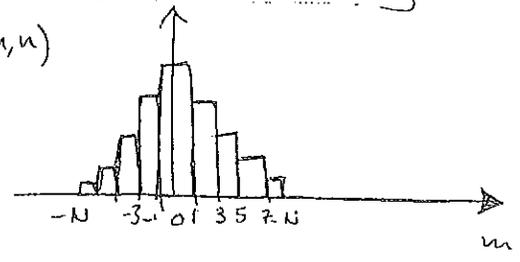


2

(vi) Thus

$$P(m, n) = \frac{N!}{\left(\frac{N+m}{2}\right)! \left(\frac{N-m}{2}\right)!} \left(\frac{i}{2}\right)^N$$

Note: If N is even/odd m is even/odd.



Clearly $\sum_{m=-N}^N P(m, n) = 1$

(vii) Usually we are interested in the answer when $N \rightarrow \infty$ but $m \ll N$.

STIRLING'S FORMULA $s \gg 1$

$$\log s! \approx \left(s + \frac{1}{2}\right) \log s - s + \frac{1}{2} \log 2\pi + \frac{1}{12s} - \dots$$

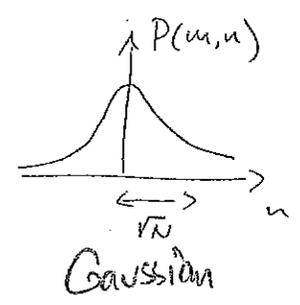
and log expansion

$$\log\left(1 \pm \frac{m}{N}\right) \approx \pm \frac{m}{N} - \frac{m^2}{2N^2} + \mathcal{O}\left(\frac{m^3}{N^3}\right) \dots$$

Using these formula we obtain after some algebra:

$N \gg 1$

$$P(m, n) \approx \left(\frac{2}{\pi N}\right)^{1/2} \exp\left\{-\frac{m^2}{2N}\right\}$$



note: $\langle m \rangle = \sum_{-N}^N P(m, n) m = 0$

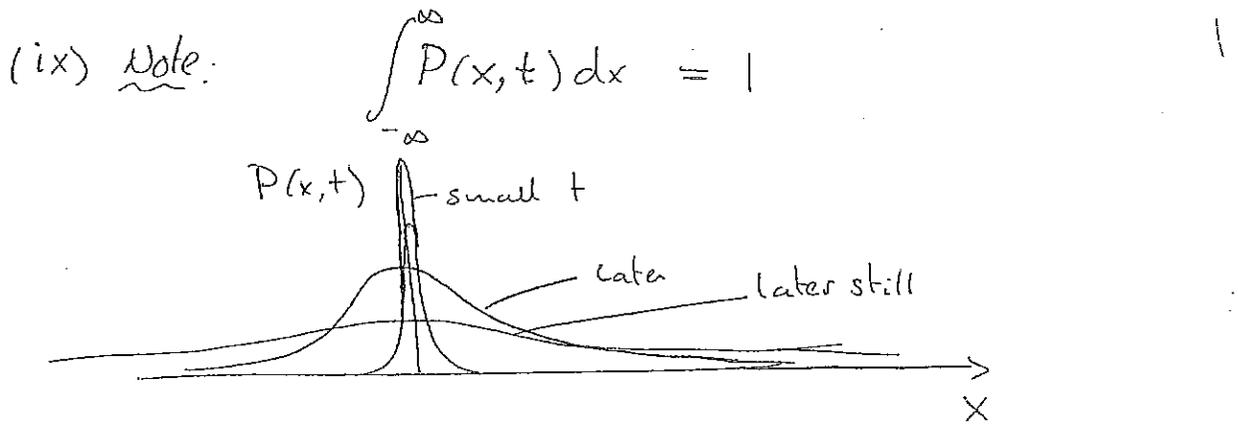
$$\langle m^2 \rangle = N$$

R.M.S. $\sqrt{\langle m^2 \rangle} = \sqrt{N}$ so typically $m \ll N$ as assumed.

(viii) Now suppose each step is length l so $x = ml$
 and there are v tosses per unit time so that $N = vt$

$$\begin{aligned}
 P(x,t) dx &= \text{Probability that } x \text{ is between } x \text{ and } x+dx \text{ at time } t. \\
 &= \text{Probability that } m \text{ is between } m \text{ and } m+1 \text{ (since it must be either even or odd)} \times \frac{dx}{2l} \\
 &= P(m, N) \frac{dx}{2l} \\
 &= \frac{1}{2(\pi Dt)^{1/2}} \exp\left\{-\frac{x^2}{4Dt}\right\} dx
 \end{aligned}$$

where $D = \frac{1}{2}vl^2$ Diffusion coefficient



(x) We can approach this in a different way. Since the probabilities are independent.

$$P(m, N) = P(-1, 1) P(m+1, N-1) + P(1, 1) P(m-1, N-1)$$

\nwarrow probability of a tail = $1/2$ \nearrow Probability of a head = $1/2$

④

So:-

$$(xi) \quad P(m, N) - P(m, N-1) = \frac{1}{2} \left\{ P(m+1, N-1) - 2P(m, N-1) + P(m-1, N-1) \right\}$$

added to both sides

Setting $m = \frac{x}{l}$ and $N = vt$ we get for small changes

per step

$$\frac{\partial P}{\partial t} = D \frac{\partial^2 P}{\partial x^2}$$

$$D = \frac{vl^2}{2}$$

• Diffusion equation.

(xii) look for solution of the form

$$P = a(t) \exp\left(-b(t) \frac{x^2}{2}\right)$$

Substituting in

$$\left\{ \frac{da}{dt} - a \frac{db}{dt} \frac{x^2}{2} \right\} = Da \left\{ -b + b^2 x^2 \right\}$$

equate constant and coefficient of x^2

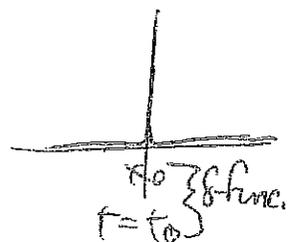
$$\Rightarrow b = \frac{1}{2Dt} \quad \frac{1}{b^2} \frac{db}{dt} = D \quad a = \frac{1}{(4\pi Dt)^{1/2}}$$

so as before.

$$P(x, t) = \frac{1}{(4\pi Dt)^{1/2}} \exp\left(\frac{-x^2}{4Dt}\right)$$

$$t' = t - t_0$$

$$= \frac{1}{[4\pi D(t-t_0)]^{1/2}} \exp\left(\frac{-(x-x_0)^2}{4D(t-t_0)}\right)$$



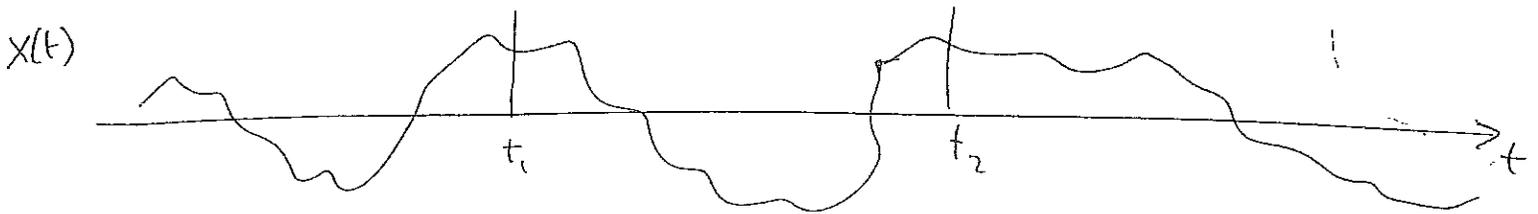
⑤

Probability distributions for random function $x(t)$

$w_1(x, t) dx \equiv$ Probability x is between x & $x+dx$ at time t .

$w_2(x_1, t_1, x_2, t_2) dx_1 dx_2 \equiv$ Joint probability that x is between x_1 & x_1+dx_1 at time t_1 AND between x_2 & x_2+dx_2 at time t_2 .

We will often assume that the system is time stationary ^{on average} so that $w_1 = w_1(x)$ and $w_2 = w_2(x_1, x_2, (t_1 - t_2))$



CONDITIONAL PROBABILITY (Time stationary for simplicity)

$P_2(x_1 | x_2, t) \equiv$ Probability that x is in $x_2 - x_2+dx_2$ at time t given that it is at x_1 at $t=0$.

MARKOV PROCESS ~~is~~ Probability of finding x ~~at~~ in a given interval does not depend on history ~~only~~ on previous time in a sequence of discrete times. Therefore.

$$P_2(x_1 | x_2, t) = \int dx P(x, |x, t_0) P(x | x_2, t - t_0)$$

222e. lecture #2. Electron ion collisions.

(i) In a plasma with almost equal ion and electron temperatures.

$$v_i \sim \sqrt{\frac{T_i}{m_i}} \quad v_e \sim \sqrt{\frac{T_e}{m_e}} \Rightarrow v_e \gtrsim 60 v_i$$

to most electrons the ions appear stationary.

(ii) We wish to calculate the changes in the electron distribution due to reasonably close encounters with ions. We will consider electron-electron collisions later.

(iii) First let's ask the question, what is the probability of finding an ion in a given volume of plasma d^3r near the colliding electron. If the ion does not know the electron is there we expect:-

$$\text{Probability of an ion in } d^3r = n_i(r) d^3r$$

Independent of the electron position. Actually of course the electron position is affected by the ion (the ion is less affected by the electron) so there is a correlation between the positions.

So the conditional probability that given that the electron is at r_1 (in d^3r_1) the probability that the ion is at r_2 in d^3r_2

$$\text{is } P(r_1, r_2) d^3r_2 \neq n_i(r_2) d^3r_2 \quad \text{but it almost is}$$

since electrons are attracted to ions you expect to see more ions near electrons

(iv) let's imagine a time interval Δt which is long compared to an individual collision but short compared to the time to change the distribution function. let $P(\underline{v}, \Delta \underline{v}; \Delta t) d^3 \Delta \underline{v}$ be the probability that an electron will scatter from $\underline{v} \rightarrow \underline{v} + \Delta \underline{v}$ in the time Δt . Then:

$$f(\underline{v}, t) = \int d^3 \Delta \underline{v} f(\underline{v} - \Delta \underline{v}, t - \Delta t) P(\underline{v} - \Delta \underline{v}, \Delta \underline{v}, \Delta t)$$

IF MOST OF THE CHANGES IN Δt ARE SMALL - i.e. NO LARGE ANGLE COLLISIONS THEN WE CAN TAYCOR EXPAND.

$$\begin{aligned} (v) \quad f(\underline{v}, t) = \int d^3 \Delta \underline{v} \left[f(\underline{v}, t) P(\underline{v}, \Delta \underline{v}, \Delta t) - \Delta t \frac{\partial f}{\partial t} P(\underline{v}, \Delta \underline{v}, \Delta t) \right. \\ \left. - \Delta \underline{v} \cdot \frac{\partial}{\partial \underline{v}} \left\{ f(\underline{v}, t) P(\underline{v}, \Delta \underline{v}, \Delta t) \right\} \right. \\ \left. + \frac{1}{2} \Delta \underline{v} \Delta \underline{v} \cdot \frac{\partial}{\partial \underline{v}} \left\{ f(\underline{v}, t) P(\underline{v}, \Delta \underline{v}, \Delta t) \right\} \right] \\ + \text{H.O.T.} \dots \end{aligned}$$

(vi) Now $\int d^3 \Delta \underline{v} P(\underline{v}, \Delta \underline{v}, \Delta t) = 1$ by definition as a probability.

DEFINE

$$\langle \Delta \underline{v} \rangle = \frac{1}{\Delta t} \int d^3 \Delta \underline{v} \Delta \underline{v} P(\underline{v}; \Delta \underline{v}) \quad \langle \Delta \underline{v} \Delta \underline{v} \rangle = \frac{1}{\Delta t} \int d^3 \Delta \underline{v} \Delta \underline{v} \Delta \underline{v} P(\underline{v}, \Delta \underline{v}, \Delta t)$$

Average velocity change per unit time

Average change per unit time in $\Delta \underline{v} \Delta \underline{v}$.

(vii) Thus dropping the cubic terms we obtain

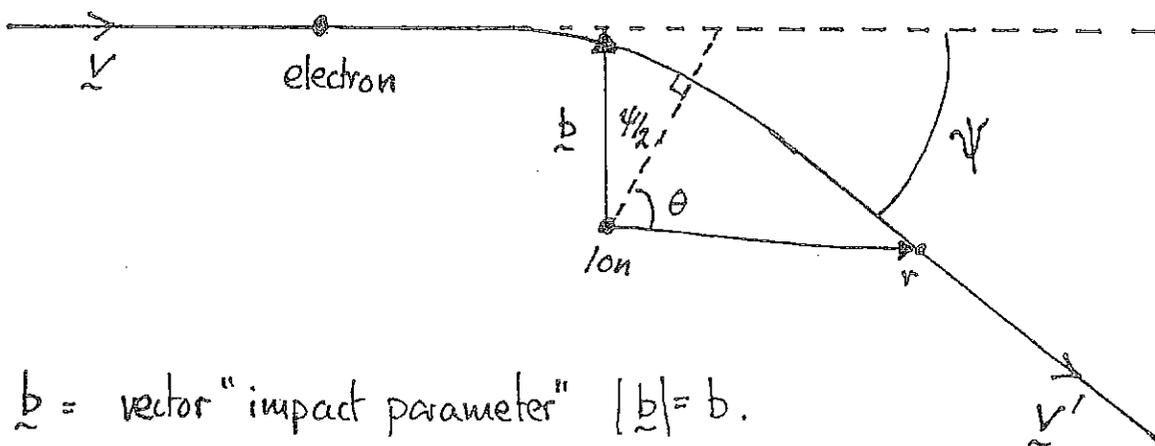
$$\left(\frac{\partial f}{\partial t} \right)_{\text{COLLISIONS}} = - \frac{\partial}{\partial \underline{v}} \cdot \left[\langle \Delta \underline{v} \rangle f(\underline{v}, t) \right] + \frac{1}{2} \frac{\partial}{\partial v_i \partial v_k} \left[\langle \Delta v_i \Delta v_k \rangle f(\underline{v}, t) \right]$$

This is called the Fokker-Planck equation. Note that

in general \mathcal{P} may depend on f (although we expect it not to be very dependant on f) so $\langle \Delta \underline{v} \rangle$ and $\langle \Delta \underline{v} \Delta \underline{v} \rangle$ also depend on f . We will now calculate $\langle \Delta \underline{v} \rangle$ and $\langle \Delta \underline{v} \Delta \underline{v} \rangle$ in electron-ion collisions.

(viii) let us now examine a collision between an electron and ion

RUTHERFORD SCATTERING:



\underline{b} = vector "impact parameter" $|\underline{b}| = b$.

ψ = scattering angle. position of electron, $[r(t), \theta(t)]$

As usual ANGULAR MOMENTUM IS CONSERVED = $v b = \left(\frac{d\theta}{dt} \right) r^2$

$$\Rightarrow \frac{d}{dt} \equiv \frac{d\theta}{dt} \frac{d}{d\theta} \equiv \frac{v b}{r^2} \frac{d}{d\theta}$$

RADIAL FORCE: - $m_e \left\{ \frac{d^2 r}{dt^2} - \underbrace{\left(\frac{d\theta}{dt} \right)^2 r}_{\substack{\uparrow \\ \text{centripetal} \\ \text{accel.}}} \right\} = - \frac{q_e}{r^2}$

Substituting for $\frac{d}{dt} \propto \frac{d\theta}{dt}$

$$\Rightarrow \frac{d}{d\theta} \left[\frac{1}{r^2} \frac{dr}{d\theta} \right] - \frac{1}{r} = - \left\{ \frac{q_e}{v^2 b^2 m_e} \right\} = - \frac{b_0}{b^2}$$

$u = \frac{1}{r} \Rightarrow \boxed{\frac{d^2 u}{d\theta^2} + u = \frac{b_0}{b^2}}$ $b_0 = \frac{q_e}{v^2 m_e}$

$u = \frac{1}{r} = \frac{b_0}{b^2} + A \cos \theta$ ← phase chosen so that at $\theta=0$ $\frac{dr}{d\theta} = 0$.

as $\theta \rightarrow \pm \psi/2$ $\left. \begin{matrix} r \rightarrow \infty \\ u \rightarrow 0 \end{matrix} \right\} -\frac{b_0}{b^2} = A \cos(\psi/2)$

$\theta = -\psi/2$ $\frac{dr}{dt} = -v = \frac{d\theta}{dt} \frac{dr}{d\theta} = \frac{vb}{r^2} \frac{dr}{d\theta} = vb \frac{d(1/r)}{d\theta}$

$v = -vb A \sin(\psi/2)$

COMBINING WE GET

$\boxed{\frac{b_0}{b} = \tan \psi/2}$ RUTHERFORD SCATTERING FORMULA.

Then since $|v'| = v$

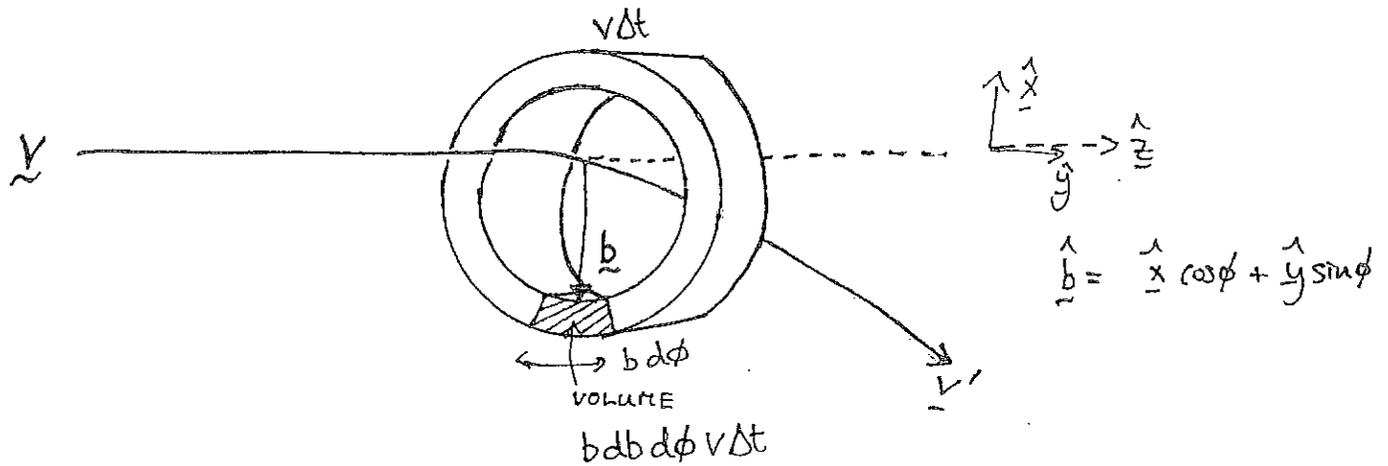
$\underline{v}' = \underline{v} \cos \psi - \hat{b} v \sin \psi$

$\underline{v}' = \left(\frac{b^2 - b_0^2}{b^2 + b_0^2} \right) \underline{v} - \hat{b} v \left(\frac{2bb_0}{b^2 + b_0^2} \right)$

Using $\sin x = \frac{2 \tan x/2}{1 + \tan^2 x/2}$
 $\cos x = \frac{1 - \tan^2}{1 + \tan^2 x/2}$

Not Correct! Needs from class are correct!

statistics: The probability $P(\underline{v} : \Delta \underline{v}) d^3 \underline{v}$ is the same as the probability there is an ion at impact parameter b in the volume $b db d\phi v \Delta t$.



$$P(\underline{v} : \Delta \underline{v}) d^3 \underline{v} = n_i b db d\phi v \Delta t$$

$$\langle \Delta \underline{v} \rangle = \langle \frac{\underline{v}' - \underline{v}}{\Delta t} \rangle = v n_i \int_0^{2\pi} d\phi \int_0^{b_{\max}} b db \left[\left(\frac{-2b_0^2}{b^2 + b_0^2} \right) \underline{v} + \frac{v (\hat{x} \cos \phi + \hat{y} \sin \phi) 2bb_0}{b^2 + b_0^2} \right]$$

← we terminate at $b = b_{\max}$
integrates to zero

$$\langle \Delta \underline{v} \rangle = \pi n_i b_0^2 v^2 \ln \left(\frac{b_{\max}^2 + b_0^2}{b_0^2} \right)$$

Similarly

$$\langle \Delta \underline{v} \Delta \underline{v} \rangle = \pi n_i b_0^2 v^3 \frac{1}{2} \left(\underline{\underline{I}} - \frac{\underline{v} \underline{v}}{v^2} \right) \ln \left(\frac{b_{\max}^2 + b_0^2}{b_0^2} \right)$$

$$\frac{\underline{v} \underline{v}}{v^2} = \frac{\underline{v}}{v}$$

2222, lecture #3. Lorentz Collision Operator.

(i) last time we derived the Fokker-Planck Collision term for electron ion collisions.

$$\left(\frac{\partial f}{\partial t}\right)_{e,i} = \frac{\partial}{\partial \underline{v}} \{ \langle \Delta \underline{v} \rangle f \} + \frac{1}{2} \frac{\partial}{\partial v_i \partial v_k} \{ \langle \Delta v_i \Delta v_k \rangle f \}$$

We derived (with some algebra)

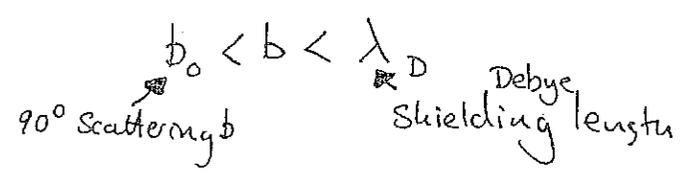
$$\langle \Delta \underline{v} \rangle = -4\pi b_0^2 n_i v \ln \Lambda \underline{v} = -\nu_{ei} \underline{v} \quad \underline{\text{DRAG}}$$

$$\langle \Delta \underline{v} \Delta \underline{v} \rangle = 4\pi b_0^2 n_i v^3 \ln \Lambda \left\{ \underline{\underline{I}} - \frac{\underline{v} \underline{v}}{v^2} \right\} = \nu_{ei} v^2 \left\{ \underline{\underline{I}} - \frac{\underline{v} \underline{v}}{v^2} \right\} \quad \underline{\underline{\text{DIFFUSION}}}$$

where $\ln \Lambda = \ln \left(\frac{\lambda_D}{b_0} \right)$ Debye length "Coulomb logarithm." typically $\ln \Lambda \sim 10-15$.

$$\nu_{ei} = \frac{4\pi n_i q^2 e^2}{m_e^2 v^3} \ln \Lambda \quad \text{"Electron-ion collision rate:"}$$

(ii) Most of the collisions come from small angle scattering with impact parameters



At $b = b_0$ scattering is 90° scattering.

