referring to some
stirring scale, l_0
and velocity v_0 .

$$Rm = \text{Magnetic Reynolds \#} = \frac{v_0 l_0}{\eta}$$

In Astrophysics both these numbers are large e.g.

CLUSTER PLASMA (BETWEEN GALAXIES IN CLUSTERS)

$$Re \sim 10^2 - 10^3 \quad Rm \sim 10^{29} !$$

the ratio $\frac{\nu}{\eta} = \frac{Rm}{Re}$ is called the Magnetic Prandtl number
 "Pm" [Sodium $Pm = 10^{-4}$
 Cluster $Pm \sim 10^{26-27}$]

WE ARE CONCERNED WITH PLASMAS WITH $Re \gg 1$ $Rm \gg 1$
 SO ALL DISSIPATION HAPPENS AT SMALL SCALES.

(vii) INERTIAL RANGE [$l \ll l_0$ but not small enough to be dissipative]

No dissipation gives:

$$E = \int d^3 r \left\{ \frac{1}{2} \rho v^2 + \frac{\underline{B}^2}{2 \mu_0} \right\} = \text{constant.}$$

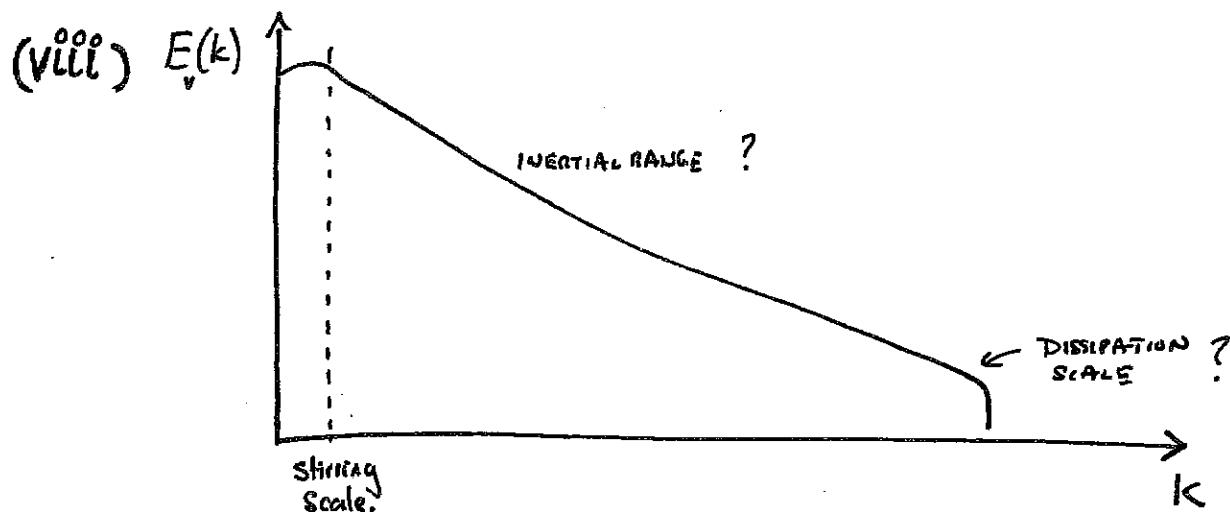
AND

$$I = \int d^3 r \underline{v} \cdot \underline{B} \quad \text{"cross helicity"}$$

or in Elsasser variables: $\frac{d}{dt} \int d^3 r |w_+|^2 = \frac{d}{dt} \int d^3 r |w_-|^2 = 0$

without dissipation energy in ~~top~~ left and right going waves separately conserved.

WAVE PACKET COLLISIONS ARE ELASTIC.



We wish to find the scaling in the inertial range of

$$\underline{V}_L = \sqrt{\langle (\underline{v}(r+\ell) - \underline{v}(r))^2 \rangle} \quad \text{velocity @ scale } \underline{\ell}.$$

$$\delta \underline{B}_L = \sqrt{\langle (\delta \underline{B}(r+\ell) - \delta \underline{B}(r))^2 \rangle} \quad \text{magnetic field @ scale } \underline{\ell}.$$

(ix) CASCADE: like Kolmogorov we assume

- (A) Magnetic and Velocity energy are passed from scale to scale until reaching the dissipation scales.
- (B) Energy at scale ℓ is passed to scale $\frac{\ell}{2}$ by interaction with Eddies of roughly the same scale.

we make extra assumptions:-

- (C) $\delta B_\ell \approx V_\ell$ Alfvén wave like.
- (D) cascade by perpendicular interaction \Rightarrow cascade time $\tau_{e_\perp} \approx \left(\frac{V_{e_\perp}}{l_\perp}\right)^{-1}$ - chopped up in a "shearing time"
- (E) Assume isotropy in plane perpendicular to \underline{B}_0 . but anisotropic w.r.t. parallel direction.

