

# APR J81CS 222b Plasma Physics:

Sorry I cannot be with you for the first lecture, enjoy it.

Book: Hazeltine & Waelbroeck "Framework of Plasma Physics", ~~see~~ My Notes and Handouts.

Lecture: Tuesday - Thursday 1 - 2:30 pm.

Kundsen I-130N (not as aetfized)

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Homework: Every week. please do it every week.

Final: Takehome 8hr exam.

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## PLAN

### 5 Weeks PLASMA WAVES

#### (i) COLD PLASMA WAVES.

a) Unmagnetized Plasma.

b) Magnetized Plasma.

c) Wave propagation  $\begin{cases} \text{WKB,} \\ \text{Mode Conversion,} \\ \text{Energy.} \end{cases}$

#### (ii) HOT PLASMA WAVES

a) Unmagnetized waves,

- Landau Damping

- Quasilinear Theory.

b) Magnetized Waves.

#### (iii) INSTABILITIES

a) Two Stream.

b) Bump on Tail.

c) Drift-Waves

### 4 Weeks COLLISIONS etc.

#### (i) Coulomb Collisions

a) Fokker-Planck Equation.

b) Lenard-Balescu

c) Radiation

- Bremsstrahlung  
- Synchrotron

d) Chapman-Enskog  
derivation of fluid  
equations.

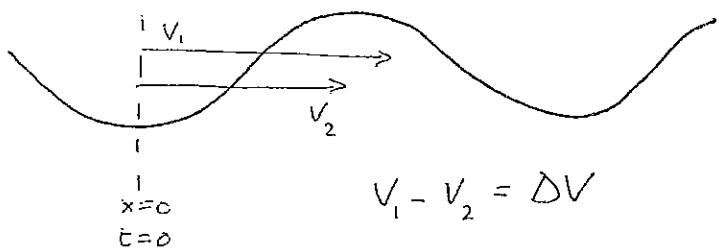
(i) COLD PLASMA APPROXIMATION.

Many plasma waves are well described by the cold plasma equations where the thermal spread in electron and ion velocities is ignored. First let us understand the criterion for this in the simplest case.

"—————"

CONSIDER 2 electrons in an electrostatic 1D wave. They start at the same position with slightly different velocities.

$$\frac{d^2x}{dt^2} = -\frac{eE}{m_e} \cos(kx - \omega t)$$



LINEARIZING WITH E VERY SMALL  $x_1(t) = v_1 t + \delta x_1(t)$

$$\frac{d^2\delta x_1}{dt^2} \approx -\frac{eE}{m_e} \cos(kv_1 t - \omega t) \Rightarrow \ddot{\delta x}_1 = \frac{eE}{m_e} \frac{\cos(kv_1 t - \omega t)}{(kv_1 - \omega)^2}$$

and clearly  $x_2(t) = v_2 t + \delta x_2(t)$   $\ddot{\delta x}_2 = \frac{eE}{m_e} \frac{\cos(kv_2 t - \omega t)}{(kv_2 - \omega)^2}$

IF  $\left(\frac{\omega - kv}{k}\right) \gg \Delta V$  then  $\delta x_1 \sim \delta x_2$

PHASE VELOCITY seen by electrons.  
THUS: if the spread in electron velocities is small compared to the wave's phase velocity <sup>as</sup> seen by the electrons then all electrons respond to the wave in the same way. A similar criterion holds for ions but since they are usually much slower it is usually a less stringent criterion.

→ WHEN THE COLD PLASMA CRITERION HOLDS WE CAN USE THE FLUID EQUATIONS WITHOUT THE PRESSURE (TENSOR) TERM SINCE IT INVOLVES THE SPREAD IN PARTICLE VELOCITIES. REMEMBER  $P = \int d^3V (\underline{V} - \underline{V}_{\text{fluid}})(\underline{V} - \underline{V}) f(V, r, t)$

(ii) COLD PLASMA EQUATIONS $\alpha$  labels species i.e. electrons & ions

CONTINUITY: -  $\frac{\partial n_\alpha}{\partial t} + \nabla \cdot n_\alpha \underline{v}_\alpha = 0$

GET USED TO THE NOTATION

MOMENTUM: -  $m_\alpha n_\alpha \left[ \frac{\partial \underline{v}_\alpha}{\partial t} + \underline{v}_\alpha \cdot \nabla \underline{v}_\alpha \right] = n_\alpha q_\alpha \left[ \underline{E} + \frac{\underline{v}_\alpha \times \underline{B}}{c} \right]$

CHARGE DENSITY:  $\rho = \sum_\alpha q_\alpha n_\alpha$  CURRENT DENSITY:  $\underline{J} = \sum_\alpha q_\alpha n_\alpha \underline{v}_\alpha$

and, of course, Maxwell's Equations: -

$$\textcircled{1} \quad \frac{\partial \underline{B}}{\partial t} = -c \nabla \times \underline{E} \quad \& \quad \nabla \cdot \underline{B} = 0$$

$$\textcircled{2} \quad \frac{\partial \underline{E}}{\partial t} = c \nabla \times \underline{B} - 4\pi \underline{J} \quad \& \quad \nabla \cdot \underline{E} = 4\pi \rho$$

COMPLETE SET OF EQUATIONS.

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(iii) LINEARIZED WAVES IN A COLD UNMAGNETIZED PLASMA

It is very important to understand the basic waves supported by the plasma. To this end we linearize about an undisturbed state - here we take two <sup>homogeneous</sup> species electrons and ions and no "equilibrium"  $\underline{B}$ .

EQUILIBRIUM  $\left\{ \begin{array}{l} e n_{e0} = q n_{i0} = \text{constant} \Rightarrow \rho_0 = 0 \\ \underline{v}_{e0} = \underline{v}_{i0} = 0 \\ \underline{B}_0 = \underline{E}_0 = 0 \end{array} \right.$

- we linearize - i.e. keep only linear terms in  $\delta \underline{v}_e, \delta \underline{v}_i, \delta \underline{B}$  &  $\delta \underline{E}$  and  $\delta n_e$  and  $\delta n_i$ .

(iv) As is common in these cases we search for solutions of the form,

$$\delta_{\underline{V}_\alpha} = \bar{\delta}_{\underline{V}_\alpha} e^{i\underline{k} \cdot \underline{r} - i\omega t}$$

$$\delta_{\underline{B}} = \bar{\delta}_{\underline{B}} e^{i\underline{k} \cdot \underline{r} - i\omega t}$$

$$\delta_{\underline{E}} = \bar{\delta}_{\underline{E}} e^{i\underline{k} \cdot \underline{r} - i\omega t}$$

$$\delta n_\alpha = \bar{\delta} n_\alpha e^{i\underline{k} \cdot \underline{r} - i\omega t}$$

The Real parts can be taken at the end to find the real solution

and we note that acting on these quantities

$$\nabla = i\underline{k} \quad \frac{\partial}{\partial t} = -i\omega$$

→ Momentum Equation :  $m_\alpha n_\alpha (-i\omega \delta_{\underline{V}_\alpha}) = n_\alpha q_\alpha \delta_{\underline{E}}$       ③

① :  $-i\omega \delta_{\underline{B}} = -i\underline{k} \times \delta_{\underline{E}}$       ④

② :  $-i\omega \delta_{\underline{E}} = i\underline{k} \times \delta_{\underline{B}} - \frac{4\pi}{c} \sum_\alpha q_\alpha n_\alpha \delta_{\underline{V}_\alpha}$       ⑤

QUESTION TO PONDER: WHY DON'T WE USE THE OTHER EQUATIONS?

Eliminating  $\delta_{\underline{V}_\alpha}$  and  $\delta_{\underline{B}}$  we obtain

$$\boxed{\frac{1}{c^2} \left[ \omega^2 - (\omega_{pe}^2 + \omega_{pi}^2) \right] \delta_{\underline{E}} = -\underline{k} \times (\underline{k} \times \delta_{\underline{E}})} \quad - ⑥$$

where  $\omega_{pe}^2 = \frac{4\pi e^2 n_{eo}}{m_e}$  and  $\omega_{pi}^2 = \frac{4\pi q^2 n_{io}}{m_i}$

" plasma frequency"      " ion plasma frequency"

Except when ions are positive  $\omega_{pi}^2 \ll \omega_{pe}^2$  and we ignore it.

Note       $\delta_{\underline{J}} = \sigma \delta_{\underline{E}}$  and  $\sigma = \frac{i}{4\pi} \left( \frac{\omega_{pe}^2 + \omega_{pi}^2}{\omega} \right)$  the conductivity.

We solve ⑥ by splitting  $\delta E$  into longitudinal & transverse parts.

$$\underline{\text{let}} \quad \delta \underline{E} = \delta \underline{E}_L \hat{k} + \delta \underline{E}_T \quad \delta \underline{E}_T \cdot \underline{k} = 0$$

**Longitudinal Wave**

dot ⑥ with  $\underline{k}$

$$(\omega^2 - \omega_{pe}^2) \delta E_L = 0 \rightarrow \omega = \pm \omega_{pe}$$

"PLASMA WAVE"

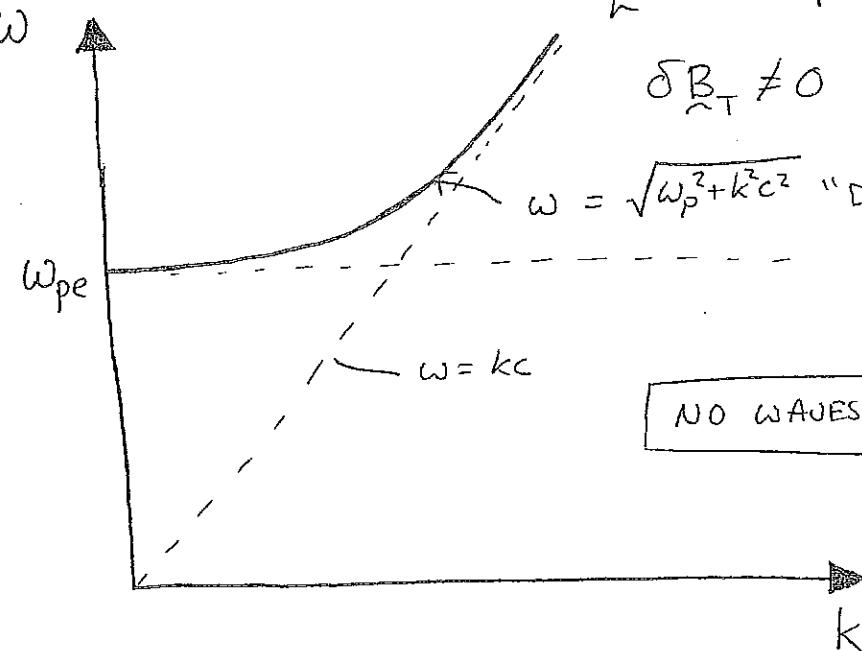
no  $k$  dependence

note  $\delta \underline{B}_L = 0$  "electrostatic" wave since we can write  $\delta \underline{E} = i \underline{k} \phi = \nabla \phi$ .

**Transverse Wave**

$$[\omega^2 - \omega_{pe}^2 - k^2 c^2] \delta E_T = 0$$

$\delta \underline{B}_T \neq 0$  Modified light wave.



- Waves with  $\omega < \omega_p$  will not propagate in a cold unmagnetized plasma (remember this it's important)

- Phase velocity =  $\frac{\omega}{k} = c \sqrt{\frac{\omega_p^2}{k^2 c^2} + 1} > c$

- Group velocity =  $\frac{\partial \omega}{\partial k} = \frac{c}{\sqrt{\frac{\omega_p^2}{k^2 c^2} + 1}} < c$



## ~~Waves in plasmas~~ #2 222b. Waves in homogeneous cold magnetized plasma

Homework: Do the questions in these notes.

(i) Last time Ben told you about waves in a <sup>cold</sup> unmagnetized plasma - there are essentially three -

$$\text{PLASMA WAVE: } \omega^2 = \omega_{pe}^2 \quad E = -\nabla\phi \quad \text{LONGITUDINAL.}$$

$$\text{MODIFIED LIGHT WAVE: } \omega^2 = \omega_{pe}^2 + k^2 c^2 \quad E, B \text{ transverse.}$$

2 polarizations  $\therefore$  2 waves "degenerate".

(ii) In a magnetized plasma the situation is much richer. Taking  $\{\underline{E}, \underline{B}\} \sim \{\underline{E}, \underline{B}\} e^{ik \cdot r - i\omega t}$  and taking Real parts at the end.

(iii) GENERAL NOTATION:

$$\underline{\underline{J}} = \underline{\underline{\sigma}} \cdot \underline{\underline{E}}$$

$$\text{from } \frac{\partial \underline{B}}{\partial t} = -c \nabla \times \underline{E} \Rightarrow$$

$$\boxed{\frac{c}{\omega} \underline{k} \times \underline{\underline{E}} = \underline{\underline{B}}} \quad (1)$$

$\underline{\underline{\sigma}}$  = conductivity tensor.

contains plasma response to be calculated later.

$$\nabla \times \underline{\underline{B}} = \frac{4\pi}{c} \underline{\underline{J}} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t} \Rightarrow$$

$$\boxed{\frac{c}{\omega} \underline{k} \times \underline{\underline{B}} = -i \frac{4\pi}{\omega} \underline{\underline{J}} - \underline{\underline{E}}} \quad (2)$$

we introduce the dielectric tensor again it is just notation

$$\underline{E} + i \frac{4\pi}{\omega} \underline{\underline{J}} = \underline{\underline{\epsilon}} \cdot \underline{\underline{E}}$$

Note. in conventional dielectric theory  $\underline{\underline{\epsilon}} \cdot \underline{\underline{E}} = \underline{\underline{D}}$

$$\Rightarrow \underline{\underline{\epsilon}} = \underline{\underline{I}} + i \frac{4\pi}{\omega} \underline{\underline{\sigma}} \quad (3)$$

$\underline{\underline{I}}$  = unit tensor.

note

$\underline{\underline{\epsilon}}$  is a function of  $\omega, B_0$  and  $\underline{k}$  in general but for cold plasma there is no  $\underline{k}$  dependence

eliminating  $\tilde{B}$  we obtain

$$\frac{c^2}{\omega^2} \underline{k} \times (\underline{k} \times \tilde{\underline{E}}) = - \underline{\underline{\epsilon}} \cdot \tilde{\underline{E}}$$

or

$$(iv) \quad \left[ \frac{c^2 k^2}{\omega^2} \left( \underline{\underline{k}} \cdot \underline{\underline{k}} - \underline{\underline{I}} \right) + \underline{\underline{\epsilon}} \right] \cdot \tilde{\underline{E}} = 0 \quad (4)$$

$$\underline{\underline{k}} = \frac{\underline{k}}{k}$$

and sometimes we write  $n^2 = \frac{c^2 k^2}{\omega^2}$  the "index of refraction".