(iii) Fokker-Planck term treats small angle scattering (- not 90° scattering) but since the effect of small angle collisions is dominant it

gives accurate answers $\nu_{90} = \frac{4\pi n_i q^2 e^2}{m_e^2 v^3} \ll \nu_{ei}$

(iv) Also note a good check is that since e-i collisions conserve electron energy

$$\langle v'^2 \rangle = v^2 = \langle (v + \Delta v)^2 \rangle = v^2 + 2v \cdot \langle \Delta v \rangle + \langle \Delta v \cdot \Delta v \rangle$$

substituting in confirms this.

(v) After some algebra we can rewrite the Fokker-Planck equation as:-

$$\left(\frac{\partial f}{\partial t} \right)_{e-i} = \frac{\nu_{ei}}{2} v^2 \frac{\partial}{\partial v} \cdot \left[\left(\frac{\mathbf{I}}{v} - \frac{v \mathbf{v}}{v^2} \right) \cdot \frac{\partial f}{\partial \mathbf{v}} \right] \quad \text{"Lorentz" collision operator.}$$

(vi) It is instructive to write this in terms of spherical polar angles.

$$\frac{\partial}{\partial \mathbf{v}} = \hat{\mathbf{v}} \frac{\partial}{\partial v} + \frac{\hat{\boldsymbol{\theta}}}{v} \frac{\partial}{\partial \theta} + \frac{\hat{\boldsymbol{\phi}}}{v \sin \theta} \frac{\partial}{\partial \phi} \quad \hat{\mathbf{v}} = \frac{\mathbf{v}}{v}$$

again after some algebra

$$\left(\frac{\partial f}{\partial t} \right)_{e-i} = \frac{\nu_{ei}}{2} \left\{ \frac{1}{\sin \theta} \frac{\partial}{\partial \theta} \left[\sin \theta \frac{\partial f}{\partial \theta} \right] + \frac{1}{\sin^2 \theta} \frac{\partial^2 f}{\partial \phi^2} \right\}$$

Angular part of ∇^2

All the stuff you know about handling ∇^2 i.e. spherical harmonics is useful with this operator.

(vii) NOTE: The collisions don't change the $|\mathbf{v}| = v$ of the particle so no $\frac{\partial f}{\partial v}$ terms appear. This is why this is sometimes called the pitch-angle scattering operator.

(viii) Axisymmetric Solutions:- $\frac{\partial}{\partial \phi} \equiv 0$ $\mu = \cos \theta$

$$\left(\frac{\partial f}{\partial t} \right)_{e-i} = \frac{v_{ei}}{2} \frac{\partial}{\partial \mu} \left[(1 - \mu^2) \frac{\partial f}{\partial \mu} \right]$$

Separable Solutions. $f = \sum_{n=1}^{\infty} f_n(v, t) g_n(\mu)$

$$\frac{\partial f_n}{\partial t} = -\lambda_n f_n \quad f_n = \bar{f}_n(v) e^{-\lambda_n t}$$

$$\frac{v_{ei}}{2} \frac{d}{d\mu} \left[(1 - \mu^2) \frac{dg_n}{d\mu} \right] = -\lambda_n g_n$$

• Legendre's Equation Solution $g_n = P_n(\mu)$ Legendre Polynomials.
 $\lambda_n = \frac{v_{ei}}{2} n(n+1)$

• As $t \rightarrow \infty$ all $n \geq 1$ decay (over a few collision times)

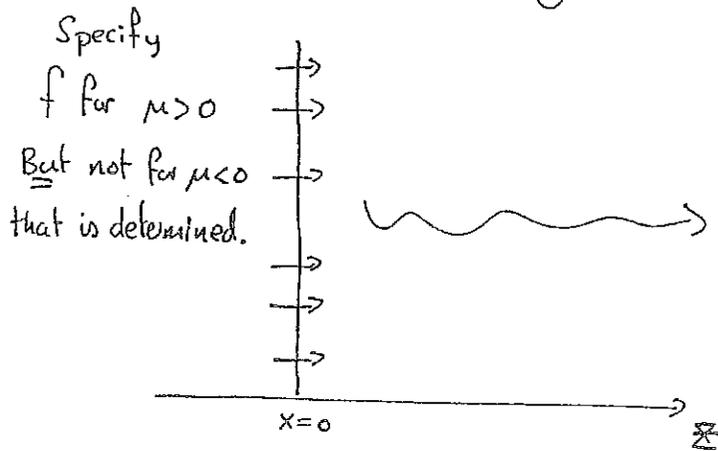
$$\text{and } \lim_{t \rightarrow \infty} f(v, \mu, t) = \bar{f}_0(v)$$

Smooths out distribution into an isotropic one.

(ix) To make the complete equation we must add the evolution of f due to the mean field etc. i.e. the VLASOV PART. So :-

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{(-e)}{m_0} \left(\underline{E} + \frac{\underline{v} \times \underline{B}}{c} \right) \cdot \frac{\partial f}{\partial \underline{v}} = \frac{\gamma e i v^2}{2} \frac{\partial}{\partial \underline{v}} \cdot \left(\underline{I} - \frac{\underline{v} \underline{v}}{v^2} \right) \cdot \frac{\partial f}{\partial \underline{v}}$$

(x) Solution for Boundaries



$$V_z = v \mu$$

$$v \mu \frac{\partial f}{\partial z} = \frac{\gamma e i}{2} \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial f}{\partial \mu}$$

New kind of Seperable solution.
 Bethe-Bothe functions.

HOMEWORK. Assume a weak electric field is applied to a

plasma (E_0) in the z direction. Assume also that

$$f_e = f_{\text{maxwellian}}^{(v)} + f_1$$

and that E_0 and f_1 are small. Given only $e-i$ collisions determine f_1 to dominant order in E_0 and calculate the current. Finally determine the resistivity. (i.e. $E_0 = \eta J$ with η the resistivity).

(i) Today we consider collisions between particles of any charge and mass. Consider the collision of two particles of charge, mass (m_1, q_1) and (m_2, q_2) .

$$(ii) \quad m_1 \frac{d\mathbf{v}_1}{dt} = \frac{q_1 q_2 (\mathbf{r}_1 - \mathbf{r}_2)}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|^3} \quad \text{and} \quad m_2 \frac{d\mathbf{v}_2}{dt} = \frac{-q_1 q_2 (\mathbf{r}_1 - \mathbf{r}_2)}{4\pi\epsilon_0 |\mathbf{r}_1 - \mathbf{r}_2|^3}$$

as usual we define the center of mass velocity \mathbf{v}_{cm} and relative velocity \mathbf{u}

$$\mathbf{v}_{cm} = \frac{m_1 \mathbf{v}_1 + m_2 \mathbf{v}_2}{m_1 + m_2}$$

$$\mathbf{r}_{cm} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2}$$

Reduced mass

$$\mathbf{u} = \mathbf{v}_1 - \mathbf{v}_2 = \frac{d}{dt} (\mathbf{r}_1 - \mathbf{r}_2)$$

$$m_{12} = \frac{m_1 m_2}{m_1 + m_2}$$

$$= \frac{d\mathbf{r}_{12}}{dt}$$

and as you know

$$m_{12} \frac{d^2 \mathbf{r}_{12}}{dt^2} = \frac{q_1 q_2}{4\pi\epsilon_0} \frac{\mathbf{r}_{12}}{r_{12}^3}$$

(iii) considering collisions on species 1 we have

$$\mathbf{v}_1 = \mathbf{v}_{cm} + \frac{m_2 \mathbf{u}}{m_1 + m_2}$$

since during a collision \mathbf{v}_{cm} is constant we get

$$\Delta \mathbf{v}_1 = \frac{m_2}{m_1 + m_2} \Delta \mathbf{u} = \frac{m_{12}}{m_1} \Delta \mathbf{u}$$

(iv) We can use our previous results for ΔV for Δu if we replace $m_e \rightarrow m_{12}$ $V \rightarrow |\underline{v}_1 - \underline{v}_2| = u$

$$\Delta \underline{u} = \frac{-2b_0^2}{b^2 + b_0^2} \underline{u} - \frac{\hat{b} u}{b_0^2 + b^2} 2bb_0$$

$$b_0 = \frac{q_1 q_2}{4\pi \epsilon_0 m_{12} u^2}$$

(v) To compute the average $\langle \Delta v_1 \rangle$ we have to add up contributions from scattering/collisions by particles of different

\underline{v}_2 . Let $P_{12}(\underline{v}_1, \underline{v}_2; \Delta \underline{v}_1) d^3 \Delta \underline{v}_1 d^3 \underline{v}_2$ be the probability of scattering from $\underline{v}_1 \rightarrow \underline{v}_1 + \Delta \underline{v}_1$ BY A PARTICLE OF MASS m_2

AND CHARGE q_2 AT VELOCITY \underline{v}_2 (IN $d^3 \underline{v}_2$). From lecture #7.

$$P_{12}(\underline{v}_1, \underline{v}_2; \Delta \underline{v}_1) d^3 \Delta \underline{v}_1 d^3 \underline{v}_2 = \int_2(\underline{v}_2) d^3 \underline{v}_2 b db d\phi u \Delta t$$

(vi) Clearly

distribution function for $m_2 q_2$ particles

$$\langle \Delta \underline{v}_1 \rangle = \frac{1}{\Delta t} \int d^3 \underline{v}_2 \int P_{12}(\underline{v}_1, \underline{v}_2; \Delta \underline{v}_1) \Delta \underline{v}_1 d^3 \Delta \underline{v}_1$$

(vii) Putting it together we get [integrating over b_1^{ϕ} and \underline{v}_2]

$$\langle \Delta \underline{v}_1 \rangle = \frac{4\pi q_1^2 q_2^2}{m_{12}^2} \ln \Lambda \frac{m_{12}}{m_1} \int \frac{(\underline{v}_1 - \underline{v}_2) f_2(\underline{v}_2)}{|\underline{v}_1 - \underline{v}_2|^3} d^3 \underline{v}_2$$

$$\langle \Delta \underline{v}_1 \rangle = \Gamma_{12} \frac{m_1}{m_{12}} \int \frac{f_2(\underline{v}_2) (\underline{v}_1 - \underline{v}_2)}{|\underline{v}_1 - \underline{v}_2|^3} d^3 \underline{v}_2$$

$$\Gamma_{12} = \frac{4\pi q_1^2 q_2^2}{m_1^2} \ln \Lambda$$

note $\Lambda = \frac{b_{max}}{\langle b_0 \rangle} \approx \frac{\lambda_D}{q_1 q_2 / T}$

we take it out of the integral for simplicity since it doesn't vary much with v .

Similarly.

(viii)

$$\langle \Delta \underline{v}_i \Delta \underline{v}_i \rangle = \Gamma_i \int \frac{f_2(\underline{v}_2)}{|\underline{v}_1 - \underline{v}_2|} \left\{ \frac{I}{\approx} - \frac{(\underline{v}_1 - \underline{v}_2)(\underline{v}_1 - \underline{v}_2)}{|\underline{v}_1 - \underline{v}_2|^2} \right\} d^3 \underline{v}_2$$

These are general formula for collisions between species 1 & 2. AND

(ix)

$$\frac{\partial f_i}{\partial t} = - \frac{\partial}{\partial \underline{v}_i} \cdot \langle \Delta \underline{v}_i \rangle f_i + \frac{\partial^2}{\partial v_i \partial v_j} \langle \Delta v_i \Delta v_j \rangle f_i$$

as before

Typically one would have to sum over interaction with several species. i.e.

$$\langle \Delta v_{\sim 1} \rangle = \sum_{i=1}^N \Gamma_{ii} \frac{m_{ii}}{m_i} \int \frac{v_{\sim 1} - v_{\sim i}}{|v_{\sim 1} - v_{\sim i}|^3} f_i(v_{\sim i}) d^3 v_{\sim i}$$

(X) ROSENBLUTH POTENTIALS.

In electrostatics
$$\underline{E}(\underline{r}) = \int \frac{\rho(\underline{r}_0) (\underline{r} - \underline{r}_0)}{|\underline{r} - \underline{r}_0|^3} d^3 r_0$$

$$= -\nabla \phi$$

and
$$\nabla^2 \phi = 4\pi \rho$$

$$\phi = \int \frac{\rho(\underline{r}_0)}{|\underline{r} - \underline{r}_0|} d^3 r_0$$

Thus we define a potential in velocity space

$$h_{12}(v_{\sim 1}) = \frac{m_1}{m_{12}} \int \frac{f_2(v_{\sim 2})}{|v_{\sim 1} - v_{\sim 2}|} d^3 v_{\sim 2}$$

Then
$$\langle \Delta v_{\sim 1} \rangle = \sum_{i=1}^N \Gamma_{ii} \frac{\partial h_{1i}}{\partial v_{\sim 1}}$$

OK, but we'll concentrate on particle species 1 & 2.

and clearly

$$\nabla_v^2 h_{12} = -4\pi \frac{m_1}{m_{12}} f_2(v_{\sim 1})$$

Can use methods of electrostatics to find h_{12} .

$$\nabla_v^2 = \frac{\partial^2}{\partial v_x^2} + \frac{\partial^2}{\partial v_y^2} + \frac{\partial^2}{\partial v_z^2} \quad \text{Laplacian in velocity space.}$$

(K) now note $\frac{\partial}{\partial v_i} \equiv \frac{\partial}{\partial u}$

$$\frac{\partial u}{\partial u} = \frac{u}{u}$$

$$\frac{\partial^2 u}{\partial u_i \partial u_j} = \frac{1}{u} \left(\delta_{ij} - \frac{u_i u_j}{u^2} \right)$$

Thus (without summation)

$$\begin{aligned} \langle \Delta v_{ii} \Delta v_{jj} \rangle &= \Gamma_{12} \int f_2(v_2) \frac{\partial^2 u}{\partial u_i \partial u_j} d^3 v_2 \\ &= \Gamma_{12} \frac{\partial^2}{\partial v_{ii} \partial v_{jj}} \int f_2(v_2) |v_1 - v_2| d^3 v_2 \end{aligned}$$

We define another potential

$$g_{12}(v_1) = \int f_2(v_2) |v_1 - v_2| d^3 v_2$$

$$\langle \Delta v_{ii} \Delta v_{jj} \rangle = \sum_{B \leq 1}^N \Gamma_{1B} \frac{\partial^2 g_{1B}}{\partial v_i \partial v_j}$$

Also note $\sum_i \delta_{ij} \frac{\partial^2 u}{\partial u_i \partial u_j} = \frac{2}{u}$ so

$$\frac{m_1}{m_{12}} \nabla_{\underline{v}}^2 g_{12}(\underline{v}) = 2 h_{12}(\underline{v})$$

again we well known methods to find g_{12} .

g_{12} & h_{12} are called the Rosenbluth Potentials.

(xi) Landau form of collision term - after some algebra we obtain an alternate form

$$\frac{\partial f_i}{\partial t} = \sum_{i=1}^N C_{ii}(f_i, f_i) \quad i \text{ labels species as before.}$$

$$C(f_i, f_i) = \frac{4\pi q_i^2 q_j^2 m_i \Lambda}{m_i} \frac{\partial}{\partial v} \cdot \mathcal{J}_{ii}$$

Collisional Flux in Velocity space

$$\mathcal{J}_{ii} = \int d^3 v_i \mathcal{U}_{ii} \cdot \left[\frac{f_i(v_i)}{m_i} \frac{\partial f_i(v_i)}{\partial v_i} - \frac{f_i(v_i)}{m_i} \frac{\partial f_i(v_i)}{\partial v_i} \right]$$

where $\mathcal{U}_{ii} = \frac{1}{|v_i - v_i|} \left\{ \frac{\mathbf{I}}{\approx} - \frac{(v_i - v_i)(v_i - v_i)}{|v_i - v_i|^2} \right\}$

This form has some distinct advantages for some calculations (for others the Rosenbluth form is better) they are of course entirely equivalent.

222c lecture #5. Collisional Relaxation.

(i) In the Landau form ^{of} the collisions,

$$\frac{\partial f_1}{\partial t} = \sum_{\text{Species}} C_{ii}(f_1, f_i)$$

↙ 2 not 4 as 'out' time.

$$C_{ii}(f_1, f_i) = \frac{2\pi q_1^2 q_i^2}{m_1} \ln \Lambda \frac{\partial}{\partial \underline{v}} \cdot \underline{J}_{ii}$$

$$\underline{J}_{ii} = \int d^3 \underline{v}_i' \underline{U}_{ii} \cdot \left[\frac{f_i(\underline{v}_i')}{m_i} \frac{\partial f_1}{\partial \underline{v}_1} - \frac{f_1(\underline{v}_1)}{m_1} \frac{\partial f_i}{\partial \underline{v}_i'} \right]$$

$$\underline{U}_{ii} = \frac{1}{|\underline{v}_1 - \underline{v}_i'|} \left\{ \underline{I} - \frac{(\underline{v}_1 - \underline{v}_i')(\underline{v}_1 - \underline{v}_i')}{|\underline{v}_1 - \underline{v}_i'|^2} \right\}$$

(ii) One would like to verify the classical results - for instance the 2nd law of thermodynamics. To do this we need a definition of entropy, Boltzmann's definition is

$$S_1 = - \int d^3 \underline{x} d^3 \underline{v}_1 f_1 \ln f_1$$

(iii) Since we will simplify our discussion to the homogeneous case

I will suppress the spatial integration.

(iv) Lets examine the time derivative of S_1 - to simplify I will consider only collisions with the same species.

$$\frac{dS_1}{dt} = - \int d^3v_1 \log f_1 C_{11}(f_1, f_1)$$

Integrating by parts.

$$= \frac{2\pi q_1^4}{m_1^2} \iint d^3v_1 d^3v_1' \left\{ \frac{1}{f_1(v_1)} \frac{df_1}{dv_1} \cdot \underline{U} \cdot \left[f_1(v_1') \frac{df_1}{dv_1} - f_1(v_1) \frac{df_1}{dv_1'} \right] \right\} \quad (1)$$

We can symmetrize by switching labels $v_1 \leftrightarrow v_1'$ and adding half of this to half of (1) so that

$$(v) \frac{dS_1}{dt} = \frac{\pi q_1^4}{m_1^2} \iint d^3v_1 d^3v_1' f_1(v_1) f_1(v_1') \left\{ \underline{\omega} \cdot \underline{U} \cdot \underline{\omega} \right\} \quad (2)$$

where:- $\underline{\omega} = \frac{\partial}{\partial v_1} \log f_1(v_1) - \frac{\partial}{\partial v_1'} \log f_1(v_1')$

but $\underline{\omega} \cdot \underline{U} \cdot \underline{\omega} = \frac{1}{u} \left[\omega^2 - \frac{(\underline{\omega} \cdot \underline{u})^2}{u^2} \right] = \frac{\omega^2}{u} (1 - \cos^2 \theta) = \frac{\omega^2 \sin^2 \theta}{u}$

so $\Rightarrow \underline{\omega} \cdot \underline{U} \cdot \underline{\omega} \geq 0$

since $f_1 > 0$ we have

$\frac{dS_1}{dt} \geq 0$

GOOD! ENTROPY INCREASES WITH TIME.

and $\frac{dS_1}{dt} = 0$ only if $\underline{\omega} \cdot \underline{U} \cdot \underline{\omega} = 0$ for all v_1 and v_1'

(vi) Since S_i cannot become infinite (the integral is bounded); it

must increase until it reaches a limit where $\frac{dS_i}{dt} = 0$

this is equilibrium. Now we show that this means f_i is a

maxwellian.

(vii) $\underline{\omega} \cdot \underline{v} \cdot \underline{\omega} = 0$ for all $\underline{v}_i, \underline{v}'_i$ implies that $\underline{\omega}$ is parallel to $\underline{v}_i - \underline{v}'_i = \underline{u}$

$$\begin{aligned} \underline{\omega} &= \frac{\partial \log f_i}{\partial \underline{v}_i} - \frac{\partial \log f_i}{\partial \underline{v}'_i} = (\underline{v}_i - \underline{v}'_i) H(\underline{v}_i, \underline{v}'_i) \\ &= \underline{A}(\underline{v}_i) - \underline{A}(\underline{v}'_i) \end{aligned}$$

it is now easy to see that $\underline{A}(\underline{v}_i) = \underline{v}_i H(\underline{v}_i, \underline{v}'_i) \Rightarrow H = \text{constant}$
 $\underline{A}(\underline{v}'_i) = \underline{v}'_i H(\underline{v}_i, \underline{v}'_i)$

we define temperature so that $H = -\frac{m_i}{T_i}$

$$\Rightarrow \left[\frac{\partial \log f_i}{\partial \underline{v}_i} = -\frac{m_i}{T_i} \underline{v}_i \right] \Rightarrow \left[f_i = \frac{n_i}{\pi^{3/2}} \left(\frac{m_i}{2T_i} \right)^{3/2} \exp \left\{ -\frac{m_i \underline{v}_i^2}{2T_i} \right\} \right]$$

MAXWELLIAN

(viii) The thermal velocity $v_{ti} = \sqrt{\frac{2T_i}{m_i}}$

GIVEN TIME THE DISTRIBUTION RELAXES TO
 A MAXWELLIAN. BOLTZMAN'S H THEOREM.

(ix) Collisional Relaxation to a Maxwellian ...
 from (2) $v_i, u \sim v_{ti}$

$$\nu_i \sim \left(n_i \frac{\pi q_i^4 m \Lambda}{m_i^2 v_{ti}^3} \sim n_i \left(\frac{\pi q_i^4 m \Lambda}{T_i} \right) v_{ti} \right) \times \frac{1}{m_i^{1/2}}$$

↑
AVERAGE CROSS SECTION.

electrons - ν_e ions - ν_i T_i, T_e same order of magnitude.

$$\frac{\nu_e}{\nu_i} \sim \sqrt{\frac{m_i}{m_e}} \sim \frac{40}{60} \text{ for hydrogen.}$$

60 for deuterium

Electrons establish a Maxwellian faster than ions.

(x) Energy exchange in an electron-ion collision.

Since $v_{CM} = \text{constant}$ $\Delta v_i \sim -\frac{m_e}{m_i} \Delta v_e \sim -\frac{m_e}{m_i} \Delta u$

$\Delta E = \text{CHANGE IN ION KE} \approx m_i v_i \cdot \Delta v_i \sim m_e v_i \cdot \Delta u$
IN 1 COLLISION

$$\frac{\Delta E}{E} = \frac{\text{Energy loss by electron per collision}}{\text{KE OF ELECTRON}} = \sqrt{\frac{m_e}{m_i}}$$

• Energy exchange rate slower than time to establish a Maxwellian.