(x) ENERGY FLUX TO SMALL SCALES

$$\epsilon = \frac{1}{2} \rho \frac{V_{e_\perp}^2}{\tau_{e_\perp}} = \text{constant by (A)}.$$

→ $V_{e_\perp} \sim V_0 \left(\frac{l_\perp}{l_0} \right)^{1/3}$

Gives $E(k_\perp) \sim k_\perp^{-5/3}$

see Fig. 3 of Narayan & Goldreich.

like Kolmogorov - almost, since we have l_\perp not l .

(xi) PARALLEL STRUCTURE.

when $\frac{k_{\perp}v_{\perp}}{\omega} \ll 1$, weak turbulence k_{\perp} cascades (increases)
but k_{\parallel} does not. (lecture 11&12)

$\omega \sim k_{\parallel}V_A$ so as k_{\perp} increases so does $\frac{k_{\perp}v_{\perp}}{\omega} \rightarrow 1$

when $\frac{k_{\perp}v_{\perp}}{\omega} \sim 1$, strong turbulence k_{\perp} cascades (see above)
 k_{\parallel} can change:

Goldreich and Sridhar postulated that turbulence would obtain

CRITICAL BALANCE

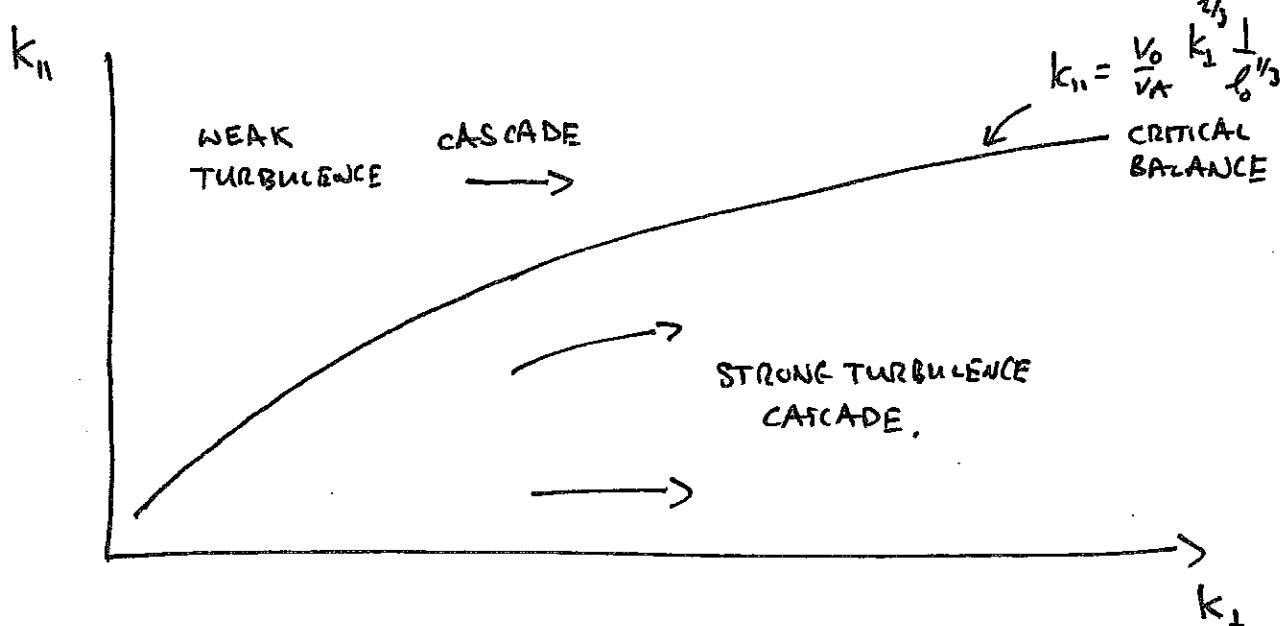
$$k_{\perp}v_{\perp} \sim \omega \sim k_{\parallel}V_A$$

SCALE BY SCALE



$$k_{\parallel} = \left(\frac{V_0}{V_A} \right)^{2/3} k_{\perp}^{2/3} \frac{1}{l_0^{1/3}}$$

CASCADE IS
ANISOTROPIC.