CONDITION THAT A NON ZERO SOLUTION FOR  $\tilde{\underline{E}}$  EXIST IS :-

$$\text{Det} \left\{ \frac{c^2 k^2}{\omega^2} \left( \underline{\underline{k}} \cdot \underline{\underline{k}} - \underline{\underline{I}} \right) + \underline{\underline{\epsilon}} \right\} = 0 \quad (5)$$

this will yield a dispersion relation

$$\omega = \omega(k, B_0)$$

Polarization of these "modes" or "eigenmodes" of the plasma are then obtained from (4) and (1).

(v) our goal is then to calculate  $\underline{\epsilon}$  and therefore  $\underline{\underline{\epsilon}}$  for a cold magnetized plasma.

(vi) we consider a plasma with electrons & one species of ion-charge  $z_i e$  and mass  $m_i$ . The equilibrium is homogeneous with  $B_0 = \text{constant}$ ,  $E_0 = 0$  and no zero order flows. Charge neutrality gives :-

$$n_{ie} = n_{io} z_i e$$

we calculate the perturbed velocities of ions and electrons since

$$\underline{J} = n_{oi} z_i e \delta v_i - n_{oe} e \delta v_e \quad (6)$$

(vii) Now consider the electrons - the ion current will be written down by analogy.

$$\frac{\partial \delta V_e}{\partial t} = -\frac{e}{m_e} \left[ \underline{E} + \frac{\delta V_e \times \underline{B}_0}{c} \right]$$

again taking

$$\delta V_e = \tilde{\delta V}_e e^{ik_r r - i\omega t}$$

$$-i\omega \tilde{\delta V}_e + \Omega_e \tilde{\delta V}_e \times \underline{b} = -\frac{e}{m_e} \underline{E}$$

$$\Omega_e = \frac{e B_0}{m_e c}$$

CYCLOTRON FREQUENCY

this equation is solved in many ways - one way is to choose  $\underline{b} = \frac{1}{2} \underline{z}$  and break into  $(x, y, z)$  coordinates another is to use the basis  $\underline{e}_{\pm} = \frac{1}{2} (\underline{x} \pm i\underline{y})$  I prefer a coordinate free solution. In general  $\underline{E}$ ,  $\underline{b}$  and  $\underline{E} \times \underline{b}$  are independent vectors so we write  $\delta V_e = p \underline{E} + q \underline{b} + r \underline{E} \times \underline{b}$  and solve for  $p$ ,  $q$  and  $r$  to get.

homework Q.1.

give this  $\rightarrow \delta V_e = \frac{-ie\omega}{m_e(\omega^2 - \Omega_e^2)} \left\{ \underline{E} - \frac{\Omega_e^2 (\underline{E} \cdot \underline{b}) \underline{b}}{\omega^2} - \frac{i\Omega_e}{\omega} \underline{E} \times \underline{b} \right\}$

Substituting this and an equivalent expression for ions into (6) identifying  $\underline{g}$  and constructing  $\underline{\xi}$  we get

homework Q.2.

give this  $\rightarrow$

$$(Viii) \quad \frac{\underline{E} \cdot \underline{E}}{\underline{\xi} \cdot \underline{\xi}} = \epsilon_1 \left( \underline{I} - \underline{b} \underline{b} \right) \cdot \underline{E} + \epsilon_{||} \underline{b} \underline{b} \cdot \underline{E} - ig \underline{b} \times \underline{E} \quad (7)$$

where

$$\epsilon_1 = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} = \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}$$

$$\epsilon_{||} = 1 - \frac{\omega_{pe}^2 + \omega_{pi}^2}{\omega^2}$$

$$g = \frac{\omega_{pe}^2 \Omega_e}{\omega(\omega^2 - \Omega_e^2)} - \frac{\omega_{pi}^2 \Omega_i}{\omega(\omega^2 - \Omega_i^2)}$$

$$\epsilon_1 = \frac{4\pi n_{e0} e^2}{m_e}$$

$$\epsilon_{||} = \frac{4\pi n_{i0} Z_i^2 e^2}{m_i}$$

$$g = \frac{2ieB_0}{m_i c}, \Omega_e = \frac{eB_0}{m_e c}$$

(ix) We note that (for real  $\omega$ )  $\hat{\underline{\epsilon}}$  is hermitian and independent of  $\hat{k}$ . We shall treat  $c^2 k^2 / \omega^2 = n^2$  as the quantity to be determined from ⑧ FOR A GIVEN  $\omega$  AND  $\hat{k}$ . Using the fact that  $\hat{\underline{\epsilon}}$  and  $(\hat{k}\hat{k} - \underline{I})$  are hermitian one can easily show that  $n^2$  is real. (see homework). From ⑧ we get after some algebra (Homework)

Homework Q.3.  
Derive this  $\rightarrow$

$$A \left( \frac{kc}{\omega} \right)^4 + B \left( \frac{kc}{\omega} \right)^2 + C = 0$$

"BOOKER QUARTIC"

$$\text{where: } A = \epsilon_{\perp} \sin^2 \theta + \epsilon_{\parallel} \cos^2 \theta = A(\omega, \theta, B_0)$$

$$B = -\epsilon_{\perp} \epsilon_{\parallel} (1 + \cos^2 \theta) - (\epsilon_{\perp}^2 - g^2) \sin^2 \theta = B(\omega, \theta, B_0)$$

$$C = \epsilon_{\parallel} (\epsilon_{\perp}^2 - g^2) = C(\omega, \theta, B_0)$$

and  $\hat{k} \cdot \hat{b} = \cos \theta$ , thus we can find  $k = k(\omega, \theta, B_0)$  from 1.

$$n^2 = \left( \frac{kc}{\omega} \right)^2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

Two roots in each direction.

+ sign is called the "Fast Wave".

- sign is called the "Slow Wave".

Rearranging the terms we can solve for  $\tan^2 \theta$

Homework  
Q.4.  
Obtain this  $\rightarrow$

$$\tan^2 \theta = - \frac{\epsilon_{\parallel} (n^2 - (\epsilon_{\perp} + g)) (n^2 - (\epsilon_{\perp} - g))}{(\epsilon_{\perp} n^2 - (\epsilon_{\perp}^2 - g^2)) (n^2 - \epsilon_{\parallel})}$$

This gives  
 $\theta = \theta(n, \omega, B_0)$

## 222b. Lecture #3 Parallel Propagating Waves

(i) last time we arrived at the equation for cold plasma waves.

$$\textcircled{1} \quad \left\{ \frac{c^2 k^2}{\omega^2} \left( \hat{k} \hat{k} - \hat{\mathbf{I}} \right) + \epsilon_{\perp} \left( \hat{\mathbf{I}} - \hat{\mathbf{b}} \hat{\mathbf{b}} \right) + \epsilon_{\parallel} \hat{\mathbf{b}} \hat{\mathbf{b}} - i g \hat{\mathbf{b}} \times \hat{\mathbf{I}} \right\} \cdot \hat{\mathbf{E}} = 0$$

with

$$\epsilon_{\perp} = 1 - \frac{\omega_{pe}^2}{\omega^2 - \Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2 - \Omega_i^2}$$

$$\epsilon_{\parallel} = 1 - \frac{\omega_{pe}^2 + \omega_{pi}^2}{\omega^2} \xrightarrow{\text{small}} g = \frac{\omega_{pe}^2 \Omega_e}{\omega(\omega^2 - \Omega_e^2)} - \frac{\omega_{pi}^2 \Omega_i}{\omega(\omega^2 - \Omega_i^2)}$$

and:  $\omega_{pe}^2 = \frac{4\pi n_e e^2}{m_e}$ ,  $\omega_{pi}^2 = \frac{4\pi n_i Z^2 e^2}{m_i}$

$$\Omega_e = \frac{eB}{m_e c}, \quad \Omega_i = \frac{zeB}{m_i c}$$

and note charge neutrality gives: -  $\frac{\omega_{pi}^2}{\Omega_i^2} = \frac{\omega_{pe}^2}{\Omega_e^2}$

also note that  $\frac{\omega_{pi}^2}{\Omega_i^2} = \left( \frac{4\pi n_i m_i}{B^2} \right) c^2 = \frac{c^2}{V_A^2} \leftarrow \text{Alfvén velocity.}$

(ii) From last time's notes (Q3 of homework therein) the determinant of  $\{\dots\}$  in  $\textcircled{1}$  yields:

$$\boxed{A \left( \frac{kc}{\omega} \right)^4 + B \left( \frac{kc}{\omega} \right)^2 + C = 0} \quad \textcircled{2}$$

"Booker Quartic"

$$\hat{\mathbf{k}} \cdot \hat{\mathbf{b}} = \cos \theta$$

$$A = A(\omega, \theta) = \epsilon_{\perp} \sin^2 \theta + \epsilon_{\parallel} \cos^2 \theta$$

$$B = -\epsilon_{\perp} \epsilon_{\parallel} (1 + \cos^2 \theta) - (\epsilon_{\perp}^2 - g^2) \sin^2 \theta$$

$$C = \epsilon_{\parallel} (\epsilon_{\perp}^2 - g^2)$$

$\textcircled{2}$  yields

$\left( \frac{kc}{\omega} \right)^2$  as a function of  $\omega$  and  $\theta$ .

(iii) Solving the quartic yields:

$$n = \frac{c}{v_{\text{phase}}} = \frac{kc}{\omega}$$

$$n^2 = \frac{k^2 c^2}{\omega^2} = - \frac{B \pm \sqrt{B^2 - 4AC}}{2A}$$

+ sign is called the "fast wave"

- sign is called the "slow wave"

(iv) Resonance occurs where  $A \rightarrow 0$  then  $k \rightarrow \infty$   
also occurs when  $\epsilon_{\perp} \rightarrow \infty$ .

(v) Another useful formula is (Q4 from last time)

$$\tan^2 \theta = - \frac{\epsilon_{\parallel}(n^2 - (\epsilon_{\perp} + g)) (n^2 - (\epsilon_{\perp} - g))}{(\epsilon_{\perp} n^2 - (\epsilon_{\perp}^2 - g^2)) (n^2 - \epsilon_{\parallel})}$$

"  
(vi) Parallel Propagation       $\underline{k} = \underline{b}$        $\cos \theta = 1$ "

from ①

$$\left[ \left( \epsilon_{\perp} - \frac{k^2 c^2}{\omega^2} \right) \left( \underline{\underline{\epsilon}} - \underline{\underline{b}} \cdot \underline{\underline{b}} \right) + \epsilon_{\parallel} \underline{\underline{b}} \cdot \underline{\underline{b}} - i g \underline{b} \times \underline{\underline{\epsilon}} \right] \cdot \underline{\underline{E}} = 0 \quad ③$$

rather than work with the Booker Quartic let's work from ③

(vii) Dot with  $\underline{b}$  to obtain

$$\epsilon_{\parallel} \underline{\underline{E}} \cdot \underline{b} = 0 \Rightarrow \text{either } \underline{\underline{E}} \cdot \underline{b} = 0 \text{ or } \epsilon_{\parallel} = 0$$

(viii) So first root is :- PLASMA WAVE.  $\epsilon_{\parallel} = 0 \Rightarrow \omega^2 = \omega_{pe}^2$

No  $\underline{k}$  dependence       $\underline{\underline{E}} = E_{\parallel} \underline{b} = - \nabla \phi$  electrostatic

Familiar charge oscillations along lines

(ix) Therefore other roots must have  $\underline{E} \cdot \underline{b} = 0$  and ③ becomes.

$$\left( \epsilon_{\perp} - \frac{k^2 c^2}{\omega^2} \right) \underline{E} - i g \underline{b} \times \underline{E} = 0 \quad \text{--- ④}$$

taking  $\underline{b} \times \underline{④}$  and substituting back in we get

$$\left( \epsilon_{\perp} - \frac{k^2 c^2}{\omega^2} \right)^2 = g^2 \Rightarrow \frac{k^2 c^2}{\omega^2} = \epsilon_{\perp} \pm g$$

$$\frac{k^2 c^2}{\omega^2} = 1 - \frac{\omega_{pe}^2}{\omega(\omega \pm \Omega_e)} - \frac{\omega_{pi}^2}{\omega(\omega \mp \Omega_i)}$$

where top signs give  $\underline{E} + i \underline{b} \times \underline{E} = 0$  Left handed circular polarization  
 $\underline{E} - i \underline{b} \times \underline{E} = 0$  Right handed circular polarization.

Defining:  $\omega_L = -\frac{\Omega_e}{2} + \frac{1}{2} \sqrt{-\Omega_e^2 + 4(\omega_{pe}^2 + \Omega_e \Omega_i)}$

$$\omega_R = \omega_L + \Omega_e \quad \text{note: } \omega_L \omega_R = \omega_{pe}^2 + \Omega_i \Omega_e$$

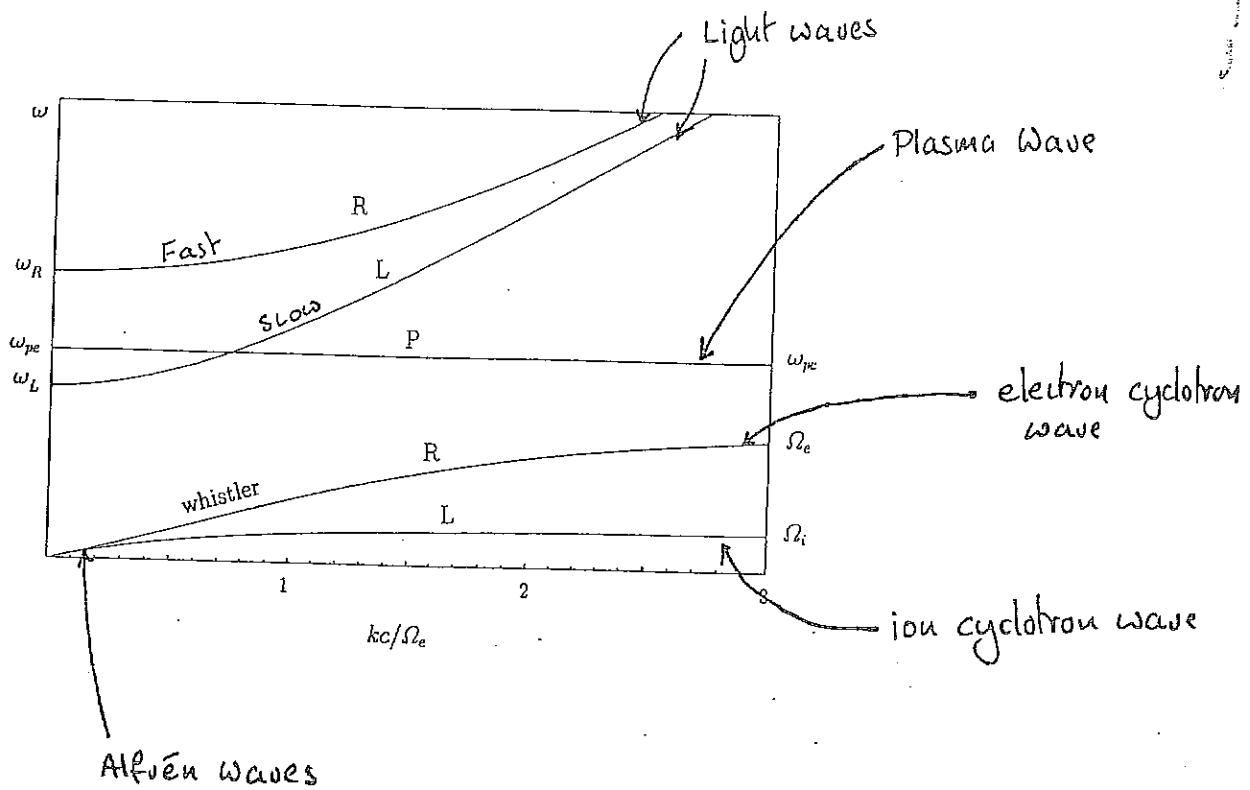
LEFT HANDED:

$$\frac{k_L^2 c^2}{\omega^2} = \frac{(\omega - \omega_L)(\omega + \omega_R)}{(\omega + \Omega_e)(\omega - \Omega_i)}$$

RIGHT HANDED:

$$\frac{k_R^2 c^2}{\omega^2} = \frac{(\omega + \omega_i)(\omega - \omega_R)}{(\omega - \Omega_e)(\omega + \Omega_i)}$$

$\omega_{pe} > \Omega_e$ .