• But colliding with a Maxwellian makes Δu random in direction, so $\langle v_i \cdot \Delta u \rangle \approx v_i^2 \Rightarrow \langle \frac{\Delta E}{E} \rangle = \frac{m_e}{m_i} \approx \frac{1}{2000}$

(xi) Time for electron-electron collisions to make electrons a Maxwellian

$$\tau_e \sim \frac{1}{\nu_e} \quad \text{Temperature } T_e$$

Time for ions to become Maxwellian at $T = T_i$

$$\tau_i \sim \frac{1}{\nu_i} \sim \left(\frac{m_i}{m_e}\right)^{1/2} \tau_e$$

Time for ion-electron collisions to make $T_e \sim T_i$

$$\tau_{ei}^E \sim \left(\frac{m_i}{m_e}\right) \tau_e \gg \tau_i \gg \tau_e.$$

(xii) We can compute the equation for Temperature equilibration

by computing

$$\frac{d}{dt} \int d^3v_e \frac{1}{2} m_e v_e^2 f_e = n_e \frac{dT_e}{dt} = \int d^3v_e \frac{1}{2} m_e v_e^2 C(f_e, f_i)$$

$$\text{with } f_e = n_e \left(\frac{m_e}{2\pi T_e}\right)^{3/2} \exp\left(-\frac{1}{2} \frac{m_e v_e^2}{T_e}\right) \quad f_i = n_i \left(\frac{m_i}{2\pi T_i}\right)^{3/2} \exp\left(-\frac{1}{2} \frac{m_i v_i^2}{T_i}\right)$$

After some integrations/algebra we get :-

$$\frac{dT_e}{dt} = - \frac{(T_e - T_i)}{\tau_{ei}^E}$$

$$\tau_{ei}^E = \frac{T_e^{3/2} m_i}{8 n_i q_i^2 e^2 \ln \Lambda (2\pi m_e)^{1/2}} = \frac{m_i}{m_e} \frac{1}{\nu_e}$$

222c. lecture #6. Scattering from a Maxwellian

(i) Often we have to consider the behaviour of a small group of test particles in a plasma. These particles scatter off the other particles (sometimes called the field particles). If the field particles are in thermal equilibrium we can give them a Maxwellian distribution.

(iii) Maxwellian:-

$$f_2(\underline{v}) = \frac{n_2}{\pi^{3/2}} \frac{1}{v_{t2}^3} \exp\left\{-\frac{v^2}{v_{t2}^2}\right\} \quad v_{t2} = \sqrt{\frac{2T_2}{m_2}}$$

(iv) Now we calculate the Rosenbluth potentials for f_2 - with the scattering (field) particles labelled 2 and the scattered particles labelled 1.

$$\nabla_v^2 h_{12} = -\frac{4\pi m_1}{m_{12}} f_2(\underline{v}) = -\frac{4\pi m_1}{m_{12}} f_{2\text{Max}}(\underline{v})$$

$$\langle \Delta \chi \rangle = \Gamma_{12} \frac{\partial h_{12}}{\partial v} \quad \text{: Drag on 1 by 2.}$$

$$\Gamma_{12} = \frac{4\pi q_1^2 q_2^2 m_1 \Lambda}{m_1^2}$$

(v) Note f_2 is spherically symmetric so h_{12} is also - using the analogy of electrostatics we get h_{12} by Gauss's theorem.

$$\Gamma_{12} \frac{\partial h_{12}}{\partial v} = \langle \Delta V_{11} \rangle \hat{v} = -v_{12} v_{t2} \left(1 + \frac{m_1}{m_2}\right) \left\{ \frac{\phi(x) - x \phi'(x)}{2x^2} \right\} \hat{v}$$

Only particles with $v_2 < v$ cause drag.

where $x = \frac{v}{v_{t2}} \quad v_{12} = \frac{2n_2 \Gamma_{12}}{v_{t2}^3}$

$$\phi(x) = \frac{2}{\pi^{1/2}} \int_0^x e^{-y^2} dy \quad \text{"Error function"}$$

(vi) You may easily verify that $h_{12}(v) = \frac{n_2}{v_{t2}} \left(1 + \frac{m_1}{m_2}\right) \frac{\phi(x)}{x}$

(vii) from $\nabla_r^2 g_{12} = \frac{m_{12}}{m_1} 2 h_{12}(v)$ spherical symmetry.

Again using Gauss's theorem:-

$$\frac{\partial g_{12}}{\partial v} = \frac{n_2}{2} \frac{1}{x^2} \left\{ (2x^2 - 1) \phi(x) + x \phi'(x) \right\} \hat{v} \quad \phi' = \frac{d\phi}{dx}$$

After some algebra

$$\begin{aligned} \text{(viii)} \quad \langle \Delta v_{\parallel} \Delta v_{\parallel} \rangle &= \Gamma_{12} \frac{\partial g_{12}}{\partial v} = \gamma_{12} v_{t2}^2 \left\{ \frac{\phi(x) - x \phi'(x)}{2x^3} \right\} \hat{v} \hat{v} \\ &+ \gamma_{12} v_{t2}^2 \left\{ \frac{\phi(x)}{x} - \frac{(\phi(x) - x \phi'(x))}{2x^3} \right\} \left\{ \hat{I} - \frac{\hat{v} \hat{v}}{v^2} \right\} \end{aligned}$$

Usually we write

$$\langle \Delta v_{\parallel}^2 \rangle = \gamma_{12} v_{t2}^2 \left\{ \frac{\phi(x) - x \phi'(x)}{2x^3} \right\} \quad \text{Parallel diffusion.}$$

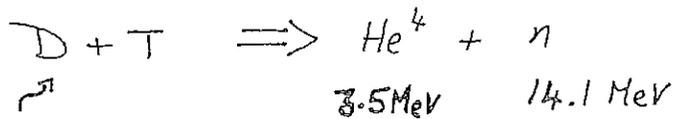
$$\langle \Delta v_{\perp}^2 \rangle = \gamma_{12} v_{t2}^2 \left\{ \frac{\phi(x)}{x} - \frac{\phi - x \phi'}{2x^3} \right\} \quad \text{Perpendicular diffusion.}$$

(ix) You will find these formula in the plasma formulary.

and the various limits $v \gg v_{t2}$, $v \ll v_{t2}$ etc.

Slowing down of Fusion α Particles in a Plasma.

(i) In the usual fusion reaction we get 3.5 MeV α particles



$\sim 20 \text{ keV}$ typically.

- 3.5 MeV α particle has velocity, $V_{\alpha 0} \approx 10^9 \text{ cm s}^{-1}$ "Birth velocity of α particles."
- at 20 keV electrons have the velocity, $V_{te} \approx 6 \times 10^9 \text{ cm s}^{-1}$
- at 20 keV ions, $V_{ti} \approx 10^8 \text{ cm s}^{-1}$

So

$$V_{ti} < V_{\alpha 0} < V_{te}$$

(xi) So for electrons $\frac{v_{\alpha} x}{v_{te}} \ll 1$ and for ions $\frac{v_{\alpha} x}{v_{ti}} \gg 1$

Asymptotically $\phi(x) \approx \left(x - \frac{x^3}{3} \dots \right) \frac{2}{\pi^{1/2}} \quad x \ll 1$

$$\phi(x) = 1 \quad x \gg 1$$

Electron Drag and Diffusion

$$\langle \Delta v_e \rangle = -v_s v_{\alpha}$$

$$v_s = \frac{8\pi n_e k e^4}{v_{te}^3 m_{\alpha} m_e} \frac{1}{3\pi^{1/2}} \ln \Lambda$$

SLOWING DOWN

$$\langle \Delta v_{ii}^2 \rangle \sim \langle \Delta x \rangle \frac{m_e}{m_{\alpha}} v_{the} \quad \text{ignorable}$$

$$\langle \Delta v_e^2 \rangle \sim \langle \Delta v \rangle \frac{m_e}{m_e} v_{ti}^2 \quad \text{ignorable}$$

ELECTRONS JUST CAUSE
DRAG.

ions: $\langle \Delta v \rangle_i \approx \langle \Delta v \rangle_e \frac{v_{the}^3}{v_{\alpha}^3} \frac{m_e}{m_i}$ negligible

$\langle \Delta v_{\alpha}^2 \rangle$ negligible

$\langle \Delta v_{\perp}^2 \rangle = \gamma_{\alpha i} \frac{v_{thi}^3}{v_{\alpha}}$ Pitch angle scattering.

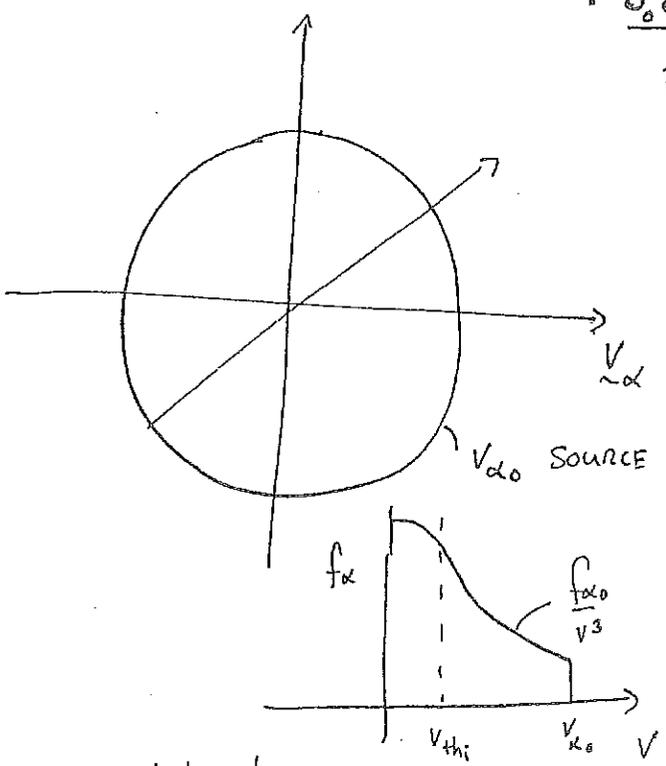
Define $v_{ck}^3 = \frac{\gamma_{\alpha i}}{\gamma_s} v_{thi}^3 \sim \mathcal{O}\left(v_{the}^3 \frac{m_e}{m_{\alpha}}\right), v_{thi} < v_{ck} < v_{the}$

$$\frac{\partial f_{\alpha}}{\partial t_c} = \gamma_s \left\{ \frac{\partial}{\partial v} \cdot \underline{v} f_{\alpha} + \frac{v_{ck}^3}{v^3} \frac{\partial}{\partial \mu} (1 - \mu^2) \frac{\partial f_{\alpha}}{\partial \mu} \right\}$$

Drag on electrons

Pitch angle scattering on ions
($\frac{\partial}{\partial \phi} = 0$ for simplicity)

$+ \frac{S_0 \delta(v - v_{\alpha 0})}{4\pi v_{\alpha 0}^2}$
Fusion source.



STEADY STATE $\frac{\partial}{\partial \mu} = 0$

$v > v_{\alpha 0} \quad f_{\alpha} = 0$

$v < v_{\alpha 0} \quad \frac{\partial}{\partial v} \cdot (v f_{\alpha}) = 0$

$$f_{\alpha} = \frac{f_{\alpha 0}}{v^3}$$

$\frac{\partial}{\partial v} v^3 f_{\alpha}$

Valid for $v > v_{thi}$

Slow down until Alphas became part of slow ion distribution.

Integrating

$$\frac{\gamma_s}{v^2} \frac{\partial}{\partial v} (v^3 f_{\alpha}) = \frac{S_0 \delta(v - v_{\alpha 0})}{4\pi v_{\alpha 0}^2} \Rightarrow$$

$$f_{\alpha 0} = \frac{S_0}{4\pi \gamma_s}$$

Physics 222c. Lecture #7: Runaway Electron Production

(i) A simple way to estimate the cross section is by balancing KE & PE

$$\frac{1}{2} m v^2 = \frac{e^2}{r_s} \quad (\text{e-e collisions})$$

$$\sigma \sim r_s^2 \sim \frac{e^4}{m^2 v^4} \ln \Lambda \quad \left[\begin{array}{l} \text{add this for small angle} \\ \text{scattering which dominates} \end{array} \right]$$

$$l_{m.f.p.} \equiv \text{MEAN FREE PATH} \sim \frac{1}{n_e \sigma} \sim \frac{m^2 v^4}{n_e e^4} \frac{1}{\ln \Lambda}$$

(ii) Now imagine an electron in an electric field E .

$$\begin{aligned} \text{Energy gain in one mean free path} &= e E l_{m.f.p.} \\ &= \frac{E m^2 v^4}{n_e e^4} \frac{1}{\ln \Lambda} \end{aligned}$$

• If this energy gain is greater than the electron's original energy ^{between} then it will keep gaining energy every collision.

(iv) In each collision it will lose some energy but typically less than half its energy. Thus the electron can and will runaway - i.e. keep gaining energy if

Energy gained between collisions $>$ $2 \times$ kinetic energy

$$\frac{E m^2 v^4}{n_e e^4 \ln \Lambda} > m v^2$$

(v) For an electron with $v > v_c$ we get RUNAWAY. where

$$v_c = \left(\frac{n_e e^3 \ln \Lambda}{E m} \right)^{1/2}$$

(vi) If $E < E_D$ then $v_c < v_{the}$ with: -

$$E_D = \left(\frac{n_e e^3 \ln \Lambda}{T_e} \right)$$

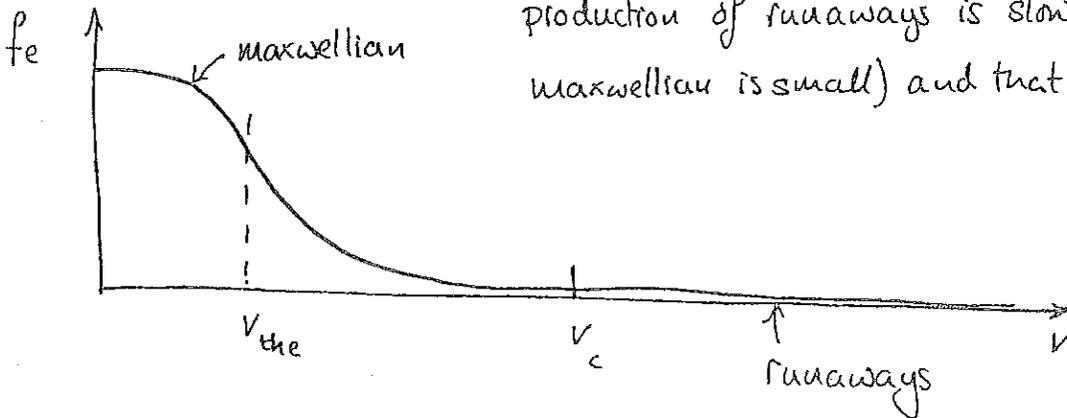
"Dreicer field"
(Dreicer 1958)

(vii) When $E > E_D$ then essentially all the electrons run away.

However this is usually a very large field _(keV/m) and we are more interested in the production of runaways at fields $E \ll E_D$.

(ix) Let us concentrate therefore in electrons with $v > v_{the}$ since we are mostly interested in $v \sim v_c$. We also assume that the

production of runaways is slow (so $\frac{\partial f}{\partial t}$ of the Maxwellian is small) and that it is steady.



• runaways are fed by the Maxwellian.

(X) Since the number of runaways is presumed small we ignore runaway-runaway collisions and consider only runaway Maxwellian collisions. Thus the electron kinetic equation is:-

Assume steady state

$$-\frac{\partial f_e}{\partial t} - \frac{eE}{m} \frac{\partial f_e}{\partial v} = - \frac{\partial}{\partial v} \cdot \langle \Delta v_{||} \rangle_{ee} \hat{v} f_e + \frac{\partial}{\partial v} \cdot \left\{ \langle \Delta v_{||}^2 \rangle_{ee} \hat{v} \hat{v} f_e \right\} + \frac{\partial}{\partial v} \cdot \left\{ \left(\frac{I - \hat{v} \hat{v}}{2} \right) \langle \Delta v_{\perp}^2 \rangle_{ee} f_e \right\} + \nu_{ei}(v) \frac{\partial}{\partial v} \cdot \left(\frac{I - \hat{v} \hat{v}}{2} \right) \frac{\partial f_e}{\partial v}$$

e-i collisions

with the coefficients $\langle \Delta v_{||} \rangle_{ee}$, $\langle \Delta v_{||}^2 \rangle_{ee}$ and $\langle \Delta v_{\perp}^2 \rangle_{ee}$ given last time.

- This is a hard equation to solve as such but we make approximations

① Average over angles - removes all pitch angle terms
(can be done by integrating over a sphere of radius v)

② Since acceleration is $\hat{u} \cdot \underline{E}$ direction

$$\underline{E} \cdot \underline{v} \sim -Ev \quad \Rightarrow \quad \int \underline{E} \cdot \hat{v} f_e v^2 d\Omega \sim 4\pi v^3 E \overline{f_e}$$

↑
solid angle

↑
 f_e integrated over spherical surface.

(Poor approximation but gives answer that is within a factor of 2)

③ Use $v \gg v_{the}$ to simplify $\langle \Delta v_{||} \rangle$ & $\langle \Delta v_{||}^2 \rangle$

$$\langle \Delta v_{||} \rangle_{ee} \hat{v} = -\gamma_{ee} v_{the}^3 \frac{2v}{v^3} \quad \langle \Delta v_{||}^2 \rangle_{ee} = \gamma_{ee} v_{the}^2 \frac{1}{2} \frac{v_{the}^3}{v^3}$$

$$\gamma_{ee} = \frac{1}{v_{the}^3} n_e e^4 \frac{8\pi}{m_e^2} \ln \Lambda$$

After some algebra we get drag $\langle \Delta v_i \rangle$ diffusion $\langle \Delta v_i^2 \rangle$

$$\frac{1}{4\pi v^2} \frac{\partial_{\ln v^2}}{\partial v} \left\{ \frac{eE \bar{f}_e}{m_e} - \gamma_{ee} \frac{v_{the}^3}{v^3} \left[v \bar{f}_e + \frac{T_e}{m_e} \frac{\partial \bar{f}_e}{\partial v} \right] \right\} = 0 = \nabla_v \cdot (S_v \hat{v})$$

RUNAWAY FLUX (INDEPENDANT OF V)

Divergence in
V space

$$S_v = \text{RUNAWAY FLUX} = 4\pi v^2 \left\{ \frac{eE \bar{f}_e}{m_e} - \gamma_{ee} \frac{v_{the}^3}{v^3} \left[v \bar{f}_e + \frac{T_e}{m_e} \frac{\partial \bar{f}_e}{\partial v} \right] \right\}$$

NORMALIZE

Define $V_0 = \left(\frac{m_e \gamma_{ee} v_{the}^3}{eE} \right)^{1/2} \approx V_c$

$$\mathcal{E} = \frac{T_e}{m_e} \frac{1}{V_0^2} \sim \frac{E}{E_D} \ll 1$$

$$c = \frac{S_v}{4\pi e E \frac{v_0^2}{m_e}} \quad \text{CONSTANT} = \frac{S_v}{4\pi \gamma_{ee} v_{the}^3}$$

$$u = \frac{v}{V_0}$$

\Rightarrow

$$\frac{\mathcal{E}}{u} \frac{d\bar{f}_e}{du} + (1 - u^2) \bar{f}_e = -c$$

$$\Rightarrow \frac{d}{du} \left\{ e^{\frac{u^2 - u^4/2}{2\mathcal{E}}} \bar{f}_e \right\} = -c e^{\frac{u^2 - u^4/2}{2\mathcal{E}}}$$

Thus with the boundary condition \bar{f}_e goes to zero as $u \rightarrow \infty$

$$\bar{f}_e(u) = C \exp\left(-\frac{u^2 - u^{3/2}}{2\varepsilon}\right) \int_u^\infty u' \exp\left(\frac{u'^2 - u'^{3/2}}{2\varepsilon}\right) du'$$

Now we apply the boundary condition $\bar{f}_e(u) \rightarrow f_{\max}$ as $u \rightarrow 0$

note $\frac{u^2}{2\varepsilon} = \frac{1}{2} \frac{m_e v^2}{T_e}$ $f_{\max} = \frac{n_e}{\pi^{3/2}} \frac{1}{v_{the}^3} \exp\left[-\frac{v^2}{v_{the}^2}\right]$

$$\Rightarrow \frac{n_e}{(2\pi T_e/m_e)^{3/2}} \exp\left(-\frac{u^2}{2\varepsilon}\right) = C \exp\left(\frac{u^{3/2} - u^2}{2\varepsilon}\right) \int_0^\infty u \exp\left(\frac{u^2 - u^{3/2}}{2\varepsilon}\right) du$$

$$\Rightarrow S_V = \gamma_{ee} n_e \left(\frac{4}{\pi^{1/2}}\right) \frac{1}{\int_0^\infty u \exp\left(\frac{u^2 - u^{3/2}}{2\varepsilon}\right) du}$$

Evaluating the integral. $\int_0^\infty u \exp\left[\frac{u^2 - u^{3/2}}{2\varepsilon}\right] du$

Integrand has a peak @ $u \approx 1$ of width $\Delta u \sim \varepsilon^{1/2}$

writing $u = 1 + \varepsilon^{1/2} s$ we get $\int_0^\infty u \exp\left(\frac{u^2 - u^{3/2}}{2\varepsilon}\right) \sim e^{\frac{1}{4\varepsilon}} \pi^{1/2} \varepsilon^{1/2}$

$$S_V = \gamma_e n_e \left(\frac{4}{\pi}\right) \cancel{\frac{1}{\pi^{1/2}}} e^{-\frac{1}{4\varepsilon}}$$

exponentially small number of runaway particles produced. Basically because there are only an exponentially small number of particles with $v \sim v_{the}$ in Maxwellian

222c. Lecture #8

Obtaining Closed Fluid Equations: Thermal Diffusion

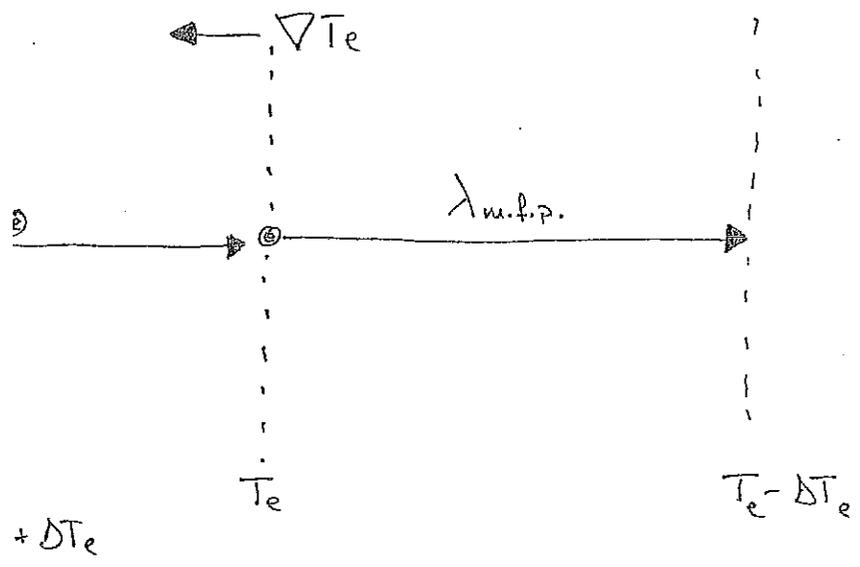
(i) Today we look at obtaining the transport terms in the fluid equations [heat flow, viscosity etc.] in the limit that the mean free path is small. The procedure is called the Chapman-Enskog expansion. It was invented around 1915 to rigorously derive the Navier Stokes equations from Boltzmann's equation.