SEE FIG. 7. MARON AND GOLDREICH.

260 Lecture #15. Gyro-kinetic-Cascade: Heating.

(i) Goldreich-Sridhar strong cascade:

$$V_{\ell_\perp} \sim V_0 \left(\frac{\ell_\perp}{\ell_0} \right)^{1/3}$$

from constant energy flux

$$\epsilon = \frac{1/2 \rho V_{\ell_\perp}^2}{\tau_{\ell_\perp}}$$

ℓ_0 = stirring scale.

CRITICAL BALANCE: $\frac{1}{\tau_{\ell_\perp}} \sim k_\perp v_\perp \sim k_\parallel v_{A0}$

$$k_\parallel \sim \left(\frac{V_0}{V_A} \right) k_\perp^{2/3} \frac{1}{\ell_0^{1/3}}$$

(ii) Anisotropic Spectrum. We want to describe anisotropy parallel and perpendicular to a perturbed field - this is best done in Lagrangian coordinates $[r = r_0 + f(r_0, t)]$

$$\hat{E}(k_\perp, k_\parallel) = \frac{1}{2} \rho \int \underline{v}(r_0, t) \cdot \underline{v}(r_0 + \Delta r_0, t) e^{ik \cdot \Delta r_0} \frac{d^3 \Delta r_0}{(2\pi)^3 \text{volume}}$$

then roughly:

$$\hat{E} k_\perp dk_\perp dk_\parallel \sim \frac{1}{2} \rho V_e^2$$

$$\hat{E} \sim \frac{1/2 \rho V_0^2 (k_\perp \ell_0)^{-2/3}}{k_\perp^2 k_\parallel}$$

(iii) We can "model" - essentially guess - a form for \hat{E} which expresses critical balance: e.g.

$$\hat{E}(k_{\perp}, k_{\parallel}) \simeq \frac{1}{2} \rho V_0^2 l_0^{2/3} e^{-\frac{k_{\parallel}^2}{k_0^2(k_{\perp})}}$$

$$\text{where } k_0(k_{\perp}) = \left(\frac{V_0}{V_A}\right)^{2/3} k_{\perp} \frac{1}{l_0^{2/3}}$$

The usual spectrum is then

$$E(k_{\perp}) = k_{\perp} \int dk_{\parallel} \hat{E}(k_{\perp}, k_{\parallel}) \simeq E_0 k_{\perp}^{-5/3}$$

$$E(k_{\perp}) k_{\perp} \sim \text{ENERGY AT SCALE } k_{\perp} \propto k_{\perp}^{-2/3} \sim l_{\perp}^{2/3}$$

(iv) ENERGY FLUX CONSTANT:

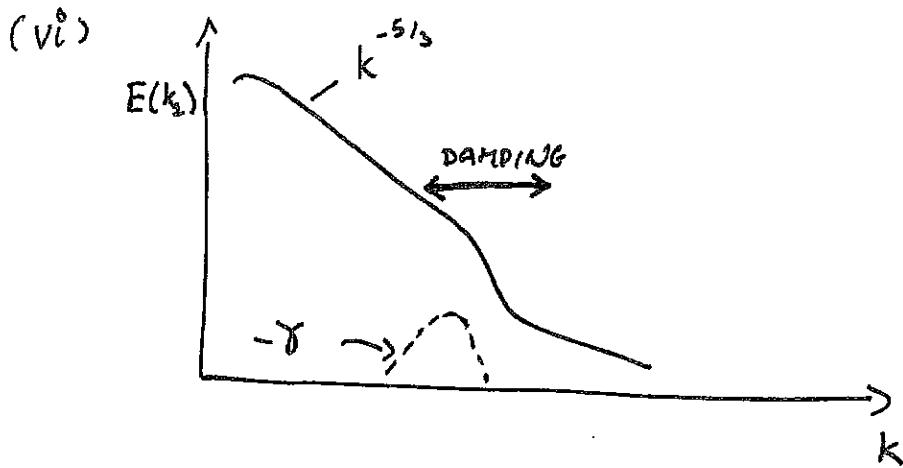
$$k_{\perp} \frac{dE}{dk_{\perp}} = 0 \quad \text{Kolmogorov}$$