## LIMITS

Alfvén waves:  $\omega^2 = k^2 V_A^2$   
 $\omega \ll \Omega_i \ll \Omega_e < \omega_{pe}$

two waves Right & Left.  
 $E \times B$  flows of ions and electrons cancel.

Ion Cyclotron:  $\omega - \Omega_i \approx -\frac{\Omega_i^3}{k^2 V_A^2}$   
 $\omega \approx \Omega_i$

left handed wave.  
electrons unimportant.

Whistler Wave:  $\Omega_i \ll \omega \ll \Omega_e$

No ion current (too fast)

$$\boxed{\omega = \frac{k^2 \Omega_e}{\omega_{pe}^2}}$$

$E \times B$  electron current.

$$J = \frac{e n_e c}{B} (\underline{E} \times \underline{B})$$

## Electron Cyclotron:

$$(\omega - \Omega_e) \approx \frac{\Omega_e}{\omega_{pe}} \left( 1 - \frac{k^2 c^2}{\omega^2} \right)^{-1}$$

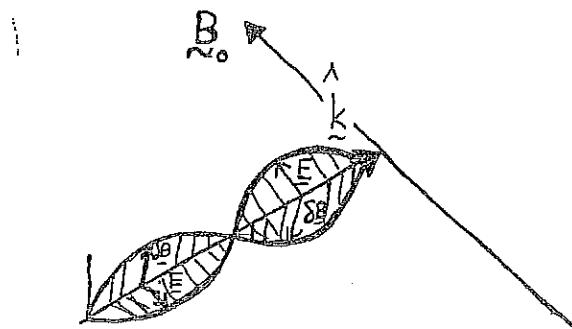
$$(i) \left\{ \frac{c^2 k^2}{\omega^2} (\hat{k} \cdot \hat{k} - 1) + \epsilon_{\perp} (1 - \frac{b \cdot b}{\omega^2}) + \epsilon_{\parallel} b \cdot b - i g \hat{k} \times \hat{B} \right\} \cdot \underline{E} = 0 \quad ①$$

This time  $\hat{k} \cdot \hat{b} = 0$

$$(ii) \text{ Dot } ① \text{ with } \hat{b}: \quad \left( \epsilon_{\parallel} - \frac{c^2 k^2}{\omega^2} \right) \hat{b} \cdot \underline{E} = 0 \quad \text{so either } \hat{b} \cdot \underline{E} = 0 \text{ or}$$

$$\text{ORDINARY MODE: } \epsilon_{\parallel} - \frac{c^2 k^2}{\omega^2} = 0 \Rightarrow \omega^2 = \omega_{pe}^2 + k^2 c^2$$

linearly polarized wave with  $\underline{E}$  along  $\hat{B}$  does not "feel" the magnetic field



(iii) Extraordinary Mode: Dot ① with  $\hat{k}$  and  $\hat{k} \times \hat{B}$  we get

$$\begin{aligned} \epsilon_{\perp} \hat{k} \cdot \underline{E} - i g (\hat{k} \times \hat{b}) \cdot \underline{E} = 0 \\ i g \hat{k} \cdot \underline{E} + \left( \epsilon_{\perp} - \frac{k^2 c^2}{\omega^2} \right) \hat{k} \times \hat{b} \cdot \underline{E} = 0 \end{aligned} \quad \left. \begin{aligned} & \left( \epsilon_{\perp}, -i g \right) \begin{pmatrix} \hat{k} \cdot \underline{E} \\ \hat{k} \times \hat{b} \cdot \underline{E} \end{pmatrix} = 0 \\ & \left( i g, \left( \epsilon_{\perp} - \frac{k^2 c^2}{\omega^2} \right) \right) \begin{pmatrix} \hat{k} \cdot \underline{E} \\ \hat{k} \times \hat{b} \cdot \underline{E} \end{pmatrix} = 0 \end{aligned} \right\} = C$$

Taking the determinant of the  $2 \times 2$  matrix

$$\Rightarrow \epsilon_{\perp} \left( \epsilon_{\perp} - \frac{c^2 k^2}{\omega^2} \right) - g^2 = 0$$

after a little algebra :-

$$\frac{k^2 c^2}{\omega^2} = \frac{(\omega^2 - \omega_R^2)(\omega^2 - \omega_L^2)}{(\omega^2 - \omega_{UH}^2)(\omega^2 - \omega_{LH}^2)}$$

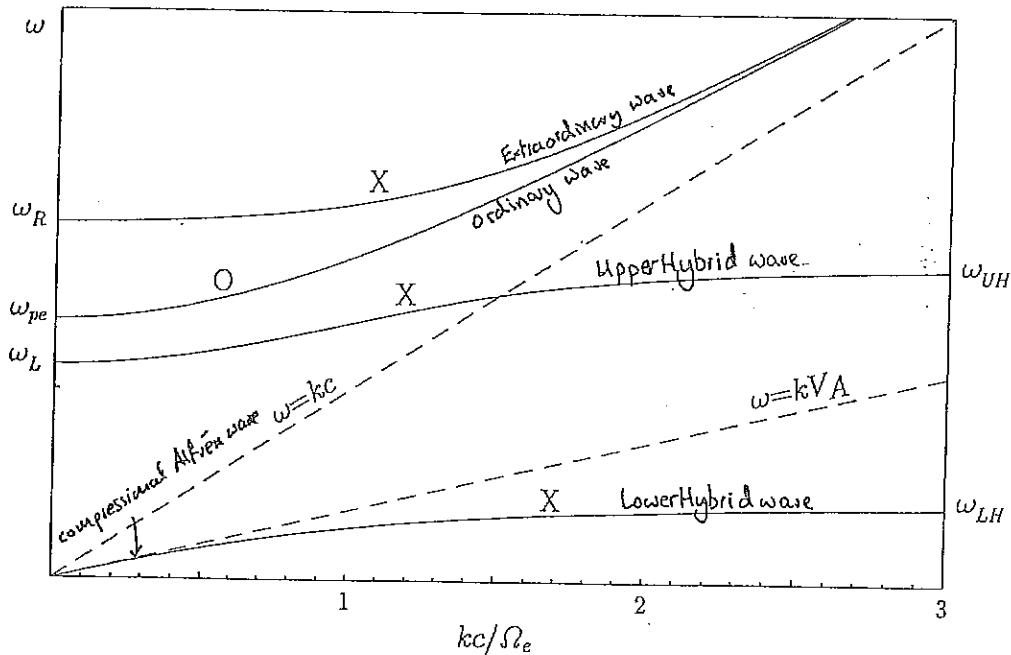
as before

$$\omega_L = -\frac{\Omega_e}{2} + \frac{1}{2} \sqrt{\Omega_e^2 + k(\omega_{pe}^2 + \Omega_e \Omega_i)} \quad \omega_R = \omega_L + \Omega_e$$

and we have two new important frequencies

$$\omega_{UH}^2 = \omega_{pe}^2 + \Omega_e^2 \quad \text{UPPER HYBRID frequency.}$$

$$\omega_{LH}^2 = \Omega_e \Omega_i \frac{(\omega_{pe}^2 + \Omega_e \Omega_i)}{(\omega_{pe}^2 + \Omega_e^2)} \quad \text{small. LOWER HYBRID frequency}$$



Perpendicular Propagation.

Physics!

LOWER HYBRID WAVES

$$\omega \sim \sqrt{\Omega_i \lambda_e}$$

Ions are unmagnetized:

$$V_i \approx \frac{1}{-i\omega} \frac{Ze}{m_i} \underline{E} + \dots \mathcal{O}\left(\frac{\omega_i^2}{\omega^2}\right)$$

Electrons are magnetized:

$$V_e \approx \frac{c \underline{E} \times \underline{B}}{B} + \frac{c (-i\omega)}{\Omega_e} \frac{\underline{E}}{B} + \mathcal{O}\left(\frac{\omega^2}{\Omega_e^2}\right)$$

" $E \times B$  drift"      "polarization drift"

Faraday's law

$$\frac{\omega \delta B}{c} = \underline{k} \times \underline{E} \quad (2)$$

↓

Maxwell's law

$$i \underline{k} \times \underline{\delta B} = \frac{4\pi}{c} \underline{j} - i \omega_r \underline{E}$$

substituting in for the currents we get:

$$\frac{i c k \times \delta B}{\omega} = \frac{i \omega_{pi}^2}{\omega^2} \underline{E} + \frac{i \omega_{pe}^2}{\Omega_e} \frac{\underline{E} \times \underline{B}}{B} - i \frac{\omega_{pe}^2}{\Omega_e^2} \underline{E} - i \underline{E} \quad (3)$$

ION CURRENT      ELECTRON CURRENT      ELECTRON POLARIZATION DRIFT.  
↓  
displacement current.

To get the behavior we set  $k \rightarrow \infty$  and let us take the power series

$$\underline{E} = \underline{E}_0 + \frac{\underline{E}_1}{k} + \frac{\underline{E}_2}{k^2} \dots \quad \underline{\delta B} = \underline{\delta B}_0 + \frac{1}{k} \underline{\delta B}_1$$

LOWEST ORDER FROM (2)

$$0 = \hat{k} \times \underline{E}_0 = 0 \Rightarrow \underline{E}_0 = \frac{1}{k} \underline{E}_0$$

LOWEST ORDER FROM (3)

$$\hat{k} \times \underline{\delta B}_0 = 0 \Rightarrow \underline{\delta B}_0 = \underline{\delta B}_0 \hat{k} \quad \text{but } \hat{k} \cdot \underline{\delta B} = 0 \text{ &} \\ \underline{\delta B}_0 = 0. \quad \underline{\text{ELECTROSTATIC}}$$

To next order Equation ③

$$\frac{ic}{\omega} \hat{k} \times \delta \underline{B}_i = \frac{i\omega_{pi}^2}{\omega^2} E_0 \hat{k} + \frac{\omega_{pe}^2 (\underline{k} \times \underline{b})}{\sqrt{\epsilon_0 \mu_0}} E_0 - \frac{i\omega_{pe}^2}{\sqrt{\epsilon_0 \mu_0}} E_0 \hat{k} - i E_0 \hat{k}$$

eliminate  $\delta \underline{B}_i$  by dotting with  $\hat{k}$  yields.

$$\left( \frac{\omega_{pi}^2}{\omega^2} - \frac{\omega_{pe}^2}{\sqrt{\epsilon_0 \mu_0}} - 1 \right) = 0 \Rightarrow$$

$$\omega^2 = \frac{\mu_0 \sigma_i \omega_{pe}^2}{\omega_{pe}^2 + \sqrt{\epsilon_0 \mu_0}}$$

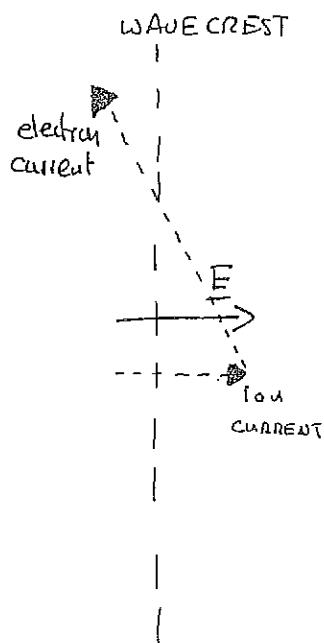
LOWER HYBRID FREQUENCY

CURRENT ALONG  $\hat{k}$  ( $\perp$  to  $\underline{B}_0$ ) CANCELS

IT IS FROM : ION CURRENT (UNMAGNETIZED)  
+ ELECTRON POLARIZATION CURRENT  
+ DISPLACEMENT CURRENT.

Note  $E \times \underline{B}$  current is largest current but it is not controlling the mode it is producing a small ( $\frac{1}{k}$ )  $\delta \underline{B}$ .

$\underline{B}$  into paper



WAVE CREST



ION CURRENT

NO CHARGE BUILD  
UP BECAUSE CURRENTS  
CANCEL.

ELECTRON  
CURRENT

## 222b. lecture #5: Vlasov Equation and Electrostatic Waves I

- (i) In 222a. we discussed the kinetic equation for the smoothed (coarse grained) distribution function. If you don't remember this discussion read the notes or the slightly different point of view at the beginning of Chapt. 3 in Hazeltine and Waelbroeck.
- (ii) Without the collision term the kinetic equation for the smoothed distribution function is the VLASOV EQUATION.

$$\frac{\partial f_\alpha}{\partial t} + \underline{v} \cdot \frac{\partial f_\alpha}{\partial \underline{r}} + \frac{q_\alpha}{m_\alpha} \left( \underline{E} + \underline{v} \times \underline{B} \right) \cdot \frac{\partial f_\alpha}{\partial \underline{v}} = 0$$

$\alpha$  = electrons or ions.

where  $\underline{E}$  and  $\underline{B}$  are the smoothed/coarse grained electric and magnetic fields. The smoothed fields satisfy Maxwell's equations.

$$\nabla \cdot \underline{E} = 4\pi \rho \quad , \quad \nabla \cdot \underline{B} = 0$$

$$\frac{\partial \underline{E}}{\partial t} = c \nabla \times \underline{B} - 4\pi \underline{J} \quad , \quad \frac{\partial \underline{B}}{\partial t} = -c \nabla \times \underline{E}$$

with smoothed charge density and current density given by

$$\rho = \sum_{\alpha} q_{\alpha} \int d^3 v f_{\alpha} \quad \text{and} \quad \underline{J} = \sum_{\alpha} q_{\alpha} \int d^3 v \underline{v}_{\alpha} f_{\alpha}$$

- (iii) In this and the next few lectures we will (for simplicity) examine an unmagnetized electron plasma in a background of stationary homogeneous background. i.e.  $f_i = f_i(v)$  - the ensures average neutrality. We will assume the electric fields involved are electrostatic so that

$$\underline{E} = -\nabla \phi \quad \underline{B} = 0.$$

(iv) Some very important properties of the equations are worth keeping in mind. Suppose we follow single electrons in the electric field - i.e. we solve the equation of motion.

$$\frac{d\mathbf{r}}{dt} = \mathbf{v} \quad \text{and} \quad \frac{d\mathbf{v}}{dt} = -\frac{e}{m} \mathbf{E}(\mathbf{r}, t)$$

to yield a solution

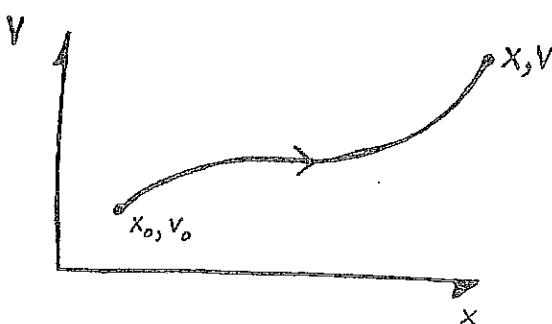
$$\mathbf{r} = \mathbf{r}(r_0, v_0, t)$$

$$\mathbf{v} = \mathbf{v}(r_0, v_0, t)$$

TRAJECTORY WITH

INITIAL CONDITIONS  $\mathbf{r} = \mathbf{r}_0, \mathbf{v} = \mathbf{v}_0$  at time  $t = 0$ .