(ii) Let's consider thermal transport by electrons - the full treatment would consider all the transport processes on electrons and ions.

(iii) First some estimates

mean free path electrons

$$\equiv \lambda_{m.f.p.e} = \frac{V_{the}}{\nu_e}$$



- Electron goes one mean free path and deposits its energy.

- $\Delta T_e \sim \lambda_{m.f.p.} \nabla T_e$

- heat flow $\approx 2n_e V_{the} \Delta T_e$

$$q_e = 2n_e \underbrace{\frac{V_{the}^2}{\nu_e}}_{K_e} \nabla T_e$$

K_e thermal conductivity.

(iv) CONSERVATION OF ENERGY \Rightarrow

$$\frac{3}{2} n_e \frac{\partial T_e}{\partial t} \approx -\nabla \cdot q_e$$

Rate of Change of Energy density

divergence of heat flux

(v) Temperature scale length $= l_T \sim \frac{T_e}{|\nabla T_e|}$

(vi) So the timescale τ_{Te} for T_e to change is:

$$\frac{1}{\tau_{Te}} \approx \left(\frac{V_{the}^2}{\gamma_e} \right) \frac{1}{L_T^2} = \gamma_e \frac{\lambda_{m.f.p.}^2}{L_T^2}$$

note: The ion thermal conduction timescale is $\frac{1}{\tau_{Ti}} \approx \frac{V_{thi}^2}{\gamma_i} \frac{1}{L_T^2}$

$$\tau_{Te} \sim \left(\frac{m_i}{m_e} \right)^{1/2} \tau_{Te}$$

THERMAL CONDUCTION BY ELECTRONS IS MUCH LARGER THAN ION THERMAL CONDUCTION.

(vii) In plasmas like the solar corona the mean free path is

short compared to the temperature scale length L_T . In this limit we can write fluid equations. Consider the Fokker-Planck equation for the electrons:

$$\frac{\partial f_e}{\partial t} + \underline{v} \cdot \nabla f_e - \frac{eE}{m_e} \cdot \frac{\partial f_e}{\partial \underline{v}} = C_{ei}^L(f_e) + C_{ee}(f_e, f_e)$$

\swarrow Lorentz operator
 \uparrow full Landau operator.

(viii) We will consider the plasma evolving on the transport timescale with $\lambda_{m.f.p.} \ll L_T$ - the electric field is introduced because it ensures a stationary density - NO FLOW but still allows heat flow. You will see below how it is determined.

(ix) Timescales :-

$$\left\{ \begin{aligned} \frac{df_e}{dt} &\sim \gamma_e \left(\frac{\lambda_{m.f.p}}{L_T} \right)^2 f_e \sim \frac{f_e}{\tau_{Te}} \\ \underline{v} \cdot \nabla f_e &\sim \frac{v_{the}}{L_T} f_e \sim \gamma_e \frac{\lambda_{m.f.p}}{L_T} f_e \sim \frac{eE}{m_e} \cdot \frac{\partial f_e}{\partial v} \\ C_{ei}, C_{ee} &\sim \gamma_e f_e \end{aligned} \right.$$

↑ We choose size of E to make it this order.

(X) Introduce a dummy small parameter ϵ to label the smallness of a term. ϵ is equivalent to the $\frac{\lambda_{m.f.p}}{L_T}$ ordering.

$$\epsilon^2 \frac{df_e}{dt} + \epsilon \left\{ \underline{v} \cdot \nabla f_e - \frac{eE}{m_e} \cdot \frac{\partial f_e}{\partial v} \right\} = C_{ei}(f_e) + C_{ee}(f_e, f_e) \quad (1)$$

- : At the end of the calculation we may simply set $\epsilon = 1$ since it is just a bookkeeping tool.

(xi) We also order f_e :-

$$f_e = f_e^{(0)} + \epsilon f_e^{(1)} + \epsilon^2 f_e^{(2)} \dots \quad (2)$$

Now we simply substitute $f_e^{(2)}$ in $f_e^{(1)}$ and equate powers of ϵ

(xii) $\mathcal{O}(\epsilon^0)$

$$0 = C_{ei}(f_e^{(0)}) + C_{ee}(f_e^{(0)}, f_e^{(0)}) \quad (3)$$

From our discussion before of Boltzmann's H theorem we know that the only solution to this equation is

$$f_e^{(0)} = f_{e \text{ MAXWELLIAN}} = \frac{n_e}{\pi^{3/2}} \left(\frac{m_e}{2 T_e(x,t)} \right) \exp \left\{ -\frac{1}{2} \frac{m_e v^2}{T_e(x,t)} \right\}$$

You will notice that since (3) is an equation in ψ alone (not r & t) we get a Maxwellian at every position and time but not necessarily the same Maxwellian - i.e. n_e and T_e can depend on r and t .

(xiii) For simplicity we will make n_e constant and adjust \underline{E} so that

there is no flow and $\frac{\partial u_e}{\partial t} = 0$. Now going to next order.

$\mathcal{O}(\epsilon')$

$$\underline{v} \cdot \nabla f_e^{(0)} - \frac{e \underline{E}}{m_e} \cdot \frac{df_e^{(0)}}{d\underline{v}} = C_{ei}^L(f_e^{(1)}) + C_{ee}(f_e^{(1)}, f_e^{(0)}) + C_{ee}(f_e^{(0)}, f_e^{(1)}) \quad (4)$$

We must solve (4) for $f_e^{(1)}$ - this is hard if we include C_{ee} and must be done numerically. So for simplicity let's keep only C_{ei}^L for this part. So with $\underline{E} = -\nabla\phi$ and C_{ei}^L from before we have after some algebra

$$\underline{v} \cdot \left\{ \frac{\nabla T_e}{T_e} \left[\frac{1}{2} \frac{m_e v^2}{T_e} - \frac{3}{2} \right] - \frac{e \nabla \phi}{T_e} \right\} f_{e \max} = \frac{\gamma_{ei}}{2} \frac{\partial}{\partial \underline{v}} \left(\frac{T_e}{v} \frac{\partial f_e^{(1)}}{\partial \underline{v}} \right)$$

where $\gamma_{ei} = \frac{4\pi n_i q_i^2 e^2 \ln \Lambda}{m_e^2 v^3} \equiv \gamma_{ei} \left(\frac{v_{the}^3}{v^3} \right)$

Solution can be obtained in spherical polars expanding in Legendre polynomials

$$(xiv) f_e^{(1)} = \frac{v^3}{v_{the}^3} \frac{v_0}{\gamma_{ei}} \left\{ \frac{\nabla T_e}{T_e} \left[\frac{1}{2} \frac{m_e v^2}{T_e} - \frac{3}{2} \right] - \frac{e \nabla \phi}{T_e} \right\} f_{e \max} + \delta f_{e \max}^{(1)}$$

where f_{max} is a perturbation of the Maxwellian, which we drop since it just redefines the Maxwellian.

(XV) $\mathcal{O}(\epsilon^2)$

$$\frac{\partial f_{\text{max}}^{(0)}}{\partial t} + \underline{v} \cdot \nabla f_e^{(1)} - \frac{eE}{m} \frac{\partial f_e^{(1)}}{\partial \underline{v}} = C_{ei}(f_e^{(2)}) + C_{ee}(f_e^{(2)}, f_e^{(0)}) + C_{ee}(f_e^{(0)}, f_e^{(2)}) + C_{ee}(f_e^{(1)}, f_e^{(1)})$$

- But we do not want to solve for $f_e^{(2)}$ we simply want to find the time evolution of $f_{\text{max}}^{(0)}$ i.e. $T_e(\underline{r}, t)$. ⑤

ANNIHILATION OF $f_e^{(2)}$

- a) We may use the fact that collisions conserve the number of particles thus

$$\int d^3 \underline{v} C_{ei}^{\downarrow}(f_e) = \int d^3 \underline{v} C_{ee}(f_e, f_e) = 0$$

- b) We may also use that e-e collisions conserve energy (in the electrons) and so does the Lorentz collision term. So

$$\int \frac{1}{2} m v^2 C_{ei}^{\downarrow}(f_e) d^3 \underline{v} = \int \frac{1}{2} m v^2 C_{ee}(f_e, f_e) d^3 \underline{v} = 0$$

(xvi) Now we annihilate $f_e^{(2)}$ in two ways:

a) First integrate $\int d^3v$ (5) yields

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \underline{V}) = 0$$

diffusive flow:-

$$n_e \underline{V} = \int \underline{v} f_e^{(1)} d^3v = \frac{2 V_{the}^2}{3 \gamma_e} \left\{ -\frac{\nabla T_e}{T_e} \frac{5}{2} I_7 + \frac{e \nabla \phi}{T_e} I_7 \right\}$$

with

$$I_n = \frac{2}{\pi^{1/2}} \int_0^\infty dx x^n e^{-x^2}$$

NO DENSITY EVOLUTION \Rightarrow
 $n_e \underline{V} = 0$

$$\frac{e \nabla \phi}{T_e} = \frac{5}{2} \frac{\nabla T_e}{T_e}$$

b) Second multiply by $\frac{1}{2} m v^2$ and integrate $\int \frac{1}{2} m v^2$ (5) d^3v

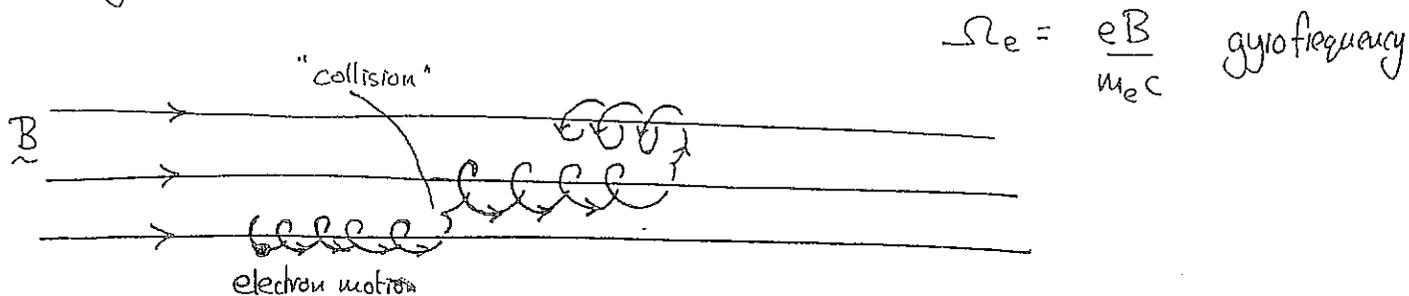
$$\Rightarrow \frac{3}{2} n_e \frac{\partial T_e}{\partial t} + \nabla \cdot \underline{q}_e + \frac{1}{2} e (\underline{E} \cdot \underline{V}) n_e = 0$$

Zero "ohmic heating"

$$\underline{q}_e = \int d^3v \frac{1}{2} m v^2 \underline{v} f_e^{(1)} = n_e \cdot \left(\frac{V_{the}^2}{\gamma_e} \right) \frac{48}{\pi^{1/2}} \nabla T_e$$

222c. Lecture #9. Transport in a magnetic field.

(i) let us consider stationary ions and electrons scattering from them in a magnetic field.



Larmor radius = $\rho_e = \frac{V_{the}}{\Omega_e}$ - RANDOM WALK STEP.

electron collision rate = γ_{ei} = decorrelation rate.

DIFFUSION RATE PERPENDICULAR TO FIELD = $\gamma_{ei} \rho_e^2 = D_{\perp}$

(ii) Compare with diffusion without field or along B .

$D_{\parallel} = \gamma_{ei} \lambda_{m.f.p.}^2$

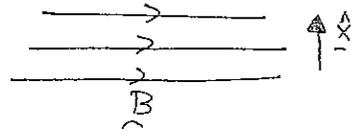
$\Rightarrow \frac{D_{\perp}}{D_{\parallel}} = \left(\frac{\rho_e}{\lambda_{m.f.p.}} \right)^2 = \left(\frac{\gamma_{ei}}{\Omega_e} \right)^2$

SOLAR PARAMETERS: $T_e = 10^2 - 10^3 \text{ eV}$ $n_e = 10^{10} - 10^{12} \text{ cm}^{-3}$
 (CORONA) $B \sim 100$

$\gamma_{ei} \sim 5 \times 10^3 \text{ s}^{-1}$ $\frac{D_{\perp}}{D_{\parallel}} \sim 5 \times 10^{-12} !$
 $\Omega_e \sim 2 \times 10^9 \text{ s}^{-1}$

- In many plasmas magnetic fields suppress transport considerably

(iii) Solving for transport in a simple \underline{B} field:



We will take $\underline{B} = B_0(x) \hat{z}$

$\nabla_n, \nabla_T, \nabla \cdot \nabla$ in \hat{x} direction:
 $\nabla_n \sim \frac{1}{L} \hat{x}$

We take $\Omega_e \gg \frac{v_{the}, v_{ei}}{L}$
 $\frac{\rho_e}{L} \sim \frac{\rho_i}{v_{ei}} \sim \delta$

(we do not order v_{ei} versus $\frac{v_{the}}{L}$)

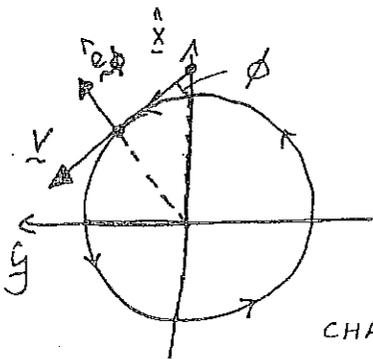
$$\frac{\partial f_e}{\partial t} + \underline{v} \cdot \nabla f_e - \frac{e}{m} \left\{ \frac{\underline{v} \times \underline{B}}{c} \right\} \cdot \frac{\partial f_e}{\partial \underline{v}} = \frac{v_{ei} v_{th}^2}{2} \frac{\partial}{\partial \underline{v}} \cdot \left(\frac{\underline{1}}{=} - \underline{\hat{v}} \underline{\hat{v}} \right) \cdot \frac{\partial f_e}{\partial \underline{v}} + C_e(f_e, f_e)$$

TIMESCALE: $\frac{v_{ei} \rho_e^2}{L^2} \sim \Omega_e \delta^3$ $\frac{v_{the} \sim \Omega_e \rho_e}{L} \sim \Omega_e \delta$ $\frac{e}{m} \frac{v_{ei} v_{th}^2}{2} \sim \Omega_e$ (LARGEST Ω_e) $\frac{v_{ei} v_{th}^2}{2} \sim \Omega_e \delta$

(iv) To solve for f_e we need to look at the term

$-\frac{e}{mc} \underline{v} \times \underline{B} \cdot \frac{\partial f_e}{\partial \underline{v}}$ more closely. This term, of course, is the largest and tends to make f_e independent of the angle of \underline{v} with respect to

the \hat{x} direction the gyro-angle ϕ .



$$\underline{v} \cdot \hat{x} = v_{\perp} \cos \phi$$

$$\underline{v} \cdot \hat{y} = v_{\perp} \sin \phi$$

$$\underline{v} \cdot \hat{z} = v_{\parallel} \quad \text{parallel velocity.}$$

CHANGE COORDS TO CYLINDRICAL POLARS FOR VELOCITY.

$$\frac{\partial}{\partial \underline{v}} \equiv 2v_{\perp} \frac{\partial}{\partial v_{\perp}^2} + \frac{\hat{e}_{\phi}}{v_{\perp}} \frac{\partial}{\partial \phi} + \hat{z} \frac{\partial}{\partial v_{\parallel}}$$

$$f_e = f_e(x, v_{\parallel}, v_{\perp}, \dots)$$

$$-\frac{e}{m} \frac{\underline{v} \times \underline{B}}{c} \cdot \frac{\partial f}{\partial \underline{v}} = \Omega_e \frac{\partial f}{\partial \phi}$$

DOING THE DUMMY PARAMETER THING:

δ as a label:

$$\delta^3 \frac{\partial f_e}{\partial t} + \delta \left\{ \underline{v} \cdot \nabla f_e - C_{ei}(f_e) - C_{ee}(f_e, f_e) \right\} = - \Omega_e \frac{\partial f_e}{\partial \phi}$$

and expanding $f_e = f_e^{(0)} + \delta f_e^{(1)} + \delta^2 f_e^{(2)} \dots \dots$ etc.

$\partial(\Omega_e)$: $\Omega_e \frac{\partial f_e^{(0)}}{\partial \phi} = 0$

$f_e^{(0)} = f_e^{(0)}(x, v_{||}, v_{\perp})$

note: $\int v_{\perp} f_e^{(0)} d^3 v_{\perp} = 0$ no perpendicular flow

$\partial(\Omega_e \delta)$: $v_{\perp} \cos \phi \frac{\partial f_e^{(0)}}{\partial x} - C_{ei}(f_e^{(0)}) - C_{ee}(f_e^{(0)}, f_e^{(0)}) = - \Omega_e \frac{\partial f_e^{(0)}}{\partial \phi}$

Average over ϕ : i.e. $\int_0^{2\pi} d\phi (\dots)$ and we that $f_e^{(1)}$ must be periodic so: $\int_0^{2\pi} d\phi \frac{\partial f_e^{(0)}}{\partial \phi} = f_e^{(0)}(2\pi) - f_e^{(0)}(0) = 0$ and of course $\int_0^{2\pi} d\phi \cos \phi = 0$

But: $\frac{1}{2\pi} \int_0^{2\pi} d\phi C_{ei}(f_e^{(0)}) = C_{ei}(f_e^{(0)})$ and $\frac{1}{2\pi} \int_0^{2\pi} d\phi C_{ee}(f_e^{(0)}, f_e^{(0)}) = C_{ee}(f_e^{(0)}, f_e^{(0)})$

THEREFORE AVERAGING TELLS US THAT

$$C_{ei}(f_e^{(0)}) + C_{ee}(f_e^{(0)}, f_e^{(0)}) = 0 \Rightarrow f_e^{(0)} = f_{max} = \frac{n_e}{\pi^{3/2} v_{the}^3} e^{-\frac{v^2}{v_{the}^2}}$$

BUT: $n_e = n_e(x) \quad T_e = T_e(x)$

so you are left with:

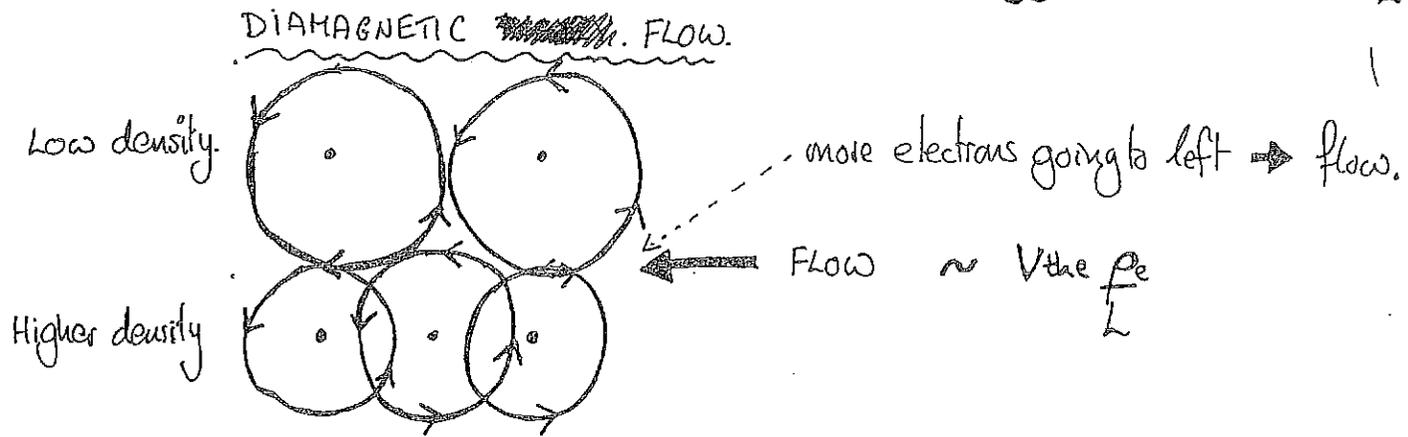
$$V_{\perp} \cos \phi \frac{\partial f_e^{(0)}}{\partial x} = -\Omega_e \frac{\partial f_e^{(1)}}{\partial \phi}$$

↙ Maxwellian

$$\Rightarrow f_e^{(1)} = \frac{-V_{\perp} \sin \phi}{\Omega_e} \left\{ \frac{1}{n_e} \frac{\partial n_e}{\partial x} + \frac{1}{T_e} \frac{\partial T_e}{\partial x} \left\{ \frac{1}{2} \frac{m_e v^2}{T} - \frac{3}{2} \right\} \right\} f_{\max}$$

This is now giving a flow velocity called the DIAMAGNETIC VELOCITY

$$\begin{aligned} \tilde{V}_{\text{DIA}} &= \frac{1}{n_e} \int d^3 \tilde{v} \tilde{v}_{\perp} f_e^{(1)} = \hat{y} \frac{T_e}{eB} \left\{ \frac{1}{n_e} \frac{\partial n_e}{\partial x} + \frac{1}{T_e} \frac{\partial T_e}{\partial x} \right\} \\ &= \frac{e \tilde{B} \times \nabla p_e}{n_e B e} \sim V_{\text{the}} \frac{\rho_e}{L} \end{aligned}$$



can be understood from electron fluid equation - ignore inertia

$$0 = -\nabla p_e - n_e \frac{e}{m_e} \frac{\tilde{V}_{\text{DIA}} \times \tilde{B}}{c}$$

Not a flow in direction of $\nabla n, \nabla T$ but perpendicular to it. So no transport.

$$\mathcal{O}(\Omega_e \delta^2)$$

$$\begin{aligned} \underbrace{\underline{v} \cdot \nabla}_{\sin\phi \cos\phi} f_e^{(1)} &= \underbrace{C_{ei}}_{\sin\phi} (f_e^{(1)}) - \underbrace{C_{ee}}_{\sin\phi} (f_e^{(1)}, f_e^{(0)}) - \underbrace{C_{ee}}_{\sin\phi} (f_e^{(0)}, f_e^{(1)}) \\ &= -\Omega_e \frac{\partial f_e^{(2)}}{\partial \phi} \end{aligned}$$

FLOW ALONG $\nabla T, \nabla n$.

$$\underline{V}_x = \frac{1}{ne} \int d^3v \underline{v}_\perp \cos\phi f_e^{(2)} = \frac{1}{ne} \int d^3v \underline{v}_\perp \sin\phi \frac{\partial f_e^{(2)}}{\partial \phi}$$

integrate by parts in ϕ

$\underline{v} \cdot \nabla f_e^{(0)}$ does not give anything since $\int d\phi \sin^2\phi \cos\phi = 0$

$$\underline{V}_x = - \left[\underbrace{D_n}_{\gamma_{ei} \rho_e^2} \nabla n + \underbrace{D_T}_{\frac{3}{2} \gamma_{eiT} \rho_e^2} \nabla T \right] \quad \text{Diffusive flow.}$$

Put into:

$$\frac{\partial n}{\partial t} = -\nabla \cdot n \underline{V}$$

\Rightarrow

$$\frac{\partial n}{\partial t} = \frac{\partial}{\partial x} (n V_x)$$

Lecture #10 Shielding Charges - Towards a better theory.