(v) Suppose we introduce some damping into the cascade then a simple nearly linear approach.

$$k_{\perp} \frac{dE}{dk_{\perp}} = \text{ENERGY LOST AT SCALE } k_{\perp} \equiv +2\gamma(k_{\perp}) \{E(k_{\perp}) k_{\perp}\}$$

↑
linear damping. (negative)

$$\epsilon = \frac{1}{2} \rho \frac{V_e^2}{\tau_e} = \left(\frac{2E k_{\perp}}{\rho}\right)^{3/2} \frac{1}{2} \rho k \quad \text{or} \quad E k_{\perp} = \left(\frac{2E}{\rho k}\right)^{2/3} \frac{1}{2} \rho$$



Integrating we get

$$\epsilon(k_{\perp}) - \epsilon_0 = \frac{2^{2/3} \rho^{1/3}}{3} \int_{k_0}^{k_{\perp}} \gamma(k_{\perp}) k_{\perp}^{-5/3} dk_{\perp}$$

Estimate of energy lost by scale k_{\perp} .

(vii) Damping happens in the gyrokinetic regime - around $k_{\perp} r_i \sim 1$

SONIC STIRRING IN $\beta = 1$ PLASMA.

Perhaps astrophysically relevant.

$$V_0 \sim C_s \sim V_A \sim v_{thi}, T_e \sim T_i$$

$$\omega = k_{\parallel} V_A \sim \frac{k_{\perp}^{2/3}}{l_0^{1/3}} \cdot v_{thi} \quad \leftarrow$$

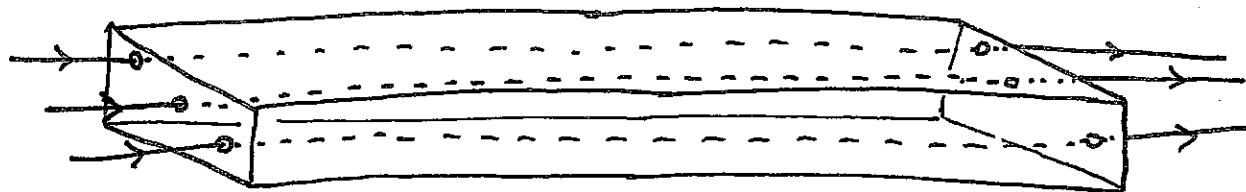
$$\frac{\omega}{\omega_{ci}} \simeq (k_{\perp} r_i)^{2/3} \cdot \left(\frac{\rho_e}{l_0} \right)^{1/3}$$

so at $k_{\perp} r_i \sim O(1)$

$$\omega \ll \omega_{ci}$$

(vii) Gyro-kinetic simulation of cascade.
 LONG THIN BOX.

Bill Dorland & SC. + Troy Carter
 Brian Brugman, Kate Dief
 Greg Hanes Eliot Quataert.



Drive system with a driven current.

$$\nabla \times \underline{B} = \mu_0 \left\{ \underline{\mathcal{J}}_{\text{PLASMA}} + \underline{\mathcal{J}}_{\text{ANTENNA}} \right\}$$

$\underline{\mathcal{J}}_{\text{ANTENNA}} \approx$ Driven by a randomly excited oscillator. At $\underline{k} = 1 \& 2$.

$$\nabla \cdot \underline{\mathcal{J}}_{\text{ANTENNA}} = 0$$

$$\frac{d^2 \mathcal{J}_{||}}{dt^2} + \omega_A^2 \mathcal{J}_{||} = F(t)$$

+ random force.

(viii)

Measure heating through Entropy Production: - see Lecture #7.

$$n_0 \frac{dT_{0i}}{dt} = - \int \frac{d^3 c}{V} \int d^3 v \frac{T}{F_0} \overline{[h_i c_i(h_i)]}$$

time average.