IN 1D



- We assume that in principle we can solve for all initial conditions,  $r_0$  and  $v_0$ .

- The inverse can also be defined

$$r_0 = r_0(r, v, t)$$

$$v_0 = v_0(r, v, t)$$

because the trajectories define a one to one mapping.

Now by the rules of partial differentiation

$$\left( \frac{\partial f_e}{\partial t} \right)_{r_0, v_0} = \left( \frac{\partial f_e}{\partial t} \right)_{r, v} + \left( \frac{dr}{dt} \right) \cdot \left( \frac{\partial f_e}{\partial r} \right)_{v, t} + \frac{dv}{dt} \cdot \left( \frac{\partial f_e}{\partial v} \right)_{r, t}$$

$$= \underbrace{\frac{\partial f_e}{\partial t} + \frac{v_0 \partial f_e}{\partial r} - \frac{eE}{m} \frac{\partial f_e}{\partial v}}_{\text{VLA SOV EQ.}} = 0$$

$$\left( \frac{df}{dt} \right)_{r_0, v_0} = 0$$

$f = \text{constant along trajectories of electrons.}$

$$\Rightarrow f(r, v, t) = f(r_0, v_0, 0) \quad \text{using: } \begin{cases} r_0 = r_0(r, v, t) \\ v_0 = v_0(r, v, t) \end{cases} \text{ then}$$

$$f(r, v, t) = f(r_0(r, v, t), v_0(r, v, t), 0)$$

"SOLUTION BY METHOD OF CHARACTERISTICS!"

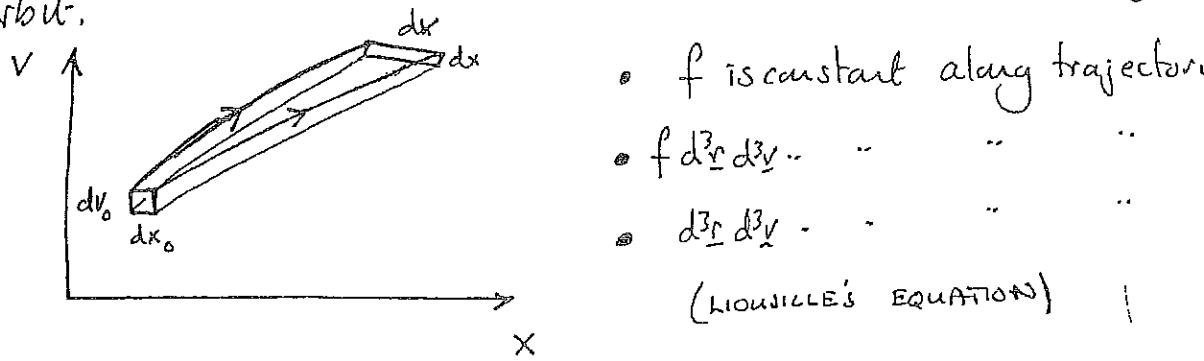
(v) It must be admitted that this formula is not very useful in practice - except in linear calculation - since  $\underline{E}$  is not given but it itself depends on  $f$  through POISSON'S EQUATION.

$$\nabla \cdot \underline{E} = -4\pi e \int d^3r f$$

but it gives an intuitive picture of the evolution of  $f$ .

(vi) Note  $f d^3r d^3V = \text{number of electrons in a little box of configuration space } d^3r d^3V$ .

and this number cannot change if we evolve the box along the orbit.



(vii)  $f$  acts like the density of an incompressible fluid in configuration space.

(viii) IMAGINE CHOPPING THE CONFIGURATION SPACE INTO LOTS OF LITTLE BOXES. THE EFFECT, OVER TIME, OF THE ELECTRIC FIELD IS TO REARRANGE THE BOXES BUT NOT TO CHANGE THE VALUE OF  $f$  IN THE BOX OR THE VOLUME OF THE BOX.

(ix) Energy Conservation. From Vlasov equation and Maxwell's equations we obtain

$$\int d^3r \int d^3V \left[ \frac{1}{2} mv^2 f \right] + \int d^3r \frac{E^2}{8\pi} = \text{constant.}$$

(\*) Suppose you ask the question - can we rearrange the box's so as to reduce the kinetic energy thereby increasing the electric field of some wave/instability in the plasma? There is an obvious case where you can't: specifically if  $f$  peaks at  $|V| = 0$  and decreases monotonically with increasing  $|V|$  and it is isotropic - (i.e. does not depend on direction of  $V$ ) then all rearrangements will increase the Kinetic Energy. This means such distributions are always stable. (Gardiner's theorem).

## Lecture #6. Landau's Solution of the Initial Value Problem.

- (i) Today we evaluate the linearized response to a small perturbation of the electron distribution function at time  $t=0$ .
- (ii) Imagine that at time  $t=0$   $f_e(\underline{r}, \underline{v}, 0) = f_0(v) + g(\underline{r}, \underline{v})$   $\leftarrow$  small.  
 for  $t > 0$  we write  $f_e(\underline{r}, \underline{v}, t) = f_0(\underline{r}, \underline{v}, t) + \delta f(\underline{r}, \underline{v}, t)$   
 with  $\delta f \ll f_0$  and  $E = -\nabla\phi$  small.

(iii) UNPERTURBED STATE:  $\frac{\partial f_0}{\partial t} + \underline{v} \cdot \frac{\partial f_0}{\partial \underline{r}} = 0 \Rightarrow f_0(\underline{r}, \underline{v}, t) = f_0(v)$

unperturbed charge =  $-e \int f_0(v) d^3v$  + ion charge = 0 by assumption.

(iv) PERTURBED STATE:  
Linearization  $\frac{\partial \delta f}{\partial t} + \underline{v} \cdot \frac{\partial \delta f}{\partial \underline{r}} = \cancel{\frac{1}{m} e \nabla \phi \cdot \frac{\partial f_0}{\partial \underline{v}}}$

POISSON'S EQ.  $\nabla^2\phi = k\pi e \int \delta f d^3v$

(v) Look for solutions that are plane waves:

$$\delta f(\underline{r}, \underline{v}, t) = \delta f_k(\underline{v}, t) e^{i \underline{k} \cdot \underline{r}}$$

$$\phi(\underline{r}, t) = \phi_k(t) e^{i \underline{k} \cdot \underline{r}}$$

$$g(\underline{r}, \underline{v}) = g_k(v) e^{i \underline{k} \cdot \underline{r}}$$

therefore:-

$$\left\{ \begin{array}{l} \frac{\partial}{\partial t} \delta f_k + ik \cdot v \delta f_k = -\frac{ie\phi_k k}{m_e} \frac{\partial f_0}{\partial v} \\ k^2 \phi_k = -4\pi e \int \delta f_k d^3 v \end{array} \right. \quad \begin{array}{l} (1) \\ (2) \end{array}$$

(vi) We solve these equations by a Laplace transform method:

defines:-

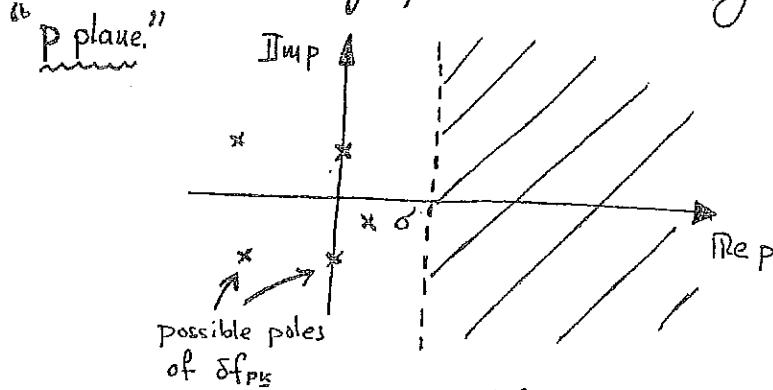
$$\delta f_{pk}(v) = \int_0^\infty e^{-pt} \delta f_k(v, t) dt$$

Laplace Transform.  
p can be complex

Suppose:  $\delta f_k(v, t)$  is finite for all  $t$  and as  $t \rightarrow \infty, \delta f_k(v, t) < e^{\sigma t}$

Then the integral exists as long as  $\operatorname{Re} p \geq \sigma$

(v) The Laplace transform  $\delta f_{pk}$  is therefore defined only for values of  $p$  which satisfy  $\operatorname{Re} p \geq \sigma$ .



since for  $\operatorname{Re} p \geq \sigma$   $\delta f_{pk}$  is finite THERE ARE NO POLES OF  $\delta f_{pk}$  TO THE RIGHT OF  $\sigma$  LINE.

Now since

$$\frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp e^{p(t'-t)} = \delta(t'-t)$$

→  $\delta f_k(v, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} dp e^{pt} \delta f_{kp}(v)$

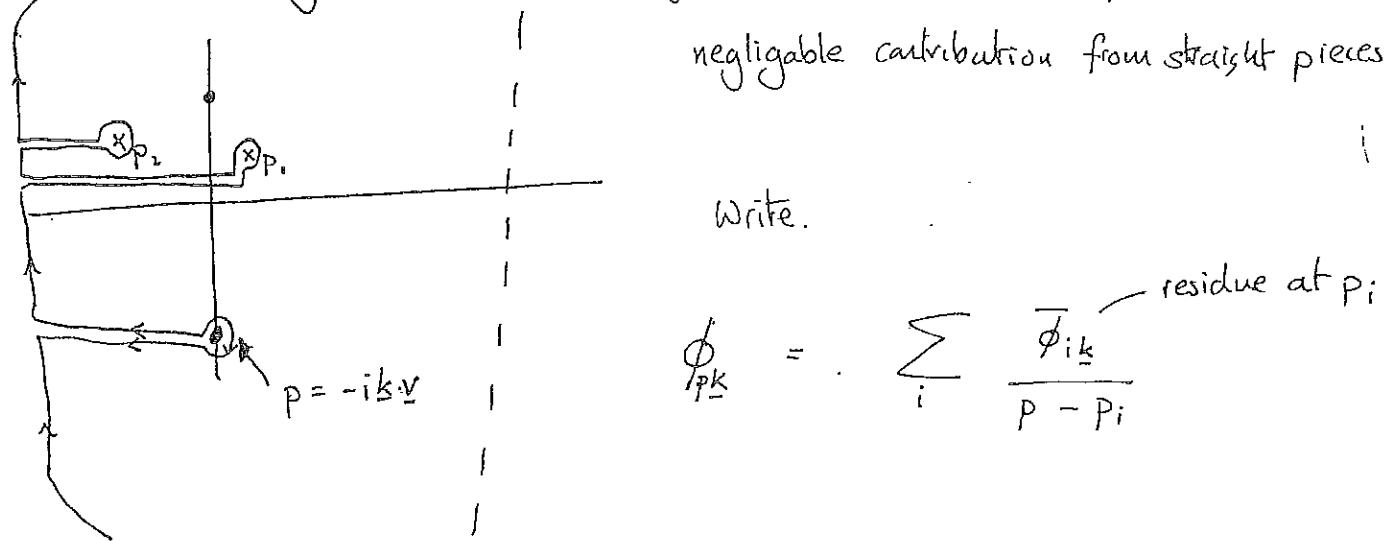
(vi) Now Laplace transform ①      NOTE:-  $\int_0^\infty e^{-pt} \frac{\partial \delta f_k}{\partial t} dt = -g_k(v) + p \delta f_{pk}$   
 (use integration by parts)

THUS:

$$\boxed{\delta f_{pk} = \frac{-ie \phi_{pk} k \cdot \frac{\partial f_0}{\partial v} + g_k(v)}{(p + ik \cdot v)}} \quad (3)$$

The poles are at the poles of  $\phi_{pk}$  and at  $p = -ik \cdot v$ .

We can do the inverse transform by distorting the contour  
 this means defining  $\phi_{pk}$  everywhere in the complex  $p$  plane as  
 the analytic continuation of it's value for  $\operatorname{Re} p > 2$ .



$$\phi_{pk} = \sum_i \frac{\bar{\phi}_{ik}}{p - p_i} \quad \text{residue at } p_i$$

$$\delta f_k(t) = -\frac{ie}{m_e} \left[ \sum_i \frac{\bar{\phi}_{ik} e^{pit}}{p_i + ik \cdot v} \right] \frac{k \cdot \frac{\partial f_0}{\partial v}}{p - p_i} - \frac{ie}{m_e} \bar{\phi}_{ik, v, k} \frac{k \cdot \frac{\partial f_0}{\partial v}}{p - p_i} e^{-ik \cdot v t}$$

From pole at  $p = -ik \cdot v$

from poles of  $\phi_{pk}$

BALLISTIC RESPONSE

(vii) Now to find the poles of  $\phi_{pk}$  we use the Laplace transform of equation ② with  $\mathcal{F}_{pk}$  from ③ this gives

$$\phi_{pk} = \frac{-i\pi e}{k^2 \epsilon(p, k)} \int \frac{g_k(v) d^3 v}{p + ik \cdot v}$$

$$\epsilon(p, k) = 1 - \frac{4\pi e^2}{mk^2} \int i k \cdot \frac{\partial f_0}{\partial v} \frac{d^3 v}{p + ik \cdot v}$$

$\Rightarrow$  Poles must be at the zeros of  $\epsilon(p, k)$  as long as the top numerator is non singular.