(i) A charge in a plasma has a shielding cloud, we would like to investigate this cloud to see how to get a better form for collisions and correlations in the plasma.

(ii) We introduce a ^{moving} test charge into the plasma - with charge

$$\text{density: } \rho_0 = q_0 \delta(\underline{r} - \underline{v}_0 t)$$

and we consider for simplicity electrostatic fields.

(iii) To compute the potential around a charge we must also take into account the shielding charges.

$$\textcircled{1} \quad \nabla^2 \phi = 4\pi \left\{ \overset{\text{sum over species}}{\sum_{\alpha} q_{\alpha}} \int \hat{f}_{\alpha} d^3 \underline{v} + q_0 \delta(\underline{r} - \underline{v}_0 t) \right\}$$

and we compute \hat{f}_{α} from the LINEARIZED VLASOV EQN. (without collisions).

$$\textcircled{2} \quad \frac{\partial \hat{f}_{\alpha}}{\partial t} + \underline{v} \cdot \nabla \hat{f}_{\alpha} = - \frac{q_{\alpha}}{m_{\alpha}} \nabla \phi \cdot \frac{\partial \hat{f}_{\alpha 0}}{\partial \underline{v}}$$

assume "equilibrium"
 $f_{\alpha 0}$ is a Maxwellian,

This is like ^{the Landau} ~~the~~ problem ^(which) we have seen before - except that there is a source, the delta function.

- Laplace

(iv) As usual we are going to solve by Fourier transforming.
 first note that

$$(3) \int d^3r \int_{0^+}^{\infty} dt e^{i\omega t - i\mathbf{k}\cdot\mathbf{r}} \delta(\mathbf{r} - \mathbf{v}_0 t) = \int_{0^+}^{\infty} dt e^{i(\omega - \mathbf{k}\cdot\mathbf{v}_0)t} = \frac{i}{i(\omega - \mathbf{k}\cdot\mathbf{v}_0 + i0)}$$

Laplace transform with $p = -i\omega$

ensures convergence and causality

$$(4) \int_{\mathbf{k}\omega}^{\sim} = \frac{q_1 \phi_{\mathbf{k}\omega}}{m_1 (\mathbf{k}\cdot\mathbf{v} - \omega + i0)} + \frac{g_{kz}(\mathbf{v})}{i(\mathbf{k}\cdot\mathbf{v} - \omega + i0)}$$

initial condition

↑ for real ω makes sure pole is on causal side

(vi) The initial condition term is going to die away due to "phase mixing" - i.e. different particles with different speeds will wash out the "wave" patterns on a time scale $\sim \partial[\mathbf{k}\cdot\mathbf{v}]^{-1}$

We discussed this when talking about Landau damping.
 So we can set $g_{kz} = 0$ henceforth.

(vii) Substituting (3) and (4) into (1) (after Fourier transforming etc.)

$$\mathbf{k}^2 \epsilon(\mathbf{k}, \omega) = \phi_{\mathbf{k}\omega} = \frac{4\pi q_0}{i(\omega - \mathbf{k}\cdot\mathbf{v}_0 + i0)}$$

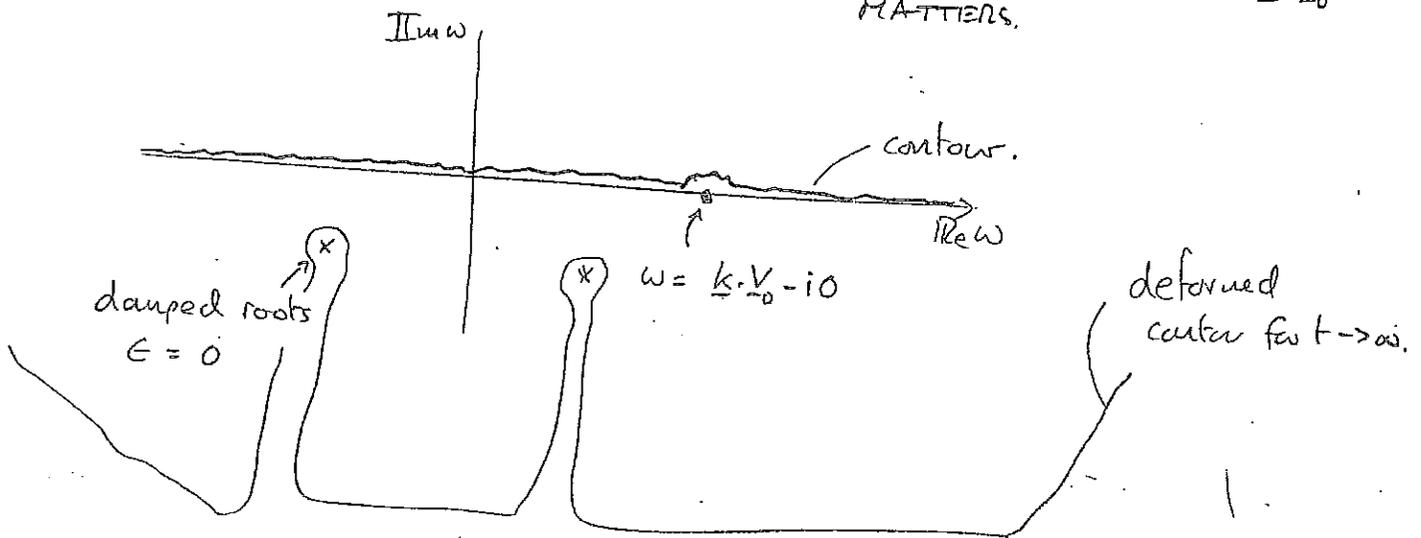
$$\epsilon(\mathbf{k}, \omega) = 1 + \sum_{\alpha} \frac{4\pi q_{\alpha}^2}{m_{\alpha} k^2} \int \frac{d^3v}{(k\cdot v - \omega + i0)} \frac{k \cdot \frac{\partial f_{\alpha 0}}{\partial v}}$$

We assume that the plasma has only damped modes so that $\epsilon(\mathbf{k}, \omega) = 0 \Rightarrow \omega = \omega_0 - i\gamma$.

(viii) Then

$$\phi(\underline{x}, t) = \frac{4\pi q_0}{(2\pi)^3} \int d^3\underline{k} e^{i\underline{k}\cdot\underline{r}} \int_{-\infty}^{\infty} \frac{d\omega e^{-i\omega t}}{i(\omega - \underline{k}\cdot\underline{v}_0 + i0) \underline{k}^2 \epsilon(\underline{k}, \omega)}$$

We can evaluate the ω integration in the limit t is large. Poles are at $\omega = \underline{k}\cdot\underline{v}_0$ and $\epsilon(\underline{k}, \omega) = 0$ but they are damped and give zero as $t \rightarrow \infty$. SO ONLY POLE AT $\omega = \underline{k}\cdot\underline{v}_0$ MATTERS.



$$(ix) \Rightarrow \phi(\underline{x}, t) = \frac{4\pi q_0}{(2\pi)^3} \int \frac{d^3\underline{k} e^{i\underline{k}\cdot(\underline{r} - \underline{v}_0 t)}}{\underline{k}^2 \epsilon(\underline{k}, \underline{k}\cdot\underline{v}_0)}$$

If we consider only electron oscillations we get the familiar forms

$$\epsilon(\underline{k}, \omega) = 1 - \frac{\omega_p^2}{\omega^2} \left\{ 1 + \frac{3}{2} \frac{k^2 v_{the}^2}{\omega^2} \right\} + \text{DAMPING } \omega \gg k v_{the}$$

$$\epsilon(\underline{k}, \omega) = 1 + \frac{1}{k^2 \lambda_D^2} \dots \quad \omega \ll k v_{the} \quad \lambda_D^2 = \frac{v_{the}^2}{\omega_0^2} \quad \text{DEBYE LENGTH.}$$

(X) STATIONARY CHARGE. $V_0 = 0$

$$k_D^2 = \frac{1}{\lambda_D^2}$$

$$\phi(\underline{r}) = \frac{q_0}{2\pi^2} \int \frac{d^3k e^{i\underline{k}\cdot\underline{r}}}{k^2 + k_D^2}$$

spherical poles about \underline{r}

$$\underline{k}\cdot\underline{r} = r k \cos\theta$$

$$\phi(\underline{r}) = \frac{q_0}{\pi} \int_0^\pi d\theta \sin\theta \int_0^\infty \frac{dk k^2 e^{i k r \cos\theta}}{k^2 + k_D^2}$$

$$= \frac{q_0}{i\pi r} \int_0^\infty \frac{dk k (e^{i k r} - e^{-i k r})}{k^2 + k_D^2}$$

$$= \frac{1}{i\pi r} \int_{-\infty}^\infty \frac{dk k e^{i k r}}{k^2 + k_D^2}$$

$$\boxed{\phi(r) = \frac{q_0 e^{-k_D r}}{r}}$$

debye shielded
charge.

- next time we investigate $V_0 \neq 0$ and two collisions of shielded charges.

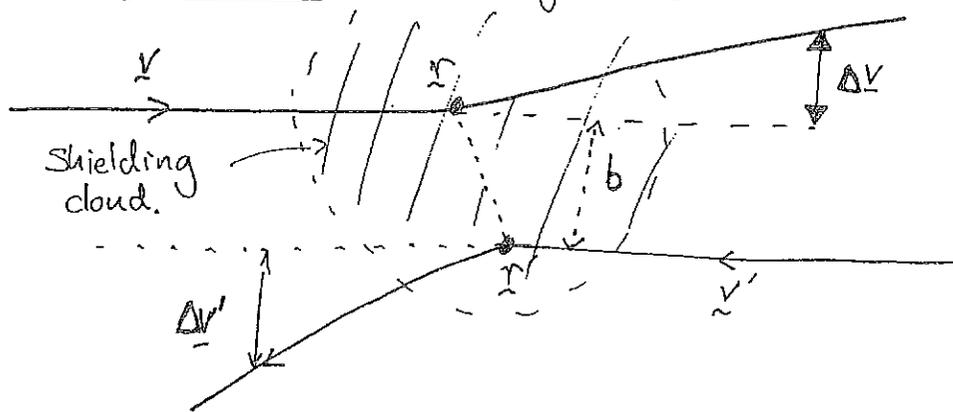
Lecture # 11: Shielded Collisions - Enhanced Scattering.

(i) Last time we calculated the shielding of a moving charge - this gave us for charges moving in a straight line a potential

$$\phi(\underline{r}, t) = \phi(\underline{r} - \underline{v}_0 t) = \frac{4\pi q_0}{(2\pi)^3} \int \frac{d^3 k}{k^2} \frac{e^{i\mathbf{k} \cdot (\underline{r} - \underline{v}_0 t)}}{\epsilon(\underline{k}, \omega)}$$

where $\omega = \underline{k} \cdot \underline{v}_0$ and ϵ is the linear dispersion function and we assume the plasma is stable. Then $\epsilon(\underline{k}, \underline{k} \cdot \underline{v}_0)$ is never zero for real \underline{k} .

(ii) Weak Collisions. Almost straight lines.



$$\frac{d\Delta V'}{dt} = -\frac{q'}{m'} \nabla \phi(\underline{r}', t) \quad \Rightarrow \quad \Delta V' = -\frac{q'}{m'} \int_{-\infty}^{\infty} \nabla \phi(\underline{r}', t) dt \text{ along orbit}$$

• If we treat the orbits as almost unperturbed - small angle scattering - we can calculate $\Delta V, \Delta V'$ perturbatively. Since the collisions are dominated by this limit we may expect to get the correct answer for all but a small subset of collisions - the large angle scattering.

$$\phi(\underline{r}', t) \approx \phi(\underline{r}' - \underline{v} t) \text{ as given above.}$$

So scattering of primed particle by SHIELDED unprimed particle

$$\Delta \underline{v}' = -4\pi \frac{q_1 q_2}{m'} i \int \frac{d^3 k}{(2\pi)^3} \left\{ \underline{k} \frac{e^{i \underline{k} \cdot (\underline{r}'(t) - \underline{v}' t)}}{\epsilon(\underline{k}, \underline{k} \cdot \underline{v})} \right\} dt$$

Now we take $\underline{r}'(t) = \underline{b} + \underline{v}' t$

where \underline{b} = impact parameter vector note: $\underline{b} \cdot (\underline{v} - \underline{v}') = 0$

$$\Delta \underline{v}' = -4\pi \frac{q_1 q_2}{m'} i \int \frac{d^3 k}{(2\pi)^3} \underline{k} \frac{e^{i \underline{k} \cdot \underline{b} - i \underline{k} \cdot (\underline{v} - \underline{v}') t}}{\epsilon(\underline{k}, \underline{k} \cdot \underline{v})} dt$$

t integral gives $2\pi \delta(\underline{k} \cdot (\underline{v} - \underline{v}'))$ so only \underline{k} perpendicular to $\underline{v} - \underline{v}'$

let $\underline{v} - \underline{v}' = \hat{\underline{v}} |\underline{v} - \underline{v}'|$

\underline{k}_\perp is perpendicular to $\underline{v} - \underline{v}'$

$$\underline{k}_\perp \cdot \hat{\underline{v}} = 0$$

$$\Delta \underline{v}' = -\frac{4\pi q_1 q_2 i}{|\underline{v} - \underline{v}'|} \int \frac{d^2 \underline{k}_\perp}{(2\pi)^2} \frac{\underline{k}_\perp e^{i \underline{k}_\perp \cdot \underline{b}}}{\epsilon(\underline{k}_\perp, \underline{k}_\perp \cdot \underline{v})} \quad \text{em velocity}$$

After lots of algebra one can calculate the Fokker-Planck terms and obtain the Balescu-Lenard collision operator.

$$\frac{df'_\alpha}{dt} = -\nabla_{\underline{v}'} \cdot \underline{S}_\alpha \Rightarrow \underline{S}_\alpha = \sum_{\text{Species } \beta} \int \left(\frac{f'_\beta(\underline{v})}{m_\alpha} \frac{\partial f'_\alpha(\underline{v}')}{\partial \underline{v}'} - f'_\alpha(\underline{v}') \frac{\partial f'_\beta}{\partial \underline{v}} \right) \cdot \underline{B} d^3 \underline{v}$$

$$\underline{B} = \frac{2q_1^2 q_2^2}{m_\alpha^2 |\underline{v} - \underline{v}'|} \int \frac{1}{k^4} \frac{\underline{k}_\perp \underline{k}_\perp d^2 k}{|\epsilon(\underline{k}_\perp \cdot \underline{v}, \underline{k}_\perp)|^2}$$

with $\epsilon = 1$ we get the LANDAU collision term.

Lecture # 12:

Acceleration - Wakefield devices

(i) To accelerate electrons to very high energies one must either make a high voltage, which is impossible, or make the electric field "keep up" with the particles. One would therefore like to make a "traveling wave" structure where the electrons keep pace with a moving electric field.

 $E \approx E_z \sin k(z - vt)$

(ii) In the last lecture and the one before we discussed the wake behind a particle - it has a phase velocity of the particle velocity. One particle cannot sustain much electric field but if we make a bunch of particles we could make a traveling wake with a large field. Then we could accelerate electrons in this WAKE FIELD.

(iii) let us take:- $\rho_{\text{BEAM}} = \frac{q_0 e}{\sqrt{\pi} a} e^{-\frac{(z - v_0 t)^2}{a^2}}$ $q_0 = \text{total charge per unit x-y area.}$

a "sheet" of charge moving in the z direction. This is produced by some conventional accelerator. The potential produced by this charge moving through a plasma is:-

$$\phi(z - v_0 t) = k\pi q_0 \int_{-\infty}^{\infty} \frac{e^{-k_z^2 a^2 / 4 + ik_z(z - v_0 t)}}{k_z \epsilon(k_z, k_z v_0)} dk_z$$

where ignoring ions in the plasma we have: $\text{since } v_0 \gg v_{th} \quad \omega \gg k \cdot v$

$$\epsilon = 1 - \frac{\omega_p^2}{k^2} \int \frac{d^3 v \cdot k \cdot \frac{df_e}{dv}}{(k \cdot v - k v_0 - i0)} \approx 1 - \frac{\omega_p^2}{(k_z v_0)^2} + \frac{i2\gamma_0}{\omega_p}$$

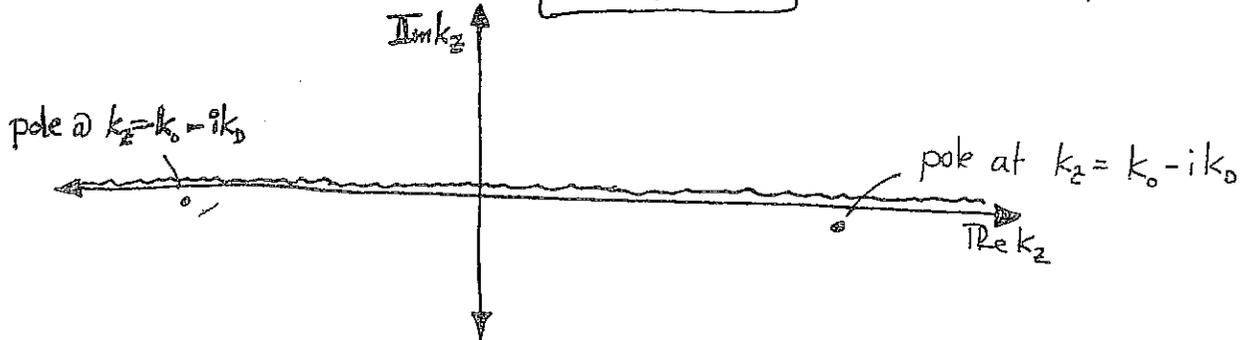
$\gamma_0 \approx \text{Landau Damping.}$

$$k_0 = \frac{\omega_p}{v_0}$$

$$\text{Thus } k_z^2 \in (k_z, k_z v_0) \approx k_z^2 - k_0^2 - i 2k_0 k_z$$

$$\phi(z - v_0 t) = k \pi q_0 \int \frac{dk_z e^{-k_z^2 a^2 / 4 + i k_z (z - v_0 t)}}{[k_z^2 - k_0^2 - i 2k_0 k_z]}$$

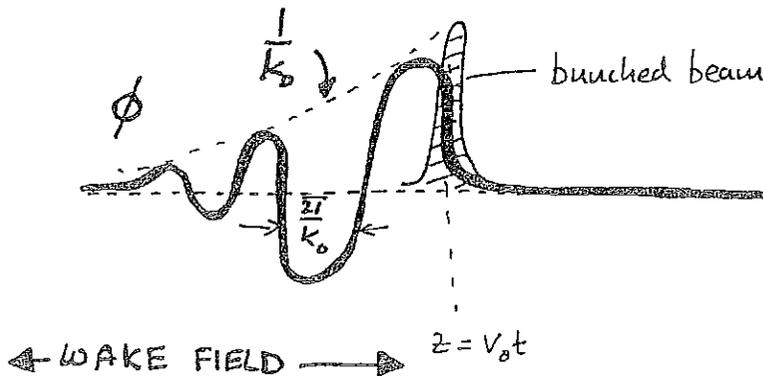
now let us look at the limit $k_0 a \ll 1$ almost a delta function blip.



- for $(z - v_0 t) \gg 0$ we push the contour up in the complex plane and we get zero because there are no poles in the upper half plane.
- for $(z - v_0 t) \ll 0$ we push the contour down in the complex plane. Then we obtain the contributions from the two poles:-

$$\phi(z - v_0 t) = -\frac{8\pi^2}{k_0} q_0 e^{+k_0 (z - v_0 t) - k_0^2 a^2 / 4} \sin k_0 (z - v_0 t)$$

$z - v_0 t \ll 0$



$$E_z \sim k_0 \phi$$

- Note we want $k_0 a \lesssim 1$ to get a reasonable size wake.

we also want q_0 to be large to make E_z large - this makes it useful to make "a" large (longer beam) so typically we choose $k_0 a \sim 1$.

(in) Now we want $v_0 \sim c$ and we want to accelerate electrons in the wave field to very high energies $\gamma = (1 - v^2/c^2)^{-1/2} \gg 1$. So we need to look at the relativistic motion in the wave field.

$$\frac{dp_z}{dt} = qE_z = \frac{d}{dt} m_0 \gamma v_z$$

$$\text{and } \frac{dE}{dt} = \frac{d}{dt} (m_0 \gamma c^2) = q v_z E_z$$

$$\text{we take } E_z = E_z(z - v_0 t) = - \frac{\partial \phi}{\partial z} = \frac{1}{v_0} \frac{\partial \phi}{\partial t}$$

$$\begin{aligned} \frac{d}{dt} [E - v_0 p_z] &= + q (v_z - v_0) E_z(z - v_0 t) \\ &= - q \frac{d\phi}{dt} \end{aligned} \quad \text{since } v_z = \frac{dz}{dt}$$

$$\Rightarrow \boxed{E - v_0 p_z + q \phi(z - v_0 t) = \text{constant for a particle.}}$$

CONSERVATION OF ENERGY IN MOVING FRAME

$$\beta = v/c \quad \beta_0 = v_0/c$$

$$\frac{1}{\sqrt{1 - \beta^2}} (1 - \beta_0 \beta) + \frac{q \phi(z - v_0 t)}{m_0 c^2} = 1 + \frac{q \phi(z_0)}{m_0 c^2}$$

clearly to get acceleration, we want $\frac{q \phi}{m_0 c^2} > 1$ to $v \sim c$

- Inject the particles/electrons to accelerate with $\beta \sim 1$ so that they stay in phase ~~phase~~ with the wake for a long time.