## Lectures #7 Landau Damping II.

we are studying the initial value problem with electrostatic perturbations

(i)  $f = f_0(v) + \delta f_k(v, t) e^{ik \cdot r}$  One plane wave.  
 "equilibrium distribution"

VLASOV EQN.

$$\Rightarrow \frac{\partial \delta f_k}{\partial t} + ik \cdot v \delta f_k = -\frac{ie}{m} \phi_k \vec{k} \cdot \frac{\partial f_0}{\partial v} \quad (1)$$

$$E = -\nabla \phi = -ik \phi_k e^{ik \cdot r}$$

Initial Condition

POISSON'S EQN.

$$\delta f_k(v, t=0) = g_k(v)$$

$$k^2 \phi_k = -4\pi e \int \delta f_k d^3 v \quad (2)$$

(ii) LAPLACE TRANSFORM:

$$\delta f_{kp} = \int_0^\infty e^{-pt} \delta f_k(v, t) dt$$

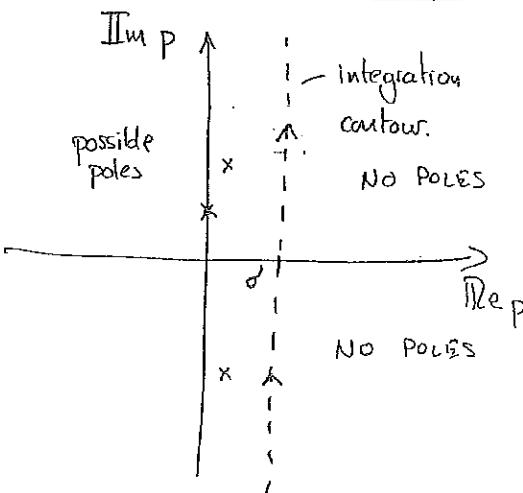
- Integrals only exist if  $\Re p$  is big enough.

$$\Re p \geq \sigma \quad \delta f_{kp}, \phi_{kp} \text{ ANALYTIC}$$

$$\phi_{kp} = \int_0^\infty e^{-pt} \phi_k(t) dt$$

FOR  $\Re p < \sigma$  WE CAN DEFINE  $\delta f_{kp}$  AND  $\phi_{kp}$  AS THE ANALYTIC CONTINUATION OF THESE FUNCTIONS.

### INVERSE TRANSFORMS



$$\delta f_k(v, t) = \frac{1}{2\pi} \int_{-\infty-i\infty}^{-\infty+i\infty} e^{pt} \delta f_{kp} dp$$

$$\phi_k(t) = \frac{1}{2\pi} \int_{-\infty-i\infty}^{-\infty+i\infty} e^{pt} \phi_{kp} dp$$

(iii) Laplace Transforming ① yields:-

$$\delta f_{k,p}(v) = \frac{1}{(p + ik \cdot v)} \left\{ g_k(v) - \frac{i e}{m} \phi_{kp} k \cdot \frac{\partial f_0}{\partial v} \right\} \quad ③$$

and ② yields

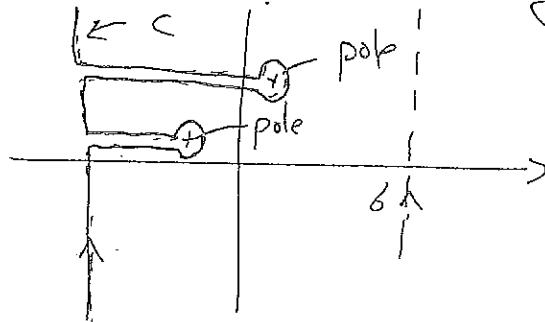
$$\phi_{kp} = \frac{k \pi e}{k^2 \epsilon(p, k)} \int \frac{g_k(v) d^3 v}{(p + ik \cdot v)} \quad ④$$

where  $\epsilon = 1 - \frac{4 \pi e^2}{k^2} \int k \cdot \frac{\partial f_0}{\partial v} \frac{d^3 v}{p + ik \cdot v}$  in ④ & ⑤ ⑤

(iv) In principle we now compute the integrals over  $v$ , for  $\Re p \geq \delta$  note from ③ there is a pole at  $p = -ik \cdot v$  so  $\delta$  must be greater than zero. We then put ④ into the inverse Laplace transform formula and get  $\phi_{kp}(t)$ . However this is a inconvenient way to find the long time behaviour.

Suppose we define  $\phi_{kp}$  for  $\Re p < \delta$  to be analytic continued from  $\phi_{kp}$  for  $\Re p > \delta$  - note this is simply a choice albeit a good choice. Then the inverse transform can be evaluated along a deformed contour  $C \rightarrow$

$$\phi_{kt} = \frac{1}{2\pi} \int_C e^{pt} \phi_{kp} dp$$



BUT WON'T

(v) WE COULD, DEFINE  $\phi_{kp}$  NON ANALYTICALLY IN THE REGION  $\Re p < 6$  BUT THEN THE CONTOUR WOULD "HANG UP" ON ANY POINT WHERE OUR DEFINED FUNCTION WAS <sup>WHERE IT</sup> NON ANALYTIC - FOR INSTANCE <sub>N</sub> HAD A JUMP.

$$(vi) \text{ Clearly } \phi_{kt} = i \sum_{n=1}^N \text{Residue}(\phi_{kp} \text{ at } n\text{th pole}) e^{pn t}$$

SUM OVER N POLES

SO "ALL WE NEED TO DO" NOW IS FIND THE POLES OF THE ANALYTICALLY CONTINUED  $\phi_{kp}$  GIVEN BY (4).

(v) DIFFICULTY How do we analytically continue integrals like

$$I(p, k) = \int \underline{k} \cdot \frac{d\underline{f}_0}{d\underline{v}} \frac{d^3 v}{p + ik \cdot \underline{v}}$$

?

To simplify we can integrate out velocities  $\perp$  to  $\underline{k} = k \hat{\underline{z}}$

Let

$$F_0 = \int d\underline{v}_x d\underline{v}_y f_0$$

$$\bar{I}(p, k) = k \int \frac{\partial F_0}{\partial v_z} \frac{dv_z}{p + ikv_z} = -i \int \frac{\partial F_0}{\partial v_z} \frac{dv_z}{v_z - \frac{ip}{k}}$$

note we take  $k > 0$  for simplicity.

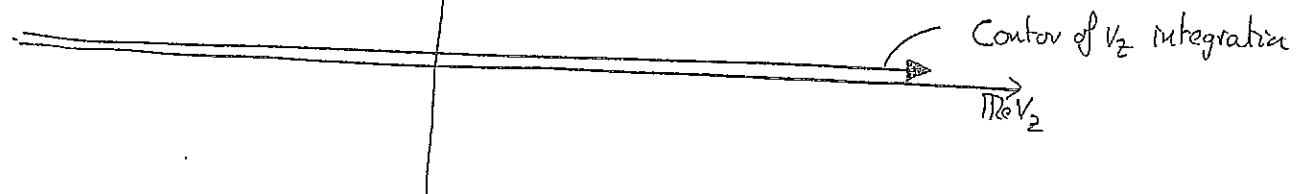
(vi) Consider a complex  $V_2$  space: with  $\operatorname{Re} p \geq \delta > 0$

Integral is along the real  $V_2$  line. Suppose  $\frac{\partial F_0}{\partial V_2}$  is analytic everywhere.

$$V_2 = \frac{ip}{k} \Rightarrow X$$

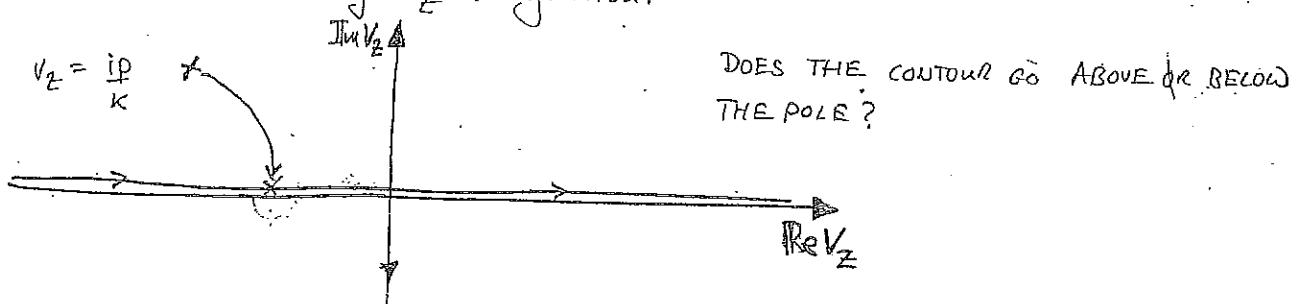
$\operatorname{Im} V_2 \uparrow$

and  $\rightarrow 0$  as  $|V_2| \rightarrow \infty$ .



The integral  $I$  is then finite for all  $\operatorname{Re} p \geq \delta$ .

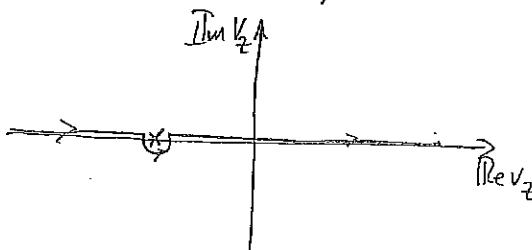
As we analytically continue the integral for  $\operatorname{Re} p < \delta$  we encounter a difficulty at  $\operatorname{Re} p = 0$ . The pole at  $V_2 = \frac{ip}{k}$  lies on the contour of  $V_2$  integration.



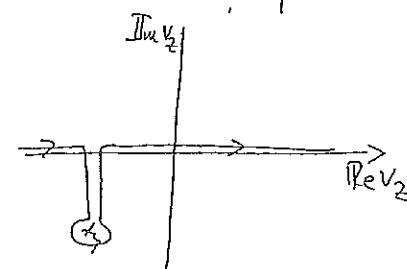
at  $p = ip_0 \pm i\epsilon$

To make  $I(p, k)$  analytic, we must have  $I(ip_0 + \epsilon, k) \Rightarrow I(ip_0 - \epsilon, k)$  as  $\epsilon \rightarrow 0$

Thus the pole can never cross the  $V_2$  contour so for  $\operatorname{Re} p = 0$



and for  $\operatorname{Re} p < 0$



WITH THIS DEFINITION OF  $I(p, k)$  IT IS ANALYTIC EVERYWHERE.

(vii) If we further define  $\int \frac{g_k(v) dv}{p + ik \cdot v}$  for  $\operatorname{Re} p < 0$  in

a similar way we can (as long as  $g_k(v)$  is analytic) treat this integral as analytic everywhere.

(viii) Clearly then the poles of  $\phi_{kp}$  come from the zeros of  $E(p, k)$  with  $I(p, k)$  defined in the way given above.

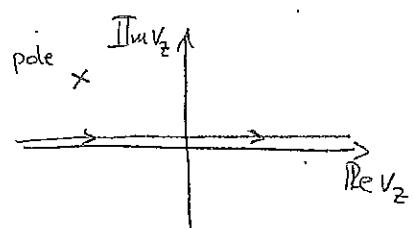
THUS TO FIND THE "MODES" OF THE PLASMA WE SOLVE

$$E(p, k) = 0 = \frac{4\pi e^2}{m k^2} \int_C \frac{\partial F_0}{\partial v_z} \frac{dv_z}{v_z - ip/k}$$

LANDAU CONTOUR

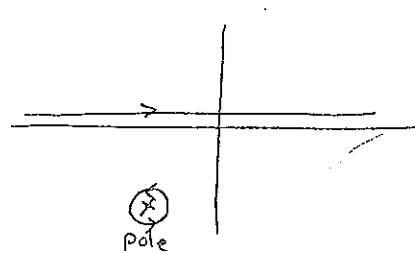
where for  $\operatorname{Re} p > 0$

$$\int_C \frac{\partial F_0}{\partial v_z} \frac{dv_z}{v_z - ip/k} = \int_{-\infty}^{\infty} \frac{\partial F_0}{\partial v_z} \frac{dv_z}{v_z - ip/k}$$



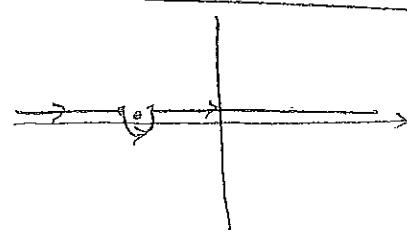
for  $\operatorname{Re} p < 0$

$$\int_C \frac{\partial F_0}{\partial v_z} \frac{dv_z}{v_z - ip/k} = \int_{-\infty}^{\infty} \frac{\partial F_0}{\partial v_z} \frac{dv_z}{v_z - ip/k} + 2\pi i \left( \frac{\partial F_0}{\partial v_z} \right)_{v_z = ip/k}$$



for  $\operatorname{Re} p = 0$  "PRINCIPLE PART"

$$\int_C \frac{\partial F_0}{\partial v_z} \frac{dv_z}{v_z - ip/k} = \int_{-\infty}^{\infty} \frac{\partial F_0}{\partial v_z} \frac{dv_z}{v_z - ip/k} + \pi i \left( \frac{\partial F_0}{\partial v_z} \right)_{v_z = ip/k}$$





## Lecture #8. Langmuir Waves and Landau Damping.

RECALL.

- (i) Finding the behavior for long times  $\phi \sim \sum \phi_n e^{P_n t}$  means we have to solve for the  $P_n$  from  $\epsilon(p_n, k) = 0$

$$\epsilon(p, k) = 0 = 1 - \frac{4\pi e^2}{m_e k^2} \int_{\text{LANDAU CONTOUR}} \frac{dF_0}{dV_z} \frac{dv_z}{V_z - \frac{ip}{k}}$$

$V_z$  contour  
always below  
pole at  $V_z = \frac{ip}{k}$

- (ii) Most people write  $p = -i\omega$  because often the solution has  $\omega = \omega_R + i\omega_I$  with  $\omega_I \ll \omega_R$  (weakly growing or damping)

NOTE: If  $\omega_I > 0$  GROWING MODE UNSTABLE.

If  $\omega_I < 0$  DAMPED MODE STABLE.

- (iii) One of the most important cases is the Maxwellian plasma [since it is the distribution in thermal equilibrium].

$$f_0 = \frac{n_0}{(2\pi)^{3/2}} \frac{1}{V_{th}^3} e^{-V_z^2/2V_{th}^2}$$

$$V_{th} = \sqrt{\frac{T_e}{M_e}}$$

thermal velocity

$$F_0 = \frac{n_0}{(2\pi)^{1/2}} \frac{1}{V_{th}} e^{-V_z^2/2V_{th}^2}$$

phase velocity

$$\text{Defining } t = \frac{V_z}{\sqrt{2}V_{th}}, \quad \omega_p^2 = \frac{4\pi n_0 e^2}{m_e} \text{ and } \xi = \frac{\omega}{k} \frac{1}{\sqrt{2}V_{th}}$$

- (iv) then we can write

$$\epsilon = 1 + \frac{\omega_p^2}{(k V_{th})^2} \int_{\text{LANDAU CONTOUR}} \frac{dt}{4\pi} \frac{te^{-t^2}}{t - \xi} = 0$$

(v) It is common to introduce the "Plasma Dispersion Function"

$$Z(\xi) = \int_{\text{LANDAY CONTOUR}} \frac{dt}{\sqrt{\pi}} \frac{e^{-t^2}}{t - \xi}$$

TABULATED IN FRIED AND CONINE  
see Plasma Formulary Pg. 30.

In terms of this function:

$$\epsilon(k, \xi) = 1 + \frac{\omega_p^2}{(k v_{th})^2} [1 + \xi Z(\xi)] = 0 \quad (1)$$

This is a transcendental equation and there are two ways to solve it: a) Numerically on a computer b) by looking for solutions in some range of  $k$  so we can simplify  $Z(\xi)$ .

(vi) COLDISH PLASMA: We look for solutions where  $\omega \sim \omega_p$  this is the cold plasma "plasma wave" and  $\frac{\omega}{k} \gg v_{th}$  this is equivalent to  $k \lambda_D \ll 1$   $\lambda_D = \frac{v_{th}}{\omega_p} \approx \text{DEBYE LENGTH}$ .

We want the corrections to  $\omega = \omega_p$  due to the thermal motion

COLDISH LIMIT:

$$\xi \gg 1$$

AND WE SHALL SEE  
THAT FOR THE SOLUTION

$$\omega = \omega_R + i \omega_I$$

$$\omega_I \ll \omega_R \sim \omega_p$$

$$\Rightarrow \xi_I \ll \xi_R$$

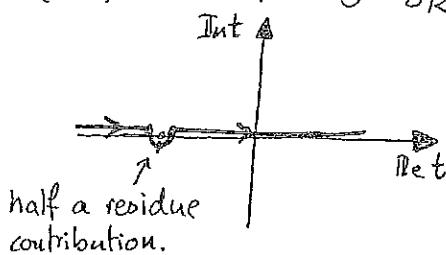
Taking Real and Imaginary parts of (1) we get expanding in  $\xi_I$

$$1 + \frac{\omega_p^2}{(k v_{th})^2} \left[ 1 + \operatorname{Re} \{ \xi_R Z(\xi_R) \} \right] = 0 \quad (2) \quad \text{Real part}$$

$$O = \frac{\omega_p^2}{(kV_{th})^2} \left[ \xi_I \frac{d}{d\xi_R} \left\{ \text{Re} \{ \bar{\zeta}_R z(\bar{\zeta}_R) \} \right\} + \text{Im} \{ \bar{\zeta}_R z(\bar{\zeta}_R) \} \right]$$

Imaginary part. ③

(vii) Now for  $\bar{\zeta} = \bar{\zeta}_R$  the Landau contour is



$$z(\bar{\zeta}_R) = \underbrace{\int_{\text{Real}} \frac{dt}{\sqrt{\pi}} \frac{e^{-t^2}}{t - \bar{\zeta}_R}}_{\text{Real}} + i\pi V_2 e^{-\bar{\zeta}_R^2}$$

$$\text{where } \int dt A \equiv \int_{-\infty}^{\bar{\zeta}_R - \delta} dt A + \int_{\bar{\zeta}_R + \delta}^{\infty} dt A \quad \begin{cases} \text{Principle part} \\ \text{Real} \end{cases}$$

Still not much progress but we now use the fact that  $\bar{\zeta}_R \gg 1$  to expand the principle part integral.

$$\text{for } t < \bar{\zeta}_R \quad \frac{1}{t - \bar{\zeta}_R} = -\frac{1}{\bar{\zeta}_R(1 - \frac{t}{\bar{\zeta}_R})} = -\frac{1}{\bar{\zeta}_R} \sum_{n=1}^{\infty} \left(\frac{t}{\bar{\zeta}_R}\right)^n$$

note that in the integral only  $t \sim \mathcal{O}(1)$  is important because of the  $e^{-t^2}$  factor so

$$\begin{aligned} \int_{\text{Real}} \frac{dt}{\sqrt{\pi}} \frac{e^{-t^2}}{t - \bar{\zeta}_R} &\approx -\frac{1}{\bar{\zeta}_R} \int_{-\infty}^{\infty} dt \sum_{n=0}^{\infty} \left(\frac{t}{\bar{\zeta}_R}\right)^n e^{-t^2} = -\frac{1}{\bar{\zeta}_R} \sum_{n=0}^{\infty} a_n \bar{\zeta}_R^{-n} \\ &\approx -\frac{1}{\bar{\zeta}_R} \left[ 1 + \frac{1}{2} \frac{1}{\bar{\zeta}_R^2} + \frac{3}{4} \frac{1}{\bar{\zeta}_R^4} \dots \right] \end{aligned}$$

Thus the real part of the dispersion relation becomes. See ②

$$(viii) 1 + \frac{\omega_p^2}{(kV_{th})^2} \left\{ -\frac{(kV_{th})^2}{\omega^2} - 3 \frac{(kV_{th})^4}{\omega^4} \dots \right\} = 0$$

small term.

Since  $\omega \sim \omega_p$  we can approximate  $\omega$  in the small term

$$\omega^2 = \omega_p^2 \left( 1 + \frac{3k^2 V_{th}^2}{\omega_p^2} \right) \quad \begin{matrix} \text{VALID WHEN } kV_{th} \ll p \\ \text{"LANGMUIR WAVES"} \end{matrix}$$

$$\text{GROUP VELOCITY} = 6 V_{th} \left( \frac{kV_{th}}{\omega} \right)$$

### (ix) Damping rate (due to Landau)

$$\xi_I = - \frac{\text{Im} \{ \xi_R Z(\xi_R) \}}{\frac{d}{d\xi_R} \text{Re} \xi_R Z(\xi_R)} = -\pi^{1/2} \xi_R^4 e^{-\xi_R^2} \quad \text{DAMPING}$$

$$\boxed{\frac{\gamma}{\omega_p} = \frac{-\pi^{1/2}}{\sqrt{8}} \left( \frac{\omega_p}{kV_{th}} \right)^3 \exp \left\{ -\frac{1}{2} \left( \frac{\omega_p}{kV_{th}} \right)^2 - \frac{3}{2} \right\}}$$

Note:  $\gamma \ll \omega_p$  because exponential is tiny our treatment breaks down when  $kV_{th} \sim \omega_p$ .

DAMPING COMES FROM THOSE ELECTRONS MOVING WITH THE WAVE  
i.e. Those with  $v_z = \frac{\omega}{k}$ . These electrons must be absorbing energy from the wave.

$$(X) \quad \omega^2 = \omega_p^2 + 3k^2 V_{th}^2$$

$$\begin{array}{c} \text{electrostatic} \\ \text{repulsion} \\ \text{of electrons} \end{array} \quad \begin{array}{c} \uparrow \\ \text{pressure} \\ \text{repulsion of} \\ \text{electrons.} \end{array}$$

$$+ \nabla p_e - \nabla p_e + \nabla p_e$$

$$+ - +$$

$$+ -eE - -eE + -eE$$

$$+ - +$$

$$+ - +$$

## 222b. Lecture # 9. Landau Damping - Some Physics

(i) Recall that for electrostatic waves

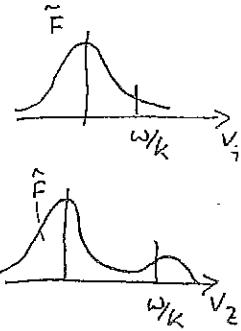
$$\sigma = 1 - \frac{\omega_p^2}{k^2} \int_{\text{LANDAU CONTOUR}} \frac{\partial \tilde{F}}{\partial V_z} \frac{dz}{V_z - \omega/k} \quad (1)$$

$$\tilde{F} = \frac{F_0}{n_0}$$

For weakly growing or damped modes  $\gamma \ll |\omega_0|$  the imaginary part of (1) yields:-

$$(ii) \quad \gamma = \frac{-\pi \frac{\omega_p^2}{k^2} \left( \frac{\partial \tilde{F}}{\partial V_z} \right)_{\omega/k}}{\frac{\partial \omega_p^2}{\partial \omega} \frac{\partial \tilde{F}}{\partial V_z} \frac{dV_z}{V_z - \omega/k}} \approx \frac{\pi}{2} \frac{\omega^3}{k^2} \left( \frac{\partial \tilde{F}}{\partial V_z} \right)_{\omega/k}$$

(iii) SO THE ELECTROSTATIC MODES DAMP IF  $\left( \frac{\partial \tilde{F}}{\partial V_z} \right)_{\omega/k} < 0$



TODAY WE WILL TRY TO UNDERSTAND THIS GROWTH/DAMPING.

(iv) Consider the motion of an electron in a weak electric field

$$E = E_0 \cos(\omega t - kz) e^{\gamma t} \quad \leftarrow \gamma \ll \omega \text{ weak growth}$$

To lowest order ignore the field to obtain:-

$$V_z = V_0$$

$$z = V_0 t + z_0$$

UNPERTURBED MOTION.  
 $V_0 = \text{const}$

$$\frac{d \delta V_z}{dt} = -\frac{e E_0}{m_e} \cos(\omega t - kz) e^{\gamma t} \approx -\frac{e E_0}{m_e} \cos[(\omega - kV_0)t - kz_0] e^{\gamma t}$$

INTEGRATING

$$\delta V_z = \frac{d \delta z}{dt} = -\frac{e E_0}{m_e} e^{\gamma t} \frac{\gamma \cos[(\omega - kV_0)t - kz_0] + (\omega - kV_0) \sin[(\omega - kV_0)t - kz_0]}{\gamma^2 + (\omega - kV_0)^2}$$

Integrating one more time:-

$$\delta z = -\frac{e E_0 e^{\gamma t}}{m} \frac{[\gamma^2 - (\omega - kv_0)^2] \cos[(\omega - kv_0)t - kz_0] + 2\gamma(\omega - kv_0) \sin[(\omega - kv_0)t - kz_0]}{[\gamma^2 + (\omega - kv_0)^2]^2}$$

(v) Now let's calculate the work done on the electron:

Power to Electron

$$= -eE(z(t), t)V(t) = -e \left[ E + \delta V + \delta z \frac{\partial E}{\partial z_0} V_0 \dots \dots \right]$$

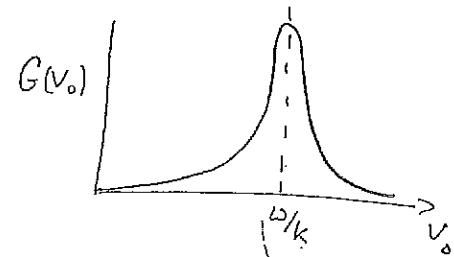
Higher order terms

WE AVERAGE OVER TIME (OR  $z_0$  IF YOU LIKE)  $\langle EV_0 \rangle = 0$

$$\langle \cos \theta \cos \theta \rangle = \frac{1}{2} \text{ etc.}$$

$$P = \frac{e^2 E_0^2}{2m} e^{2\gamma t} \frac{d}{dv_0} \left[ \frac{\gamma v_0}{\gamma^2 + (\omega - kv_0)^2} \right]$$

$G(v_0)$



So for  $v_0 < \omega/k$   $P > 0$  Electrons get energy from wave. DAMPING.

$v_0 > \omega/k$   $P < 0$  Electron gives energy to wave. GROWING.

(vi) Now consider a bunch of electrons - given by the distribution  $F_0(v_0)$  - let us sum up the energy to the electrons

$$\text{TOTAL POWER TO ELECTRONS} = \frac{e^2 E_0^2}{2m} e^{2\gamma t} \int dv_0 F_0(v_0) \frac{d}{dv_0} \left[ \frac{\gamma v_0}{\gamma^2 + (\omega - kv_0)^2} \right]$$

$$= -\frac{e^2 E_0^2}{2m} e^{2\gamma t} \int dv_0 \left( \frac{\partial F_0}{\partial v_0} \right) \frac{\gamma v_0}{\gamma^2 + (\omega - kv_0)^2}$$

$$\text{as } \gamma \rightarrow 0 \quad \frac{\gamma v_0}{\gamma^2 + (\omega - kv_0)^2} \rightarrow \pi \frac{\omega}{k^2} \delta(v_0 - \frac{\omega}{k})$$

$$\text{Power to Electrons} = -\frac{e^2 E_0^2}{2m} e^{2\omega t} \frac{\pi \omega}{k^2} \left( \frac{\partial F_0}{\partial V_0} \right)_{\Omega}$$

Clearly when  $\frac{\partial F_0}{\partial V_0} < 0$  DAMPING. Electrons get energy from wave.

$\frac{\partial F_0}{\partial V_0} > 0$  GROWING. Electrons give energy to wave.

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(vii) We can understand the situation in a more pictorial way.

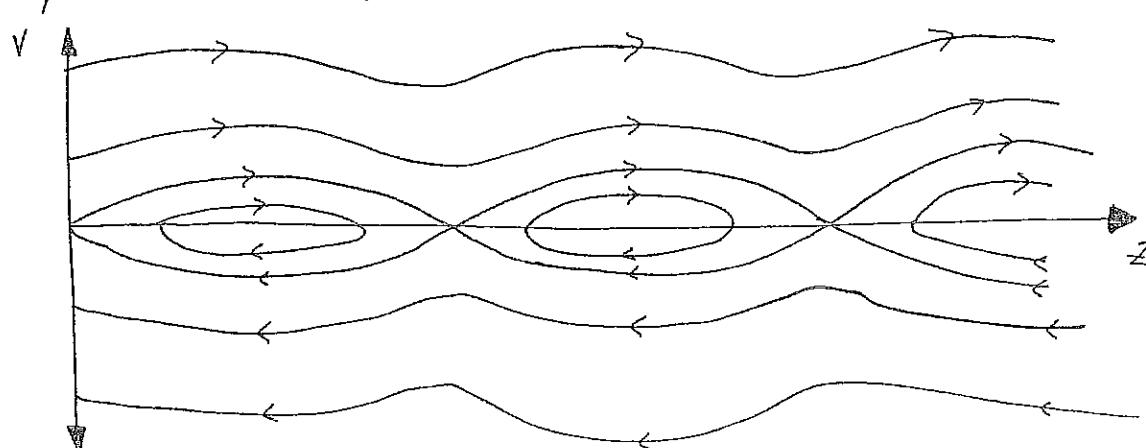
In the frame moving with the velocity  $\omega/k$  ( $z = \omega/k t$ ) =  $z'$

$$\frac{d^2 z'}{dt^2} = -\frac{e E_0}{m} \cos k z'$$

conservation of Energy for this is:-

$$\frac{1}{2} m v'^2 + \frac{e E_0}{k} \sin k z' = \epsilon$$

$$v' = \frac{dz'}{dt}$$

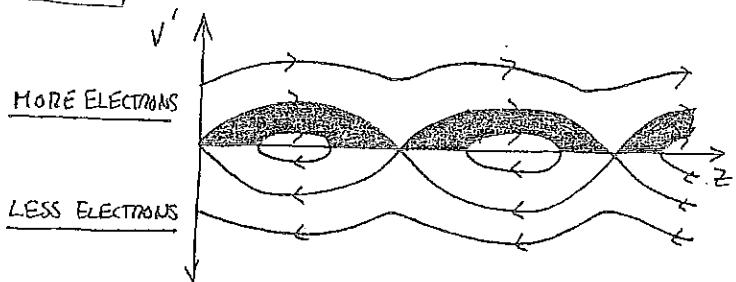


- SURFACES OF CONSTANT  $\epsilon$  IN  $V-z$  SPACE "ISLANDS" AROUND THE  $V=0$  LINE. TRAJECTORIES OF ELECTRONS ALONG  $\epsilon$  SURFACES
- $F$  constant along trajectories.