- let $\beta = 1 - \Delta\beta$ $\beta_0 = 1 - \Delta\beta_0$

The best we can do is get $\Delta\beta \ll \Delta\beta_0$ then accelerated electrons overtake the wake.

this gives: $\gamma - 1 \approx \frac{q \Delta\phi}{m_0 c^2} \frac{1}{1 - \beta_0}$

$$\Delta\phi = \phi(z - v_0 t) - \phi(z_0)$$

So what you want is $\Delta\beta_0 \ll 1$ and large ϕ .

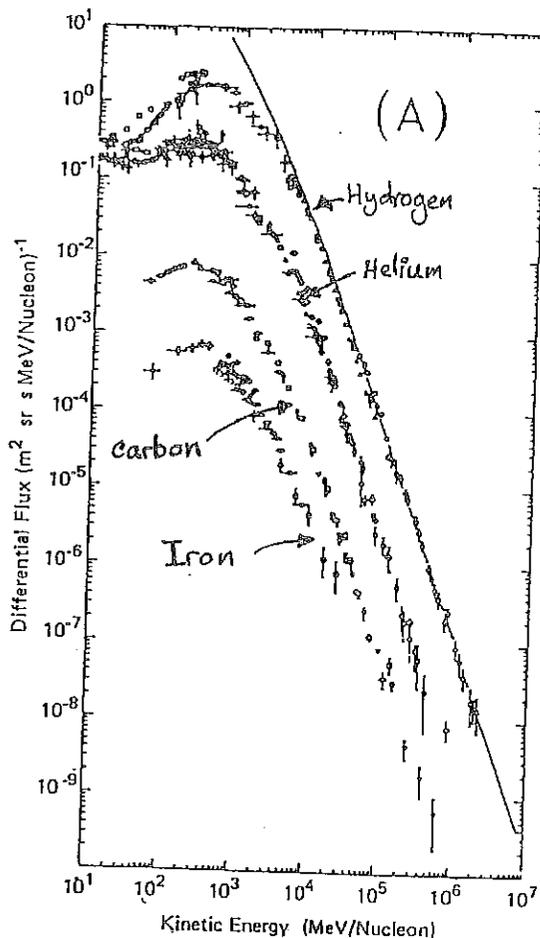
222c. Lecture #13: Acceleration of Cosmic Rays.

Comprehensive treatment in:- Blanford and Eichler, Phys. Reports, 154 #1 (1987)

(i) Cosmic Rays arriving at the earth have a intensity per unit Energy, per unit solid angle, per nucleon.

$$I(E) = E^{-2.7}$$

$$10^3 \text{ MeV} < E < 10^5 \text{ MeV.}$$



- Composed of many elements. Most common are shown.

- Abundances of elements are not the same as in Solar system.

- H, He. less than Solar system compared to Iron Fe.

- Note $m_p c^2 = 938 \text{ MeV}$

(Larmor radius)

$$r_p = 10^{12} \left(\frac{E}{m_p c^2} \right) \cdot \left(\frac{3 \mu\text{G}}{B} \right) \text{ cm.}$$

(typical galactic field.)

⇒ 10^9 MeV protons have Larmor radii of order a parsec.

(ii) We "know" that the cosmic rays are confined to the galaxy quite well. The main evidence is that secondary species _{Li, Be, B} are produced by hard collisions of C, N, O with the interstellar hydrogen - using the cross sections and the observed secondary species intensities one can deduce that a typical cosmic ray has gone $4 \times 10^5 \text{ pc}$ - on the other hand the height of the galaxy is 100 pc !

(iii) The energy density in cosmic rays is large - about the same as the thermal energy density in the galaxy. The only way to sustain this is via supernovae. In fact it is estimated that 3% of the supernova power goes to cosmic rays.

(iv) The acceleration of cosmic rays is thought to happen in shocks around supernovae. The relativistic particles are scattered by Alfvén waves backwards and forwards across the shock - they pick up energy until they are lost.

(v) We may use a relativistic version of the Fokker-Planck equation to investigate the acceleration.

$$\frac{df}{dt} + \underline{v} \cdot \frac{df}{d\underline{x}} + \dot{p} \frac{df}{dp} = \left\{ - \frac{\partial}{\partial p} \left\langle \frac{\Delta p}{\Delta t} \right\rangle f + \frac{1}{2} \frac{\partial^2}{\partial p^2} \left[\left\langle \frac{\Delta p \Delta p}{\Delta t} \right\rangle f \right] \right\}$$

where the Fokker-Planck coefficients come from the scattering off Alfvén waves.

$$\left\langle \frac{\Delta p}{\Delta t} \right\rangle = \frac{1}{\Delta t} \int d^3 \Delta p P(p, \Delta p) \Delta p$$

$$\left\langle \frac{\Delta p \Delta p}{\Delta t} \right\rangle = \frac{1}{\Delta t} \int d^3 \Delta p P(p, \Delta p) \Delta p \Delta p$$

$P(p, \Delta p)$ probability of scattering from $p \rightarrow p + \Delta p$ in time Δt .

Detailed balance when the scatter doesn't recoil gives $P(p, -\Delta p) = P(p - \Delta p, \Delta p)$

Using detailed balance
 (vi) For recoilless scatter $\langle \frac{\Delta p}{\Delta t} \rangle = \frac{1}{2} \frac{\partial}{\partial p} \cdot \left\langle \frac{\Delta p \Delta p}{\Delta t} \right\rangle$

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f = \frac{1}{2} \frac{\partial}{\partial p} \cdot \left\langle \frac{\Delta p \Delta p}{\Delta t} \right\rangle \cdot \frac{\partial f}{\partial p}$$

(vii) Fermi's Argument:-

CALCULATE CHANGE IN p DUE TO SCATTERING FROM A HEAVY NONRELATIVISTIC SCATTERER.

$$P_i \equiv \text{COVARIANT momentum 4 vector of particle} = (p, \frac{E}{c}) \leftarrow \text{COSMIC RAY}$$

$$P^j \equiv \text{CONTRAVARIANT momentum 4 vector of scatterer} = (M_0 \underline{V}, -M_0 c^2 - \frac{1}{2} M_0 V^2)$$

Magnetic Scatterer.

$P_i P^j$ is a Lorentz invariant

Let us evaluate it in two frames - the "lab" frame (unprimed) and the scatterer's frame (primed)

$$M_0 \underline{p} \cdot \underline{V} = \frac{E}{c} (M_0 c^2 + \frac{1}{2} M_0 V^2) = -\frac{E'}{c} M_0 c^2$$

$$E' = \sqrt{p'^2 c^2 + m^2 c^4}$$

$$E = \sqrt{p^2 c^2 + m^2 c^4}$$

taking $p' - p \ll p$

$p' - p = \delta p$ and $V \ll c$ we get

$$\boxed{-\frac{\underline{p} \cdot \underline{V}}{V} = \delta p}$$

$V \sim c$ for cosmic rays

After collision p' is unchanged but direction of \underline{p}' is changed. For head on collisions $p' \rightarrow -p'$. Transforming back to the lab we get a net change of p of order $2\delta p$.

(viii) Assume the scatterers are isotropic in V . And assume that

the scattering rate ν is independent of \vec{p} ($\nu = n_s \sigma c$)

then averaging over V we get.

$$\left\langle \frac{\Delta \vec{p} \Delta \vec{p}}{\Delta t} \right\rangle \approx \frac{\overline{\delta p^2} \nu}{c^2} \underline{\underline{I}} \sim \frac{\nu}{3} p^2 \frac{\langle v^2 \rangle}{c^2} \underline{\underline{I}}$$

$$\left(\langle (\vec{p} \cdot \underline{V})^2 \rangle = \frac{1}{3} p \langle v^2 \rangle \right)$$

(iv) Steady State we add a Loss term $\frac{f}{\tau_{esc}}$ to the equation

$$\frac{df}{dt} = \frac{df}{dt} = 0$$

$$\frac{1}{p^2} \frac{d}{dp} p^4 \frac{df}{\tau_{acc} dp} = \frac{f}{\tau_{esc}}$$

$$\tau_{acc}^{-1} = \frac{\nu}{3} \frac{\langle v^2 \rangle}{c^2}$$

$$\Rightarrow f = \text{const } p^{-\alpha}$$

$$\alpha = \frac{3}{2} + \frac{3}{2} \left(1 + \frac{2}{3} \frac{\tau_{acc}}{\tau_{esc}} \right)^{1/2}$$

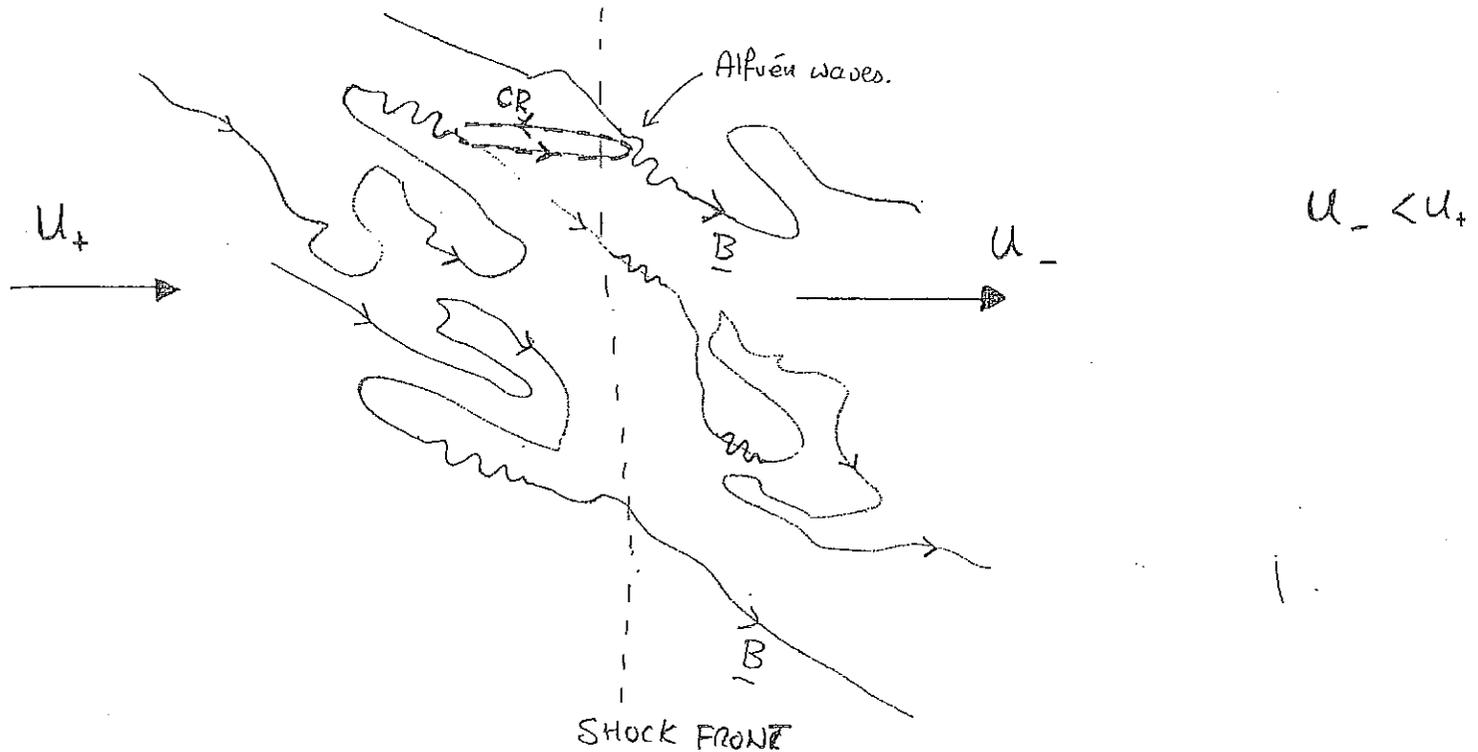
$$\frac{dN}{dE} \propto p^2 f \propto p^{-\lambda}$$

$$\lambda = \alpha - 2$$

↑
of cosmic rays per unit E .

222 c. Lecture #14: Acceleration of Cosmic Rays. II.

(i) Acceleration of cosmic rays cannot be "Fermi Acceleration" by reflection off matter since matter collisions would destroy all nuclei Fe, C etc and they are seen as part of CR abundances. Fermi argued that it must be magnetic disturbances. We now believe it is a combination of shock waves and Alfvén waves to provide the scattering.



- (ii) a) COSMIC RAYS SCATTER BACKWARDS AND FORWARDS ACROSS THE SHOCK FRONT, picking up momentum on each crossing.
- b) Alfvén waves scatter cosmic rays on each side. We take $V_A \ll u_+ - u_-$ for simplicity.
- c) Alfvén waves are, ^{then roughly} stationary in the frame moving with plasma.
- d) Eventually the CR escapes from shock region or the shock dies away.

(iii) Today we examine the reflection of cosmic rays by Alfvén waves. We will assume that these encounters cause small scatterings that add up to a diffusive behavior.

(iv) We will assume that the waves are shear Alfvén waves with

In the plasma frame: $\omega = \pm k \cdot V_A$ $\underline{k} \parallel \underline{B}_0$

$$\underline{B} = B_0 \hat{z} + \delta \underline{B}$$

$$\delta \underline{B} = \int dk \delta B(k) \left\{ \begin{array}{l} \sin(k(z - V_A t) + \delta_k) \hat{x} \\ + \cos(k(z - V_A t) + \delta_k) \hat{y} \end{array} \right\}$$

only circular polarized for simplicity.

(v) We will move to a frame moving with velocity $V_A \ll u_+ - u_- < c$.

In this frame $\delta \underline{E} = 0$ [since $\frac{\partial \delta \underline{B}}{\partial t} = 0$] and $z - V_A t \rightarrow z$.

(vi) CONSIDER CR MOTION IN THIS FRAME.

$$\frac{d\underline{p}}{dt} = q \left(\frac{\underline{v} \times \underline{B}_0}{c} \right) + q \left(\frac{\underline{v} \times \delta \underline{B}}{c} \right) \quad \text{--- (1)}$$

$$\frac{d\underline{E}}{dt} = 0 \quad \gamma = \text{const.} \\ \Rightarrow v = \text{const.}$$

(vii) Define $\mu = \frac{\underline{p} \cdot \hat{z}}{p} = \frac{\underline{v} \cdot \hat{z}}{v}$ "PITCH ANGLE"

Now let's solve (1) perturbatively, treating $\delta \underline{B} \ll \underline{B}_0$. $\underline{v} = \underline{v}_0 + \underline{v}_1 + \dots$

$$\frac{d\underline{v}_0}{dt} = \frac{q B_0}{m_0 \gamma c} (\underline{v}_0 \times \hat{z}) = \Omega (\underline{v}_0 \times \hat{z})$$

$$\Omega = \frac{q B_0}{m_0 \gamma c} = \text{relativistic cyclotron frequency.}$$

This has, of course, the usual solution:-

$$\underline{z}_0 = v\mu \hat{z} + v(1-\mu^2)^{1/2} \left[\sin(\omega t + \theta_0) \hat{x} + \cos(\omega t + \theta_0) \hat{y} \right]$$

and integrating again we obtain:

$$z(t) = v\mu t + z_0 \quad \text{etc.}$$

(viii) Now we won't solve completely to next order we will just compute the changes in μ . Dotting (i) with \hat{z} we obtain

$$\frac{d\mu}{dt} = \frac{q}{m_0 \gamma c v} (\underline{v}_\perp \times \delta \underline{B}) \cdot \hat{z}$$

$$= \frac{q(1-\mu^2)^{1/2}}{m_0 c \gamma} \int \delta B(k) \sin[(\omega - kv\mu)t + \psi_k] dk$$

$$\psi = \theta_0 - \delta_k - kz_0$$

(ix) We are going to compute the Fokker-Planck coefficient $\langle \underline{\Delta\mu} \underline{\Delta\mu} \rangle$ small changes

due to the "random" Alfvén waves acting over a time T .

keep $\mu \sim \text{const}$ to this approximation.

$$\Delta\mu \Delta\mu = \frac{q^2 (1-\mu^2)}{m_0^2 c^2 \gamma^2} \int dk \int dk' \delta B(k) \delta B(k') \int_0^T dt' \int_0^T dt'' \sin[(\omega - kv\mu)t' + \psi_k] \cdot \sin[(\omega - k'v\mu)t'' + \psi_{k'}]$$

(x) Now we make the RANDOM PHASE ASSUMPTION that the ψ_k is distributed randomly through the interval $0 - 2\pi$.

(x) We then average over the phases to obtain $\delta(k-k')$ and after integration over t' and t''

$$\left\langle \frac{\Delta \mu \Delta \mu}{T} \right\rangle = \frac{q^2 (1-\mu^2)}{m_0^2 \gamma^2 c^2} \int dk \delta B^2(k) \left\{ \frac{1 - \cos \Delta E}{\Delta^2 T} \right\}$$

$$\Delta = \Omega - kv\mu$$

(xi) In the limit $T \gg \Omega^{-1}$ we have $\frac{1 - \cos \Delta T}{\Delta^2 T}$ as a

very peaked (delta function like) function about $k = \frac{\Omega}{v\mu}$

then:

$$\left\langle \frac{\Delta \mu \Delta \mu}{T} \right\rangle \sim \frac{q^2 (1-\mu^2)}{m_0^2 \gamma^2 c^2} \frac{\delta B^2(k(\mu))}{v\mu} \pi$$

where $k(\mu) = \frac{\Omega}{v\mu}$

writing $\frac{\delta B^2(k)}{4\pi} = \mathcal{E}(k)$ energy density in k space.

$$\left\langle \frac{\Delta \mu \Delta \mu}{T} \right\rangle = \nu(\mu) (1-\mu^2)$$

$$\nu(\mu) = \frac{\pi}{2} \frac{k \mathcal{E}(k)}{(B_0^2/8\pi)} \Omega \quad k = \frac{\Omega}{v\mu}$$

putting this into the Fokker-Planck equation we get

$$\frac{\partial f}{\partial t} = \frac{1}{2} \frac{\partial}{\partial \mu} (1-\mu^2) \nu(\mu) \frac{\partial f}{\partial \mu}$$

In the frame of the waves (roughly the lab frame).

222c. Lecture #16: Acceleration of Cosmic Rays III.

(i) last time we analyzed the scattering of cosmic rays (CRs) by Alfvén waves. We worked in a stat frame that is stationary in the plasma. We obtained a scattering Fokker-Planck term.

$$\left(\frac{\partial f_{CR}}{\partial t}\right)_{WAVES} = \frac{1}{2} \frac{\partial}{\partial \mu'} \left\{ (1 - \mu'^2) \nu(\mu') \frac{\partial f_{CR}}{\partial \mu'} \right\}$$

$$\mu' = \frac{p'_z}{p'}$$

← "Pitch Angle Scattering"

$$\nu(\mu') = \frac{\pi}{2} \frac{k \delta B^2(k)}{B_0^2} \Omega$$

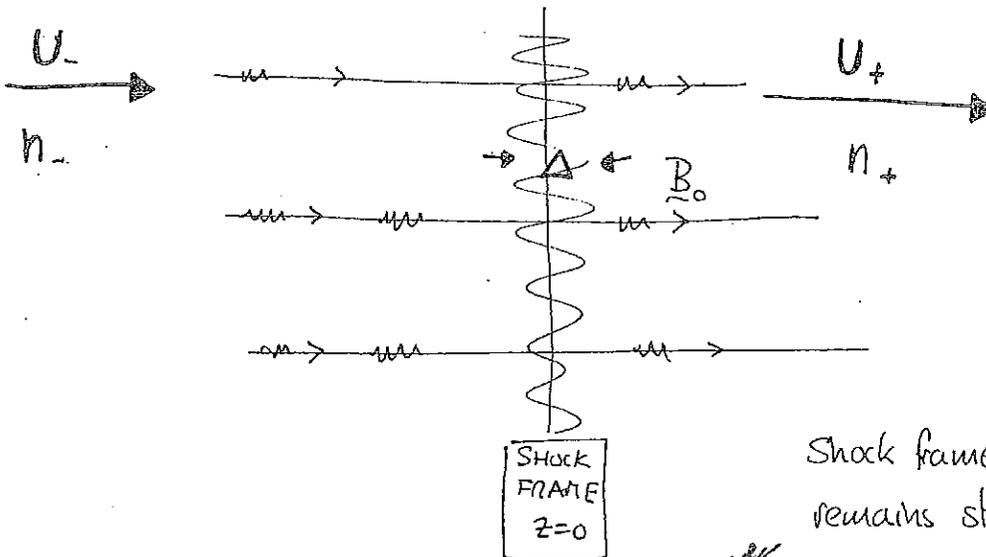
$$k = \frac{\Omega}{v_{\mu'}}$$

only resonant waves give scattering.

PRIMES DENOTE PLASMA FRAME. (NOT THE CR FRAME)

NOTE: We take V_A to be tiny so that plasma & wave frame are the same.

(ii) As a simplifying measure we will take a shock with $\underline{u} = u(z) \hat{z}$ as the background plasma flow. We also take $\underline{B}_0 = B_0 \hat{z}$



$u_+ - u_- \gg V_A$
 & Non relativistic
 $u_+, u_- \ll c$
 $\Delta =$ shock width.

Shock frame is where it remains steady.

(iii) **THE RANKINE HUGONIOU RELATIONS**

CONSERVATION OF MOMENTUM: $p_+ + \rho_+ u_+^2 = p_- + \rho_- u_-^2$

CONSERVATION OF ENERGY: $h_+ + \frac{1}{2} \rho_+ u_+^2 = h_- + \frac{1}{2} \rho_- u_-^2$

$h \equiv$ enthalpy $= \frac{\gamma p}{(\gamma - 1) \rho}$

CONSERVATION OF PARTICLES: $\rho_+ u_+ = \rho_- u_-$

$$\frac{1}{\gamma} = \frac{u_+}{u_-} = \frac{\gamma - 1}{\gamma + 1} + \frac{2}{(\gamma + 1) M^2}$$

$$M = \left(\rho_- u_-^2 / \gamma p_- \right)^{1/2}$$

Product of ...

(iv) Now consider the evolution (well steady state) of the cosmic ray distribution. We have to do a lot of work to do this so first let me tell you what these key steps are.