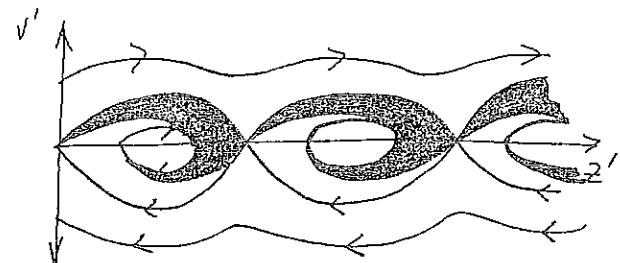
$$\frac{\partial F_0}{\partial V_0} > 0$$

At  $t=0$

SHAPED EXTRA ELECTRONS

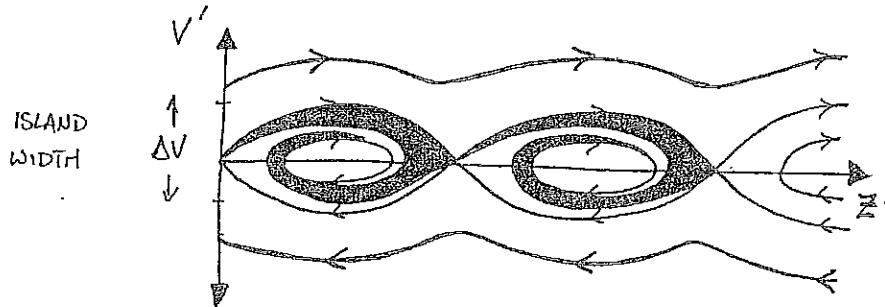


LATER



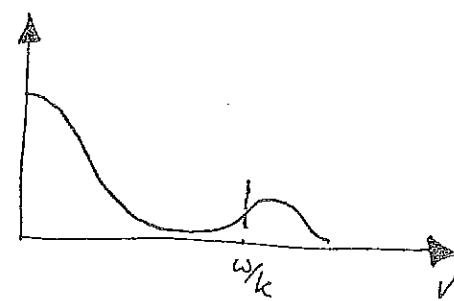
- Electrons closer to middle of island rotate around island faster.

STILL LATER

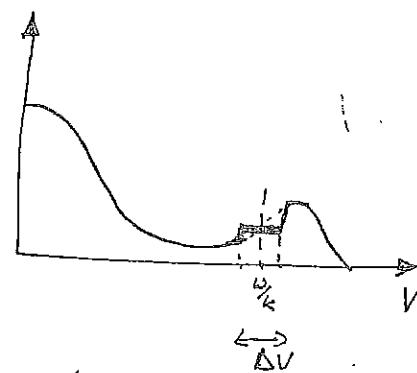


- Effectively "flattens" the distribution of extra electrons over the island - ie. over  $\Delta V$  "on coarse grain averaging."

( $t=0$ )

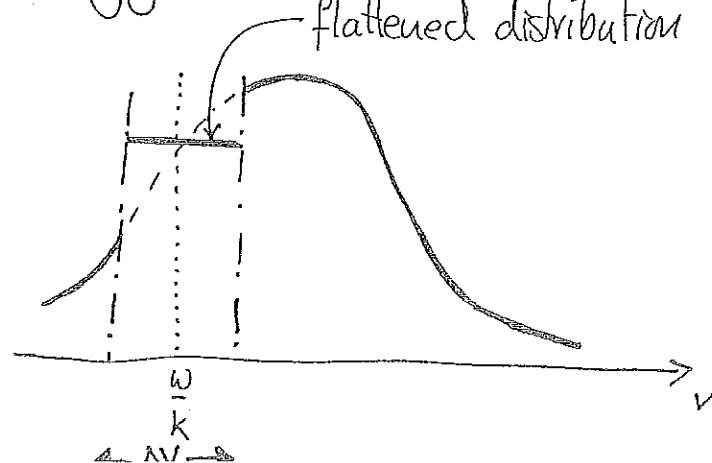


(LATER)  
AVERAGED  $f$



- Note from cartoon it is obvious that the distribution lost energy in the flattened region - this means the waves gained energy.

flattened distribution



222b. Lecture #10

Victor Cowley

## Wave Particle Interaction:- Quasi-Linear Theory

(i) Electron Motion in a single wave:-

$$m \frac{d^2z}{dt^2} = -eE_0 \sin(kz - \omega t)$$

TRANSFORM TO FRAME MOVING WITH WAVE:  $z' = z - \frac{\omega t}{k}$

$$\boxed{\frac{d^2z'}{dt^2} = -\frac{eE_0}{m} \sin kz'}$$

Pendulum equation.

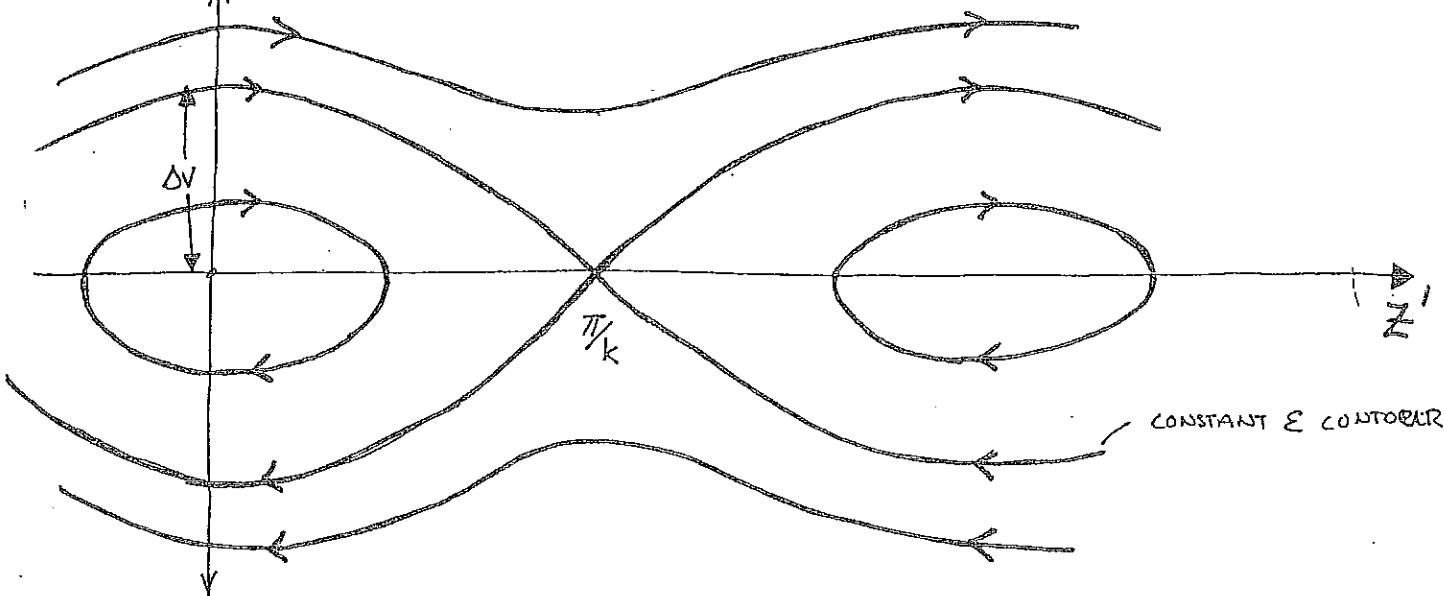
can be solved in terms of elliptic functions.

Energy Integral:

$$\frac{1}{2} m v'^2 - \frac{eE_0}{k} \cos kz' = E \equiv \text{constant}$$

$$v' = \frac{dz'}{dt}$$

$v'$



(ii)  $\Delta V$  = WIDTH OF "ISLAND" SEPARATRIX

$$\Delta V = 2 \sqrt{\frac{eE_0}{mk}}$$

$$\text{EQUATION FOR SEPARATRIX: } \frac{1}{2} m v'^2 = \frac{eE_0}{k} (1 + \cos kz')$$

(iii) DEEPLY TRAPPED. Small oscillations about  $z \approx 0$   $\frac{d^2z}{dt^2} = -\frac{eE_0 k}{m} z' \quad z' = z_0 \cos \omega_B t$

$$\omega_B \equiv \text{BOUNCE FREQUENCY} \equiv \sqrt{\frac{eE_0 k}{m}}$$

(iv) To remain in the linear regime during one growth time

$$\text{GROWTH RATE} = \gamma \gg \omega_B.$$

(iii) Multiple Waves. Consider waves in a periodic box of length  $L$ .

$$k_n = \frac{2\pi}{L} n$$

$$E(z, t) = \sum_n E_n(t) e^{ik_n z - i\omega_n t}$$

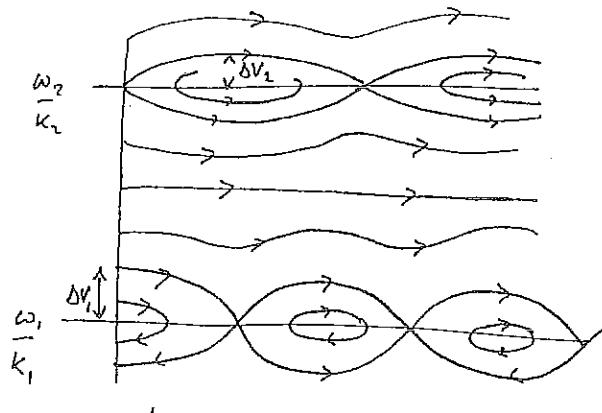
we shall consider  $E_n(t)$  to be slowly varying so that  $E_n \omega_n \gg \frac{dE_n}{dt}$

Our intention now is to develop a theory of how the multiple waves interact with the electrons.

(iv) Well separated waves

$$\text{If } \Delta V_1 + \Delta V_2 \ll \left| \frac{\omega_2}{k_2} - \frac{\omega_1}{k_1} \right|$$

: Very little interaction, behaves like 2 single wave interactions.



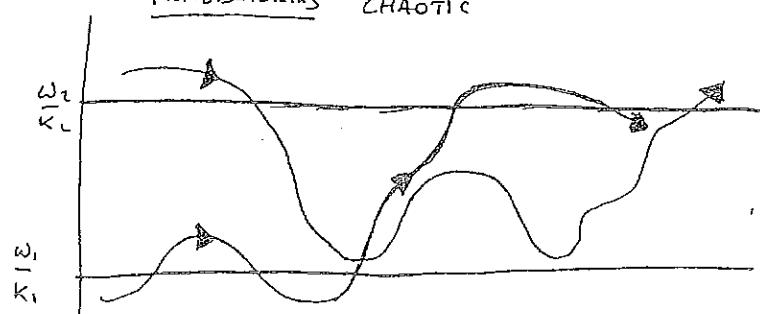
Arrows IN FRAME OR WAVE 1.

(v) OVERLAP (Chirikov)

$$\Delta V_1 + \Delta V_2 \gg \left| \frac{\omega_2}{k_2} - \frac{\omega_1}{k_1} \right|$$

-: Electron Never trapped for long in either wave. DIFFUSION IN V.

HARD TO DRAW SURFACES AS NO E CONSERVATION  
TRAJECTORIES CHAOTIC



222 b. Lecture # 11

Quasi Linear Theory Part II.

(i) Last time we derived two basic equations of Quasi-linear theory.

MULTIPLE TIMESCALES:- LONGEST TIME - SLOW EVOLUTION OF  $f_0 \equiv \tau_0$ .

$$\frac{1}{\tau_0} \sim \frac{e^2}{m^2} \frac{E^2}{\delta k} \frac{1}{v^3} \quad \begin{matrix} \delta k = \text{spectrum width} \\ (\text{see below}) \end{matrix}$$

:- THE CORRELATION TIME OF AN ELECTRON IN THE WAVES.

(TIME WAVES TO GET OUT OF PHASE)  
AS SEEN BY ELECTRON.

$$\frac{1}{\tau_c} \sim |\Delta\omega - v\delta k| \sim \delta k |v - \frac{d\omega}{dk}| \sim \delta k v \quad \text{when } \frac{d\omega}{dk} \ll v$$

:- GROWTH TIME  $\tau_k^{-1}$  OFTEN ASSUMED  $\tau_k \tau_c \ll 1$   
SO THIS IS LONG TIME - USUALLY ASSUMED  $\tau_k \tau_0 \gg 1$ .

:- PERIOD OF WAVES  $\omega_k^{-1}$  ALWAYS THE SHORTEST  
TIMESCALE

USUALLY

$$\tau_0 \gg \tau_k^{-1} \gg \tau_c \gg \omega_k^{-1}$$

and we have the  
overlap criterion

$$2 \left( \frac{e E_k}{k m} \right)^{1/2} > \left| \frac{\omega_1}{k_1} - \frac{\omega_2}{k_2} \right|$$

(ii) We are, of course, just solving:

$$\frac{df}{dt} + v \frac{df}{dz} = \frac{e}{m} E \frac{df}{dv} \quad \text{VLASOV EQN.}$$

$$\frac{\partial E}{\partial z} = -4\pi e \int f dv + 4\pi e n_0 \quad \text{"IONS"}$$

(iii) You can think of ~~the~~ Quasi-linear theory as the approximation of ~~weak~~ waves that are weakly growing but overlap a lot so that the electron feels a random force causing it to diffuse.

$$\Delta V \approx \text{random velocity step in time } \tau_c \approx \frac{e}{m} E \tau_c \sim \frac{e}{m} \frac{E}{\delta k v}$$

$$\text{Diffusion in Velocity} = D \approx \frac{(\Delta V)^2}{\tau_c} \approx \frac{e^2}{m^2} \frac{E^2}{v \delta k} \quad \boxed{\text{NOTE: } \frac{D}{V^2} \sim \frac{1}{\tau_0} \leftarrow \begin{matrix} \text{Time to diffuse} \\ \text{a velocity } V \end{matrix}}$$

# PLAN OF THE DERIVATION OF QUASI-LINEAR THEORY.

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split  $f$  into 2 parts

$$f(z, v, t) = f_0(v, t) + f_1(z, v, t)$$

SLOWLY  
VARYING  
TIME SCALE  
 $\tau_c$

FAST  
VARYING  
TIME SCALE  
 $\omega_c$

LINER PART

$$\text{SOLVE LINEARIZED VLAOU}$$

$$ikE_k = -\epsilon \frac{dE_k}{dv} dt$$

$$E_k = \sum_{\kappa} E_{\kappa} e^{i\kappa z} e^{(ikz - i\omega_k t + i\phi_k)}$$

$\Rightarrow$  DISPERSION RELATION

$$1 - \frac{\omega_c^2}{k^2} \int_v \frac{dE_k}{dv} \frac{1}{v - \omega_k} = 0$$

Treats  $f_0$  as constant  
on  $\omega_c$  timescale.

Nonlinear Part.  
Smooth  $f_1$  back into Vlaou Eqn.

Note: since  $f_0$  changes on  $\tau_c$  timescale so does  $\gamma_k = \gamma_k(t)$

Pass to continuous limit

$$\sum_{\kappa} \rho(\kappa) |E_{\kappa}|^2 \equiv \int \frac{dk}{2\pi} \epsilon(k, t) G(k)$$

for any  $k$ .

Evaluation of  $f_0$  comes from Vlaou

$\frac{df_0}{dt} = \frac{e}{m} \left\langle E \frac{df_1}{dv} \right\rangle$   
Spatial average.