- Fokker-Planck (FP)
- [A]: Transform the equation for f_{CR} to be in terms of \mathcal{L} in the shock frame and \mathcal{P}' in the plasma frame. $f_{CR} = f_{CR}(z, \mu', p', \phi)$
 - [B]: Average the FP equation over gyro angle. $f_{CR} = \bar{f}_{CR}(z, \mu', p')$
 - [C]: Reduce the FP equation for small mean free path for pitch angle scattering. $f_{CR} = f_{CR}(z, p')$ isotropic.
 - [D]: Solve this final equation for f_{CR} find the power law!
(we do [D] next time)

(v) [A] We transform from shock frame to the non-inertial plasma frame

$$(\mathcal{P}', E'/c), (\mathcal{P}, E/c) \text{ contravariant 4 vectors.}$$

Transform to local frame (velocity $u(z) \hat{z}$) Lorentz transform

$$\frac{p'_z + \frac{E'}{c} u(z)}{\sqrt{1 - u^2/c^2}} = p_z \quad p'_y = p_y \quad p'_x = p_x$$

to "Galilean order" $\mathcal{O}(u/c)$

$$\boxed{p'_z = p_z - \frac{E'}{c} u}$$

Transforming kinetic Equation gives: TO $\mathcal{O}(u/c)$

$$\frac{df_{CR}}{dt} + (\mathcal{V}' + u(z) \hat{z}) \cdot \nabla f_{CR} - (\mathcal{P}' \cdot \nabla u) \cdot \frac{df_{CR}}{d\mathcal{P}} = - \frac{\partial}{\partial p'} \left\{ f_{CR} \frac{dp'}{dt} \right\}$$

COMPRESSION TERM

$$= - \frac{q}{m\gamma} \frac{p'_x B_0}{c} \frac{df_{CR}}{dp'} + \frac{1}{2} \frac{\partial}{\partial \mu'} (1 - \mu'^2) \gamma \frac{df_{CR}}{d\mu'}$$

(vi) [B] Gyro-Averaging

$$f_{cr} = f_{cr}^0 + \epsilon f_{cr}^1 \dots \text{etc.}$$

$$\epsilon = \frac{v}{\Omega \Delta} \ll 1.$$

We treat the cyclotron frequency as large $\Omega \gg \frac{v}{\Delta} \Rightarrow$ Larmor radius is smaller than Δ .

$$\mathcal{I}(\Omega) : \frac{\partial f_{cr}^0}{\partial \phi} = 0 \Rightarrow f_{cr}^0 = f_{cr}^0(z, \mu', p')$$

Gyro-Averaging ① we obtain: [i.e. $\oint_0^{2\pi} (\dots) d\phi$]

$$\mathcal{I}(\epsilon \Omega) \left[\frac{\partial f_{cr}^0}{\partial t} + (v'\mu' + u(z)) \frac{\partial f_{cr}^0}{\partial z} - p'^2 \mu'^2 \frac{du}{dz} \frac{\partial f_{cr}^0}{\partial p'} - p'\mu'(1-\mu'^2) \frac{du}{dz} \frac{\partial f_{cr}^0}{\partial \mu'} \right]$$

$$= \frac{1}{2} \frac{\partial}{\partial \mu'} \left[(1-\mu'^2) v(\mu') \frac{\partial f_{cr}^0}{\partial \mu'} \right] \quad \text{②}$$

(vii) [C] Now we treat $v \gg \frac{v'}{\Delta} \gg \frac{u}{\Delta}$ in fact lets make an ordering

$$\frac{u}{\Delta} \sim \mathcal{O}(\delta^2 v) \quad \frac{v'}{\Delta} \sim \mathcal{O}(\delta v).$$

$$f_{cr}^0 = f_{cr}^{00} + \delta f_{cr}^{01} + \delta^2 f_{cr}^{02} \dots \text{etc.}$$

Sets diffusion rate across Shock:
 $v \left(\frac{\lambda_{mf}}{\Delta} \right)^2 \sim \frac{u}{\Delta}$

Now we "solve" ② using this ordering.

$$\mathcal{I}(v) :- 0 = \frac{1}{2} \frac{\partial}{\partial \mu'} \left\{ (1-\mu'^2) v(\mu') \frac{\partial f_{cr}^{00}}{\partial \mu'} \right\}$$

Integrating the only nonsingular solution is $f_{cr}^{00} = f_{cr}^{00}(z, p')$

$$\mathcal{I}(\delta v) :- v'\mu' \frac{\partial f_{cr}^{01}}{\partial z} = \frac{1}{2} \frac{\partial}{\partial \mu'} \left[(1-\mu'^2) v(\mu') \frac{\partial f_{cr}^{01}}{\partial \mu'} \right]$$

Can be integrated to find f_{cr}^{01}

$$\mathcal{J}(\delta^2 \gamma) :- \frac{\partial f_{cr}^{oo}}{\partial t} + v' \mu' \frac{\partial f_{cr}^{o1}}{\partial z} + u(z) \frac{\partial f_{cr}^{oo}}{\partial z} - p' \mu'^2 \frac{du}{dz} \frac{\partial f_{cr}^{oo}}{\partial p'}$$

$$= \frac{1}{2} \frac{\partial}{\partial \mu'} (1 - \mu'^2) \gamma(\mu') \frac{\partial f_{cr}^{o2}}{\partial \mu'}$$

We substitute for f_{cr}^{o1} in terms of f_{cr}^{oo} and integrate over μ' $\left[\int_{-1}^1 d\mu' \dots \right]$ to obtain an equation for f_{cr}^{oo}

$$\frac{\partial f_{cr}^{oo}}{\partial t} + u(z) \frac{\partial f_{cr}^{oo}}{\partial z} - \frac{\partial}{\partial z} \left[D \frac{\partial f_{cr}^{oo}}{\partial z} \right] = \frac{1}{3} \frac{du}{dz} p' \frac{\partial f_{cr}^{oo}}{\partial p'}$$

$$D = \frac{1}{\pi} \int_{-1}^1 d\mu' (1 - \mu'^2) \frac{v'^2}{\gamma'(\mu')}$$

spatial Diffusion coefficient.

$$\sim \mathcal{J} \left(\frac{v'^2}{\gamma'} \right)$$

COMPRESSIONAL FLOW $\frac{du}{dz} < 0$

Just 2 terms:

$$\frac{\partial f_{cr}^{oo}}{\partial t} \approx - \left(\frac{1}{3} \frac{du}{dz} \right) p' \frac{\partial f_{cr}^{oo}}{\partial p'}$$

growing separable solutions

$$f_{cr} \sim p^{-\kappa} e^{\frac{\kappa}{3} \left| \frac{du}{dz} \right| t}$$

222c. Lecture # 17. Acceleration of Cosmic Rays IV.

(i) In the last lecture we showed how if we took the relativistic equation for the cosmic rays and (a) transformed the momentum to the moving frame (b) Averaged over the gyrophase and (c) looked at the short mean free path limit we got to lowest order: -

$$f_{CR} = f_{CR}^{oo}(z, p', t)$$

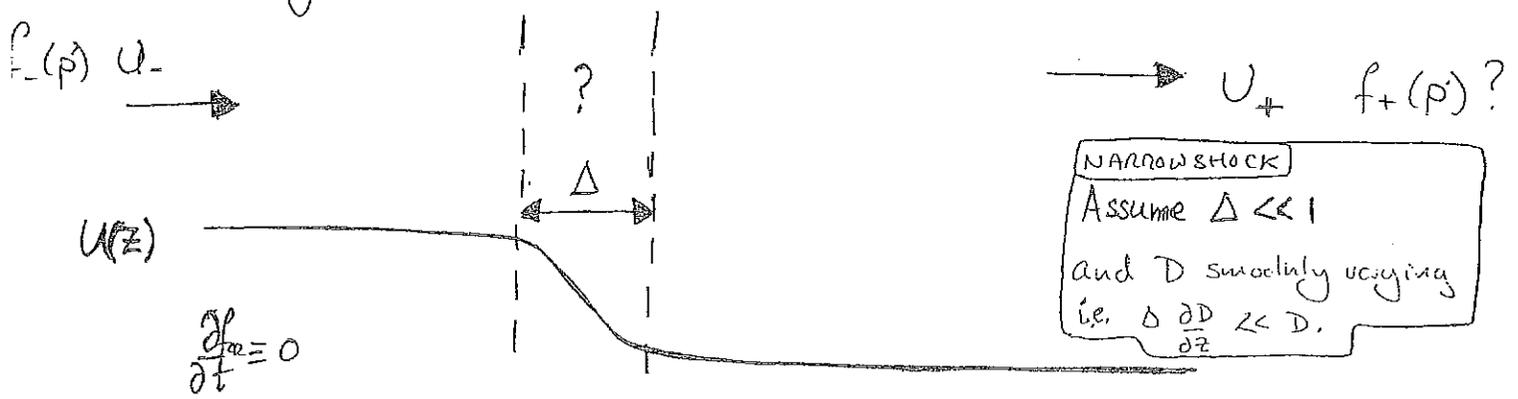
where

$$\textcircled{1} \quad \frac{\partial f_{CR}^{oo}}{\partial t} + u(z) \frac{\partial f_{CR}^{oo}}{\partial z} - \frac{\partial}{\partial z} \left[D \frac{\partial f_{CR}^{oo}}{\partial z} \right] = \frac{1}{3} \frac{du}{dz} p' \frac{\partial f_{CR}^{oo}}{\partial p'}$$

$u(z) \equiv$ local plasma velocity.

$$D(z, p') = \frac{1}{4} \int_{-1}^1 d\mu' (1 - \mu'^2) \frac{v'^2}{v(\mu')} \quad \text{spatial diffusion coefficient due to scattering from Alfvén waves.}$$

(ii) Now we want to solve the problem of cosmic ray acceleration by passage across a shock. We consider the distribution function at $z = -\infty$ to be $f_-(p')$, a given. Since on average the cosmic rays are moving with the fluid (to lowest order f_{CR} has no flux in p' space) we know that particles will be swept through the shock with the flow. We want to find $f_{CR}(z \rightarrow \infty, p') = f_+(p')$



(iii) We break up the flow into regions I, II, III where

REGION: I : $z \ll -\Delta$ $u(z) = U_-$

REGION: II : $|z| \ll \frac{D}{U}$ $u(z)$ is rapidly varying

REGION: III : $z \gg \Delta$ $u(z) = U_+$

We solve in these three regions and match them in their overlap regions

I-II: $-\frac{D}{U} \ll z \ll -\Delta$ II-III: $\Delta \ll z \ll \frac{D}{U}$

(iv) REGION I $z \ll -\Delta$

$$U_- \frac{\partial f_{CR}^{\infty}}{\partial z} - \frac{\partial}{\partial z} \left(D(z, p') \frac{\partial f_{CR}^{\infty}}{\partial z} \right) = 0$$

$$\rightarrow f_{CR}^{\infty} = f_-(p') + [f_0(p') - f_-(p')] \exp + \int_0^z \frac{U_- dz}{D(z, p')}$$

Note this solution is a competition of flow in +ve direction and diffusion back in the -ve direction. Typical length $\approx \left| \frac{D}{U_-} \right| \gg \Delta$

(v) REGION III, $z \gg \Delta$

$$U_+ \frac{\partial f_{CR}^{\infty}}{\partial z} - \frac{\partial}{\partial z} \left(D(z, p') \frac{\partial f_{CR}^{\infty}}{\partial z} \right) = 0$$

solution with f finite as $z \rightarrow \infty$ is

$$f_{CR}^{\infty} = f_+(p')$$

(vi) REGION II. In this region $u(z)$ changes rapidly and therefore

$$\frac{du}{dz} \text{ is large. We take } \frac{\partial}{\partial z} \sim \frac{1}{\Delta} \gg \frac{U}{D}$$

Expand: $f_{CR}^{\infty} = f_{CR}^{\infty 0} + \frac{\Delta U}{D} f_{CR}^{\infty 1} \dots$ etc.

To lowest order

$$\mathcal{O}\left(\frac{D}{\Delta^2}\right): \frac{\partial}{\partial z} \left(D \frac{\partial f_{cr}^{ooo}}{\partial z} \right) = 0 \Rightarrow \boxed{f_{cr}^{ooo} = f_{II}(p')}$$

Next order

$$\mathcal{O}\left(\frac{U}{\Delta}\right): -\frac{\partial}{\partial z} D \frac{\partial f_{cr}^{oo1}}{\partial z} = \frac{1}{3} \frac{du}{dz} p' \frac{\partial f_{II}}{\partial p'}$$

$$\frac{\partial f_{cr}^{oo1}}{\partial z} = -\frac{U(z)}{3} \frac{p'}{D} \frac{\partial f_{II}(p')}{\partial p'} + G(p')$$

(vii) MATCHING:

First:

$$\Delta \ll z \ll \frac{D}{U}$$

$$\text{III} \rightarrow f_{cr}^{oo} = f_+(p')$$

$$\text{II} \rightarrow f_{cr}^{oo} = f_{II}(p') + \left(\frac{U_+}{3} \frac{p'}{D} \frac{\partial f_{II}}{\partial p'} + G(p') \right) z$$

$$\Rightarrow \boxed{\text{(2)} - f_+(p') = f_{II}(p') \text{ and } G(p') = -\frac{U_+}{3} \frac{p'}{D} \frac{\partial f_{II}}{\partial p'}}$$

Second:

$$-\Delta \gg z \gg -\frac{D}{U}$$

$$\text{I} \rightarrow f_{cr}^{oo} = f_0(p') + [f_0(p') - f_-(p')] \frac{z U_-}{D}$$

$$\text{II} \rightarrow f_{cr}^{oo} = f_{II}(p') + \frac{(U_- - U_+)}{3} \frac{p'}{D} \frac{\partial f_{II}}{\partial p'} z$$

$$\Rightarrow \boxed{\text{(3)} - f_0(p') = f_{II}(p') \text{ and } \frac{(U_- - U_+)}{3} p' \frac{\partial f_{II}}{\partial p'} = [f_0(p') - f_-(p')] U_-}$$

Combining

$$\frac{(U_- - U_+)}{3} p' \frac{\partial f_+}{\partial p'} = [f_+ - f_-] U_-$$

let $r = \frac{U_-}{U_+}$ Compression ratio $r > 1$

$$f_+(p') = \frac{1}{q} p'^{-q} \int_0^{p'} dp'' f_-(p'') p''^{(q-1)} \quad - (4)$$

NOTE $f_+(p')$ does not depend on D and it is a power law p'^{-q} if f_- is fairly localized so that for large p' $\int_0^{p'} dp'' f_-(p'') p''^{(q-1)} \approx \text{constant}$

272e. Lecture # 18Reconnection of Magnetic Field lines.

(i) One of the most cherished concepts of MHD is the Frozen-in theorem

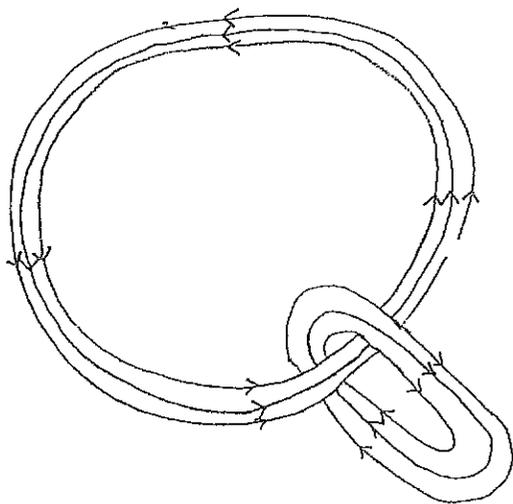
$$\underline{\underline{E}} + \frac{\underline{\underline{v}} \times \underline{\underline{B}}}{c} = 0 \quad \Rightarrow \quad \frac{\partial \underline{\underline{B}}}{\partial t} = \nabla \times (\underline{\underline{v}} \times \underline{\underline{B}})$$

we proved (some 9 months ago) that

a) Flux ($\int \underline{\underline{B}} \cdot d\underline{\underline{A}}$) is constant on any loop moving ~~through~~ ^{with} the plasma.

b) Field lines "move" with velocity $\underline{\underline{v}}$.

(ii) A simple extension of this is to say that field lines can never be broken. We often talk about the topology of field lines - which is essentially how field lines link



A simple linkage. 1

(iii) One can generalize the theorem in many ways for example the so called "Hall terms" do not break field lines

Electron Eqn.

$$n e \left\{ \underline{\underline{E}} + \frac{\underline{\underline{v}}_e \times \underline{\underline{B}}}{c} \right\} = \nabla p_e \quad \underline{\underline{v}}_e = \underline{\underline{v}} - \frac{\underline{\underline{j}}}{n e}$$

"Hall terms"

$$\Rightarrow \underline{\underline{E}} + \frac{\underline{\underline{v}} \times \underline{\underline{B}}}{c} = \frac{\nabla(n T_e)}{n e} + \frac{\underline{\underline{j}} \times \underline{\underline{B}}}{n e}$$

(iv) suppose $\nabla T = 0$ then

$$\frac{\partial \underline{B}}{\partial t} = -c \nabla \times \underline{E} = \nabla \times (\underline{v}_e \times \underline{B}) - \frac{c}{e} \frac{\nabla \times (\underline{I} \cdot \nabla \ln n)}{0}$$

$$\Rightarrow \boxed{\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v}_e \times \underline{B})} \quad \underline{\text{field lines are frozen to the electrons}}$$

This equation is relevant to the experiments of Reiner Stenzel.

(v) However certain effects will destroy the theorem and resistance is one such effect.

$$\underline{E} + \frac{\underline{v} \times \underline{B}}{c} = \eta \underline{J} \Rightarrow \frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) + \frac{\eta c^2}{4\pi} \nabla^2 \underline{B}$$

Magnetic Diffusion Coefficient = $\frac{\eta c^2}{4\pi} = 10^7 \frac{1}{T_e^{-3/2}} \text{ cm}^2 \text{ s}^{-1}$

↑
temperature in eV.

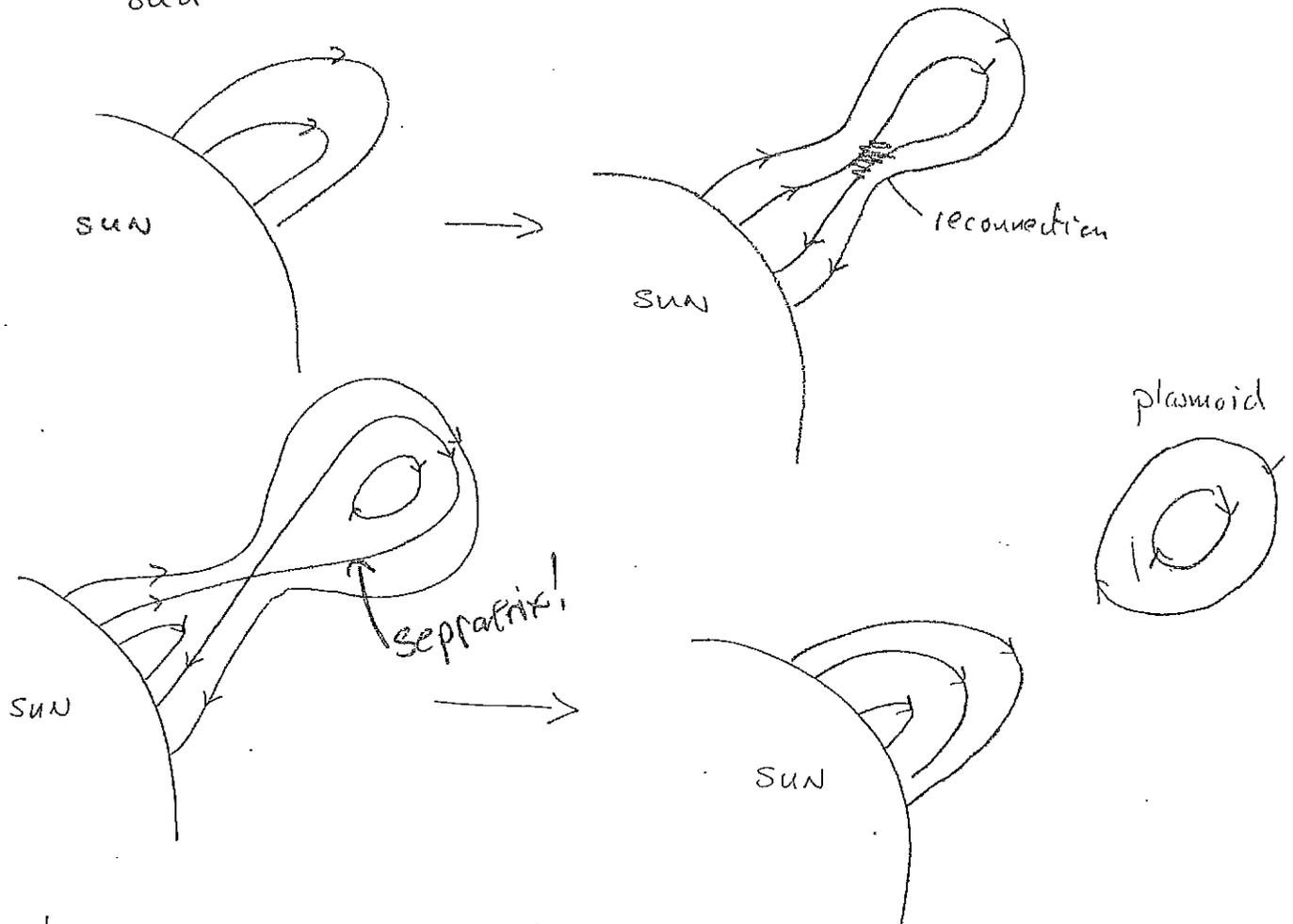
In many plasmas of interest the time for resistive diffusion, τ_D , to take place is very long

TOKAMAK: $T_e \sim 10^4 \text{ eV}$ $L \sim 100 \text{ cm}$ $\tau_D \sim \left(\frac{4\pi L^2}{\eta c^2} \right) = 10^3 \text{ seconds}$

SOLAR CORONA: $T_e \sim 10^2 \text{ eV}$ $L \sim 10^{11} \text{ cm}$ $\tau_D \sim 10^{18} \text{ seconds} \sim 3 \times 10^{10} \text{ y}$

(vi) One way to increase the effect is to make diffusion take place over very small distances. In the late 50s it was realized one could do this in boundary layers.

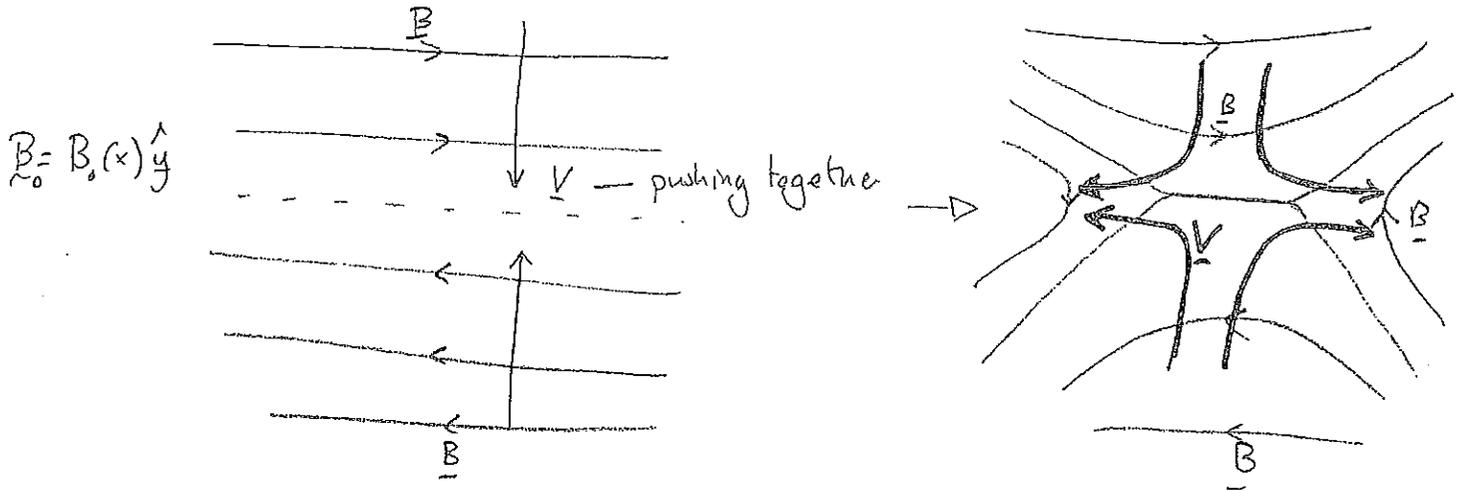
(vii) Typical reconnection problem is forming a "plasmoid" in the sun



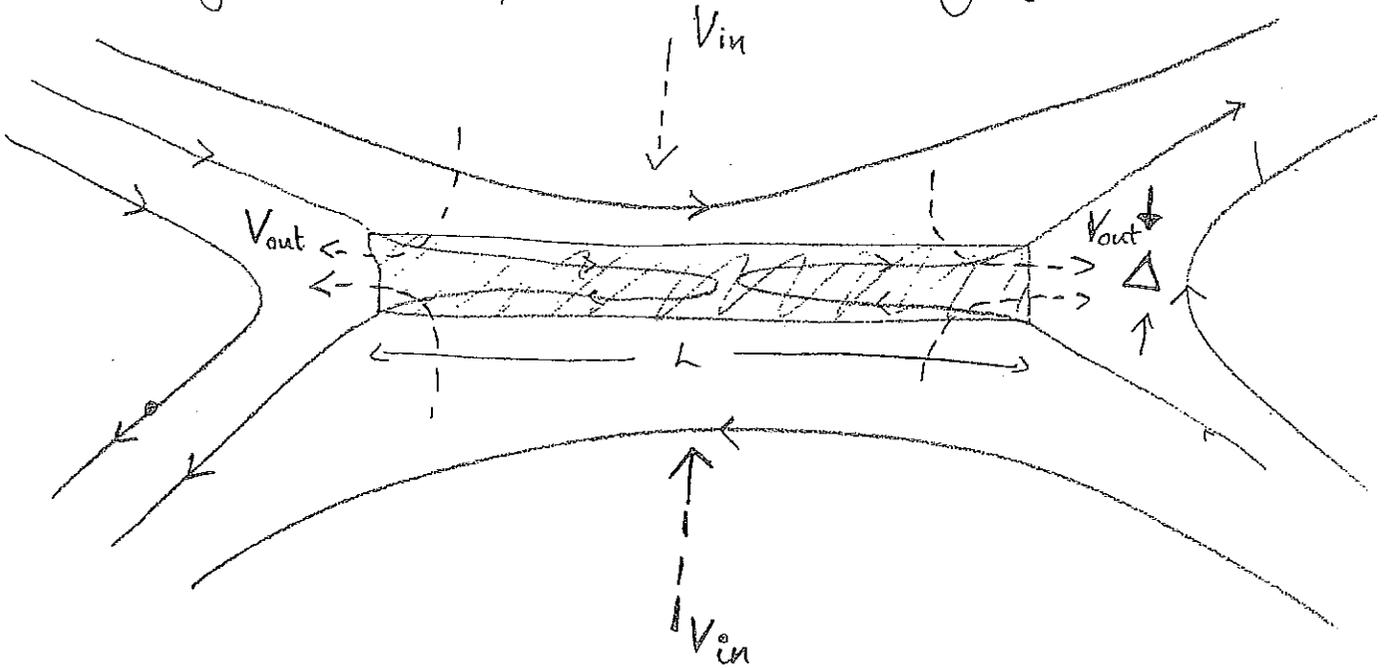
this may occur in what is called CORONAL MASS EJECTIONS.

(viii) Let us now examine how reconnection could possibly take place in a narrow layer.

(ix) Simple Situation field reversed



(x) let's try and understand what happens in the reconnection region which is presumed to be the only region where η is important.



(xi) Note $V_x \propto x$ for small x

(x) let's take the flow to be incompressible $\nabla \cdot \vec{V} = 0$

$$\Rightarrow V_{in} L = V_{out} \Delta$$

(xi) we look at a steady situation so $\frac{\partial \underline{B}}{\partial t} = 0$

Induction Eqn. $\nabla \times \underline{V} \times \underline{B} = \frac{\eta c^2}{4\pi} \nabla^2 \underline{B} \Rightarrow \frac{V_x B_y - V_y B_x}{\Delta} \sim \frac{\eta c^2}{4\pi} \frac{B_y}{\Delta^2}$

$$\Rightarrow \Delta \approx \frac{\eta c^2}{4\pi} \frac{1}{V_x} \sim \frac{\eta c^2}{4\pi} \frac{1}{V_{in}}$$

Diffusion out balances convection in (of magnetic field)

(xii) Also we can calculate the momentum changes

$$\rho \underline{V} \cdot \nabla \underline{V} = -\nabla p - \nabla \frac{B^2}{8\pi} + \underline{B} \cdot \nabla \underline{B}$$

roughly $\rho \frac{V_y^2}{L} \sim \frac{B_y^2}{L 8\pi} \Rightarrow V_y \sim \left(\frac{B_y^2}{\rho 8\pi} \right)^{1/2} = V_A$

Alfvén speed.

(xiii) Therefore from (x), (xi) and (xii) we get 1.

$$V_{in} = V_y \frac{\Delta}{L} = \frac{V_A}{L} \frac{\eta c^2}{4\pi} \frac{1}{V_{in}}$$

$$V_{in} = V_A \left(\frac{\eta c^2}{4\pi L^2} \cdot \frac{L}{V_A} \right)^{1/2} \sim V_A \left(\frac{\tau_A}{\tau_\eta} \right)^{1/2}$$

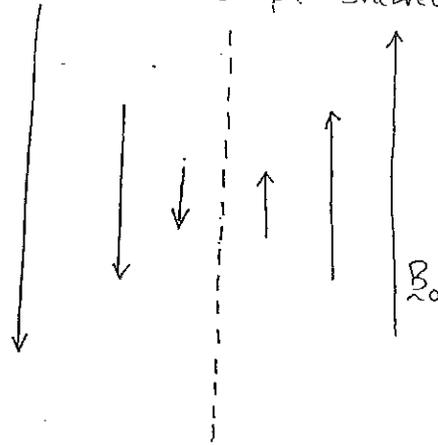
(xiv) If L is a large distance (macroscopic) then

$$V_{in} \ll V_A \quad \text{slow reconnection} \quad \text{Sweet-Parker 1957-58}$$

222c. Lecture # 19

The Tearing Mode.

(i) Consider a simple sheared field



$$\underline{B}_0 = B_0 \tan \frac{y}{L_s} \hat{y}$$

- It is easy to show that this situation is stable to ideal MHD perturbations (use δw and note $\mathbf{j}_{\parallel} = 0, \mathbf{b} \cdot \nabla \mathbf{b} = 0$)
- Is it stable when we add resistivity -- and allow reconnection. We take $\underline{E} + \underline{v} \times \underline{B} = \eta \underline{j}$

(ii) We look at simple 2D perturbations & since $\nabla \cdot \underline{B} = 0$ we also take, for simplicity,

$$\nabla \cdot \underline{B} = 0 \Rightarrow \underline{B} = \nabla \psi \times \hat{z}$$

$$\nabla \cdot \underline{v} = 0 \Rightarrow \underline{v} = \nabla \phi \times \hat{z}$$

A little Algebra yields:

$$\textcircled{1} \quad \frac{d\underline{B}}{dt} = \nabla \times (\underline{v} \times \underline{B}) + \frac{\eta c^2}{4\pi} \nabla^2 \underline{B} \Rightarrow \frac{\partial \psi}{\partial t} + \underline{v} \cdot \nabla \psi = \frac{\eta c^2}{4\pi} \nabla^2 \psi$$

and the curl of the momentum equation yields (note $\nabla \times \nabla p = 0$)

$$\rho \frac{d(\nabla \times \underline{v} \cdot \hat{z})}{dt} = \rho \frac{d \nabla^2 \phi}{dt} = -\frac{\hat{z}}{4\pi} \cdot [\nabla \psi \times \nabla (\nabla^2 \psi)] \quad \textcircled{2}$$

(iii) Now linearize $\textcircled{1}$ & $\textcircled{2}$ about $\underline{v} = 0, \underline{B} = \underline{B}_0$

$$\psi(x, y, t) = \psi_0(x) + B_0 \psi_1(x) \cos ky e^{\gamma t}$$

$$\phi(x, y, t) = \frac{\gamma}{k} \phi_1(x) \sin ky e^{\gamma t}$$

(iv) Near $x = 0$ $\psi_0 \approx -\frac{B_0 x^2}{2L_s}$

$$\psi_1 \approx \psi_1(0) + \mathcal{O}(x^2)$$

as we shall see ψ_1 is an even function.

(V) Linearizing ① & ② we obtain

$$T(x) = \tanh \frac{x}{l_s}$$

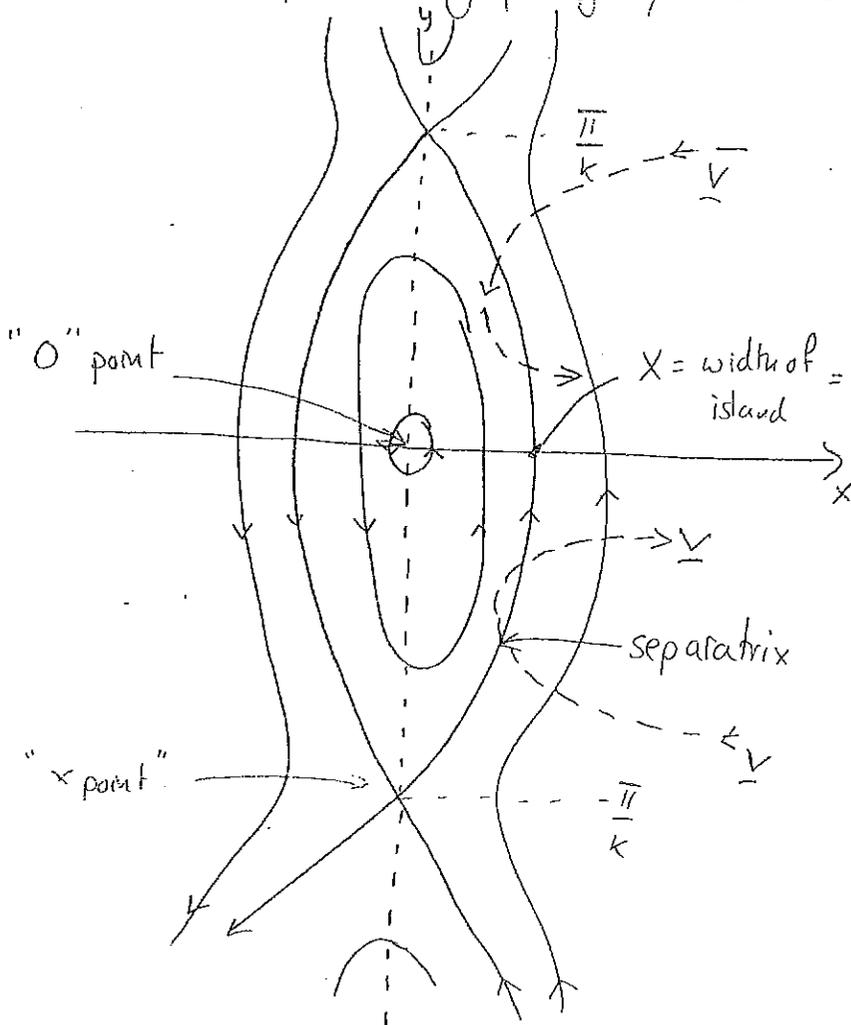
From ②:
$$-\gamma^2 \epsilon_A^2 [\phi_1'' - k^2 \phi_1] = T(x) [\psi_1'' - k^2 \psi_1] - T''(x) \psi_1 \quad \text{③}$$

From ①:
$$\psi_1 - T(x) \phi_1 = \frac{l_s^2}{\gamma \epsilon_R} [\psi_1'' - k^2 \psi_1] \quad \text{④}$$

- Note ψ_1 is even and ϕ_1 odd (or vice versa although we don't consider this case)

(vi)
$$\epsilon_A^2 = \frac{4\pi p}{k^2 B^2} = \frac{1}{k^2 v_A^2} \quad \tau_R = \frac{4\pi l_s^2}{\eta c^2} \quad \text{resistive time}$$

lets plot a rough plot of ψ near $x=0$ (note $B \cdot \nabla \psi = 0$)



$$X = \text{width of island} = 2 \left(\frac{-\psi_1(0)}{\psi_0''} \right)^{1/2}$$

As island grows $|\psi_1|$ gets bigger.

(vii) Now we search for a solution with $\delta t_A \ll 1 \ll \delta t_R$

Then dropping all small terms

$$\textcircled{3} \rightarrow T(x) [\psi_1'' - k^2 \psi_1] - T''(x) \psi_1 = 0 \quad \textcircled{5}$$

$$\textcircled{4} \rightarrow \psi_1 = T(x) \phi_1$$

Note $\textcircled{5}$ is just $\nabla_{\kappa} (\beta \kappa \beta) = 0$ (Equilibrium condition)

(viii) Equation $\textcircled{5}$ is SINGULAR AT $T(x) = X = 0$ SO WE GET SOLUTIONS ON EITHER SIDE AND EXCLUDE A BOUNDARY LAYER $X = \delta \ll 1$ at $x=0$

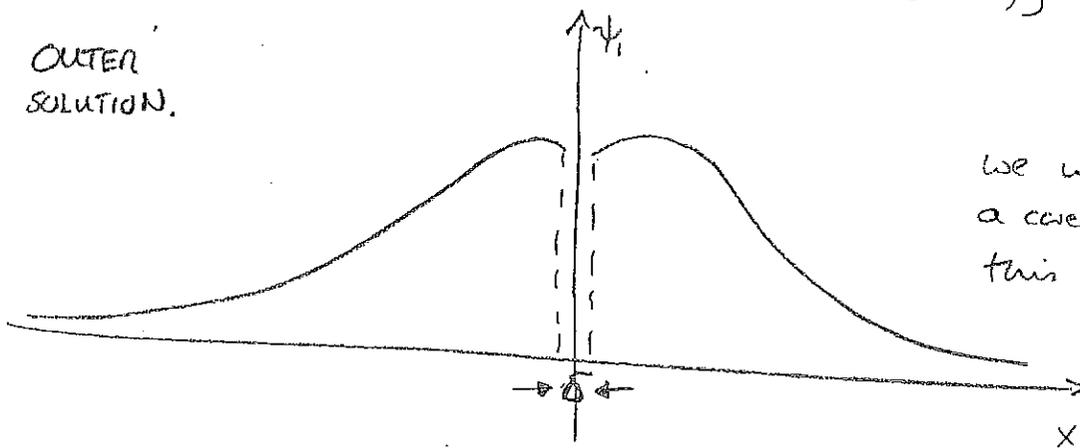
$$x \gg \Delta$$

$$\psi_1(x) = c_+ e^{-kx} \left[1 + \frac{\tanh(x/l_s)}{kl_s} \right]$$

$$x \ll -\Delta$$

$$\psi_1(x) = c_- e^{+kx} \left[1 - \frac{\tanh(x/l_s)}{kl_s} \right]$$

OUTER SOLUTION.



We will choose $c_- = c_+$
a careful analysis shows
this is correct

We define

$$\Delta' = \frac{\left(\frac{d\psi_1}{dx} \right)_{0^+} - \left(\frac{d\psi_1}{dx} \right)_{0^-}}{\psi_1(0)} = 2 \left[\frac{1}{kl_s^2} - k \right]$$

"Delta-prime"

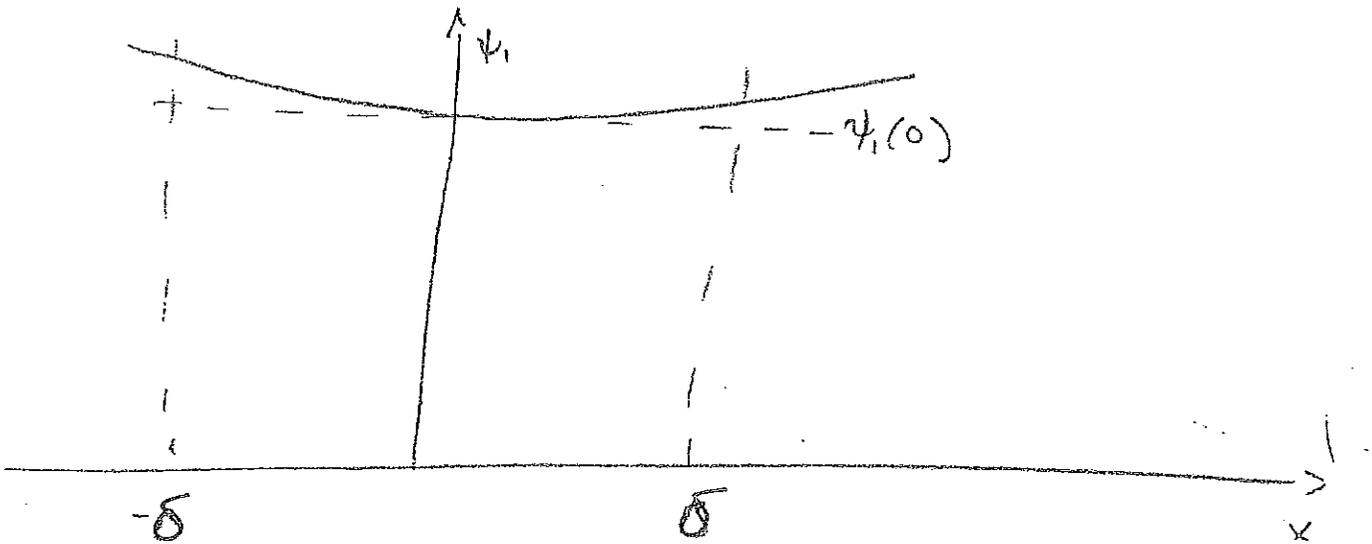
(ix) In the region $|x| \sim \delta$ we treat $\frac{d}{dx} \sim \frac{1}{\delta} \gg k$

then from (3)

$$-\gamma^2 \tau_A^2 \phi_1'' = \frac{x}{l_s} \psi_1'' \quad (6)$$

and from (4)

$$\psi_1 - \frac{x}{l_s} \phi_1 = \frac{l_s^2}{\gamma \tau_R} \psi_1'' \quad (7)$$



On the scale of δ the outer solution has almost no slope — thus ψ_1 is almost constant through the boundary "tearing layer" Then

$$\Delta' \approx \frac{1}{\psi_1(0)} \int_{-a}^a \psi_1'' dx \quad \delta \ll a \ll 1$$

from (6)

$$\approx -\frac{1}{\psi_1(0)} l_s \gamma^2 \tau_A^2 \int_{-a}^a \frac{\phi_1''}{x} dx$$

since for $x \gg \delta$

$$\phi_1 \sim \frac{\psi_1}{x} \text{ then}$$

integral converges and we set $a \rightarrow \infty$.

and

$$\psi_1(0) - \frac{x}{l_s} \phi_1(x) = -\frac{l_s^3}{\gamma \tau_R} \cdot \gamma^2 \tau_A^2 \frac{\phi_1''}{x}$$

Normalizing reduces ^{this} to a math problem

$$\dot{x} = l_s \left(\frac{\gamma \tau_A^2}{\tau_R} \right)^{1/4} z = \delta z \quad \chi = \frac{\phi_1}{\psi_1(0)} \left(\frac{\gamma \tau_A^2}{\tau_R} \right)^{1/4}$$

$$\Rightarrow \boxed{\chi'' - z^2 \chi = z} \quad \text{here } \chi' = \frac{d\chi}{dz}$$

$$\underbrace{\gamma^2 \tau_A^2 \left(\frac{\gamma \tau_A^2}{\tau_R} \right)^{-3/4}}_{\gamma^{5/4} \tau_R^{3/4} \tau_A^{1/2}} \int_{-\infty}^{\infty} \chi'' \frac{dz}{z} = \Delta' l_s$$

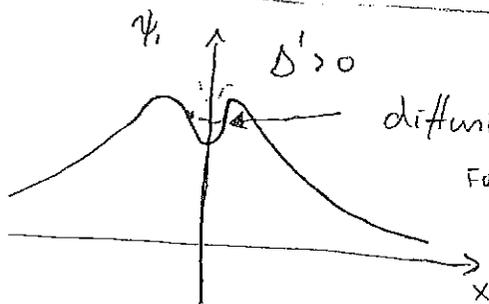
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Only if $\Delta' > 0$ otherwise must have complex z .

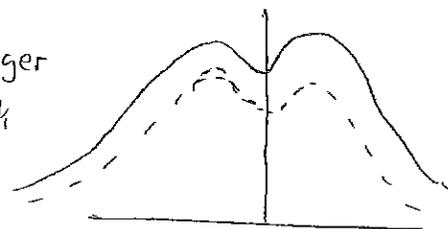
$$\gamma = \frac{-4/5}{I} \tau_R^{-3/5} \tau_A^{-2/5} (\Delta' l_s)^{4/5}$$

Grows only if
 $\Delta' > 0$

$$\delta = l_s (\Delta' l_s)^{1/5} \left(\frac{\tau_A}{\tau_R} \right)^{2/5} \frac{1}{I^{1/5}}$$



diffusion makes $\psi_1(0)$ get bigger
Force balance then makes ψ_1
bigger everywhere else



You can verify that if $\Delta' < 0$ diffusion reduces $\psi_1(0)$