Treat  $f_0$  as  
constant on  $\tau_c$   
timescale

$$\frac{df_0}{dt} = \frac{e}{\partial v} \left\{ D(v, t) \frac{\partial f_0}{\partial v} \right\}$$

$$D = \frac{e^2}{m^2} \left[ \int_0^{\infty} \frac{dk}{2\pi} \epsilon(k, t) 2\pi \delta(\omega_k - kv) - \frac{d}{dt} \int_0^{\infty} \frac{dk}{2\pi} \epsilon(k, t) \frac{\partial}{\partial k} P(\omega_k - kv)^{-1} \right]$$

To follow evolution must evaluate  $f_0(v, t)$   $\epsilon(k, t)$  and  $\gamma_k(t)$  self consistent.

Structure of energy etc. given in Hazeltine - Jaehnisch.

(v) The second term in  $D(v, t)$  is smaller by a factor  $\gamma_k / \omega_k$  and we note that  $\frac{1}{E(k,t)} \frac{\partial E(k,t)}{\partial t} \sim I(\gamma_k) \gg \frac{1}{f_0} \frac{\partial f_0}{\partial t}$

So we write  $f_0 = f_B + f_w$   $f_B \gg f_w$

$$\frac{\partial f_B}{\partial t} = \frac{\partial}{\partial v} \left\{ D_0(v, t) \frac{\partial f_B}{\partial v} \right\}$$

QUASI-LINEAR  
DIFFUSION

$$\text{where } D_0(v, t) = \left( \frac{e}{m} \right)^2 \int \frac{dk}{2\pi} E(k, t) 2\pi \delta(\omega_k - kv)$$

$$= \left( \frac{e}{m} \right)^2 \frac{E(k = \frac{\omega_k}{v}, t)}{|v - \frac{d\omega_k}{dt}|}$$

Using the delta function to evaluate integral.

Integrating Equation for  $f_w$  treating  $f_B$  as constant

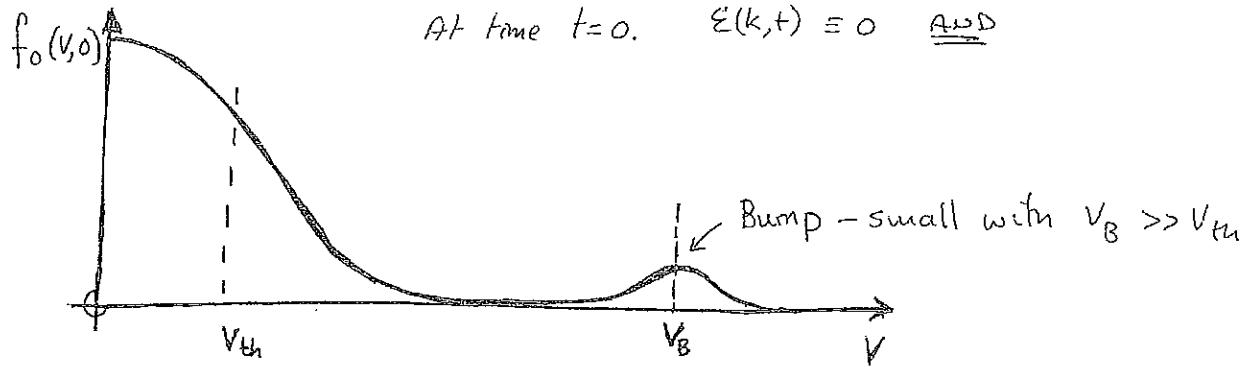
and

$$f_w = - \left( \frac{e}{m} \right)^2 \frac{\partial}{\partial v} \left\{ \int_0^\infty \frac{dk}{2\pi} E(k, t) \frac{\partial}{\partial \omega_k} P(\omega_k - kv)^{-1} \right\} \frac{\partial f_B}{\partial v}$$

$f_w$ , SMALL NEVER ACCUMULATES IN TIME.

$f_B$ , ACCUMULATES OVER TIME.

## Bump on the Tail: weak beam evolution.



- Dispersion relation simplifies since we are interested in unstable modes with  $\omega/k \sim v_B \gg v_{th}$ . Expand  $\frac{1}{v - \omega/k} \approx -\frac{k}{\omega} \left( 1 + \frac{kv}{\omega} + \dots \right)$

we get as before:

$$\omega^2 = \omega_p^2 \left( 1 + \frac{3k^2 v_{th}^2}{\omega^2} \right) \quad \text{LANGMUIR WAVES}$$

and from half pole

$$\gamma_k = \frac{\pi}{2} \omega_k \left( \frac{\omega_p}{k} \right)^2 \left( \frac{\partial f_0}{\partial v} \right)_{v=\omega/k}$$

EVOLUTION OF  $f_0$  ( $f_B$  REALLY)

TWO THINGS TO NOTE: - ① DIFFUSION HAPPENS TO RESONANT PARTICLES ONLY

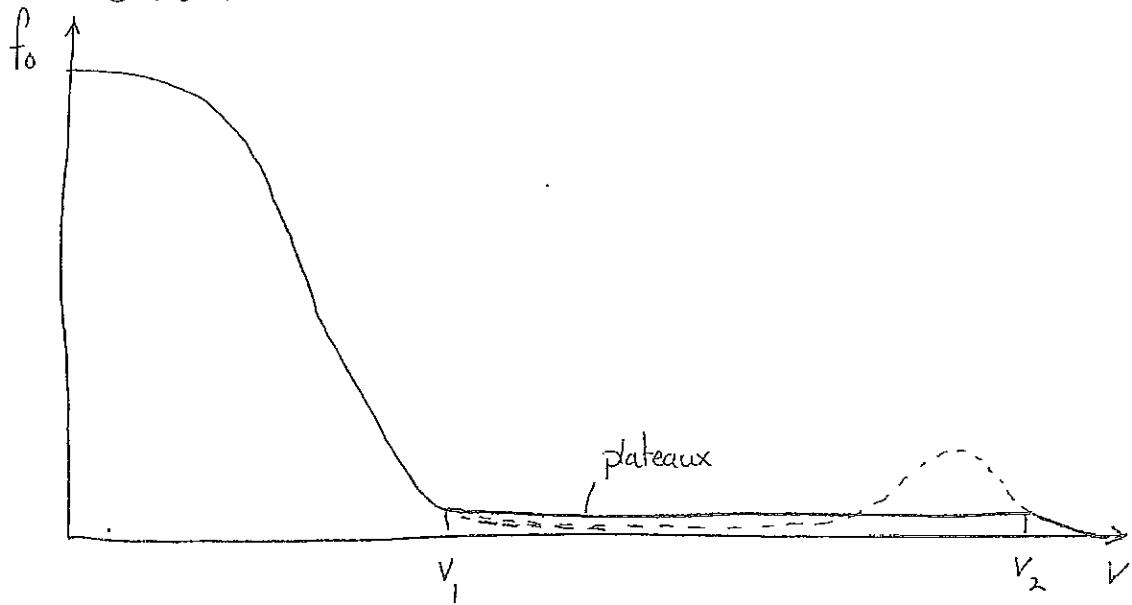
② ONLY MODES THAT HAVE  $\gamma_k > 0$  GROW SO ONLY REGIONS WHERE  $\frac{\partial f_0}{\partial v} > 0$  HAVE GROWING MODES AND  $\epsilon(k, t) > 0$ .

$\Rightarrow$  DIFFUSION OF REGIONS WHERE  $\frac{\partial f_0}{\partial v} > 0$  BUT NOT OTHER REGIONS.

so  $f_0$  MUST EVOLVE TO A STATE WITH  $\frac{\partial f_0}{\partial v} \leq 0$  EVERYWHERE

AND CONSERVING PARTICLES. THEREFORE A PLATEAU IS FORMED.

So as  $t \rightarrow \infty$



Conservation of particles yields

$$\int_{v_1}^{v_2} f_0(v, 0) dv = f_0(v_1, \infty) (v_2 - v_1) \quad \text{determines } v_2 \text{ and } v_1$$

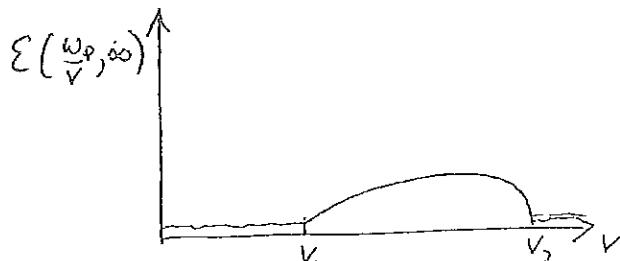
$$\text{To find } \mathcal{E}(k, \infty) \text{ we note that} \quad 2\gamma_k \mathcal{E}_k = \frac{\partial \mathcal{E}_k}{\partial t} = \mathcal{E}_k \pi \omega_k (\omega_p)^2 \left( \frac{\partial f_0}{\partial v} \right)_{v=\frac{\omega_k}{\omega_p}}$$

$$\frac{\partial f_0}{\partial t} = \frac{\partial}{\partial v} \left[ \frac{e^2}{m^2} \frac{\mathcal{E}\left(\frac{\omega_k}{\omega_p} = kv, t\right)}{V - \frac{d\omega}{dk}} \frac{\partial f_0}{\partial v} \right]$$

$$= \frac{\partial}{\partial t} \left[ \frac{\partial}{\partial v} \left[ \frac{e^2}{m^2} \frac{k^2}{\pi \omega_k \omega_p^2} \mathcal{E}(k, t) \right] \right]_{k=\frac{\omega_k}{\omega_p}}$$

using  $\omega_k \sim \omega_p$  and  $V > \frac{d\omega}{dk}$   $\mathcal{E}(k, 0) = 0$  and  $f_0(v, 0)$  given

$$\mathcal{E}(k, \infty) = \left( \frac{m}{e} \right)^2 \pi \frac{\omega_p^3}{k^3} \int_{v_1}^{\omega_p/k} [f_0(v, \infty) - f_0(v, t)] dv$$



(vi) Quasi-Linear Theory.

This is a useful way to calculate the evolution of the waves and distribution function in systems where the growth is weak and the waves overlap well.

ANSATZ FOR SOLUTION :-  $f = f_0(v, t) + f_1(z, v, t) \dots \text{H.O.T.}$

$\uparrow$   
 SLOWLY VARYING  
IN TIME  
 $\uparrow$   
 FAST VARYING  $\frac{df_1}{dt} \sim \omega_p$

Electric field comes from Poisson's equation

$$4\pi\rho = \nabla \cdot E \Rightarrow ik_h E_h = -4\pi e \int dV f_{in}$$

OUR GOAL IS TO FIND THE SLOW EVOLUTION OF  $f_0$ .

$$(vii) \quad \frac{\partial f_1}{\partial t} + v \frac{\partial f_1}{\partial z} = \frac{e}{m} E \frac{\partial f_0}{\partial v} = \frac{\partial f_1}{\partial t}_{z=z_0+vt}$$

SOLUTION  $f_1(z, v, t) = g(z-vt, v) + \frac{e}{m} \int_0^t dt' E(z-vt', t-t') f'_0(v, t-t')$

$\uparrow$   
 INITIAL  
CONDITIONS  
 $\int_0^t dt'$   
 INTEGRATION ALONG THE UNPERTURBED ORBIT.

Substituting into Poisson's Equation gives  $\omega_n$  and  $\gamma_n$  the growth rates of each  $n$ .

(viii) Substituting into Vlasov Equation

Exactly like the linear case from

$$1 - \frac{\omega_p^2}{k} \int \frac{\partial f_0}{\partial v} \frac{dz}{v - \omega/k} = 0$$

SLOW TIME  
DEPENDENCE IN  
 $f_0$ . ALSO IN  
 $\omega_n$  &  $\gamma_n$ .

$$\frac{\partial f_0}{\partial t} = \left\langle \frac{e}{m} E(z, t) \frac{\partial f_1}{\partial v} \right\rangle$$

SPATIAL AVERAGE  
OVER  $z$ .

SLOW  
VARIATION

$$= \left( \frac{e}{m} \right)^2 \frac{\partial}{\partial v} \int_0^t d\tau C(z, t; z-vt, t-\tau) f'_0(v, t-\tau)$$

Drop initial  
conditions.  
Average over  $v$ .

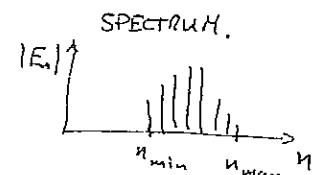
where

$$C(z, t; z-vt, t-\tau) = \langle E(z, t) E(z-vt, t-\tau) \rangle$$

"LAGRAGIAN  
CORRELATION  
FUNCTION"

Noting that  $E$  is real so that:-  $E_{-n} = E_n^*$  and  $\omega_{-n} = -\omega_n$ .

$$C = \sum_n E_n^*(t) E_n(t-\tau) e^{i(\omega_n - k_n v)\tau}$$



Terms in this sum became out of phase when

$$\tau = \tau_c \sim \left| \omega_{k_{\max}} - k_{v_{\max}} v - \omega_{k_{\min}} - k_{v_{\min}} v \right|^{-1} \sim \left| v - \frac{d\omega}{dk} \right|^{-1} \Delta k^{-1}$$

$$\Delta k = \frac{2\pi}{L} (k_{\max} - k_{\min})$$

so this provides the basic deceleration time.

LET US ASSUME THAT  $f_0$  VARIES LITTLE ON THIS TIME,  $\tau_c$ .

$$\Rightarrow \boxed{\frac{\partial f_0}{\partial t} = \left( \frac{e}{m} \right)^2 \frac{\partial}{\partial v} \left[ I(v, t) \frac{\partial f_0}{\partial v} \right]} \quad (\text{A})$$

DHWIPU Eq.

$$\text{where } I(v, t) = \int_0^\infty C(v, k, t) dk$$

To evaluate this we note that we are interested in modes that grow so we take  $\omega_n$  to have a small positive imaginary part to ensure causality. We also write

$$E_n(t-\tau) = E_n(t) - \tau \frac{\partial E_n}{\partial t} \quad \text{and we go to the limit } L \rightarrow \infty \quad \sum_n |E_n|^2 = \int_0^\infty \frac{dk}{2\pi} E(k, t)$$

DETAILS OF ALGEBRA IN HAZELINE AND WAEBROEK PGS 315 - 318.

$$I(v, t) = \int_0^\infty \frac{dk}{2\pi} E(k, t) 2\pi \delta(\omega_k - kv) - \frac{1}{2} \int_0^\infty \frac{dk}{2\pi} E(k, t) \frac{\partial}{\partial \omega_k} P \left[ \frac{1}{(\omega_k - kv)} \right]$$

WE THEREFORE MUST SOLVE (A) AND

WITH  $\gamma(k, t)$  THE LINEAR GROWTH RATE.

$$\boxed{\frac{dE_k}{dt} = 2\gamma(k, t) E_k} \quad (\text{B})$$



## Lecture #12. Propagation of light in an Inhomogeneous Plasma.

(i) There is one topic in homogeneous plasmas that I won't treat - that is deriving the hot plasma dispersion relation - this is a tedious algebraic operation and it is done in many (almost all) books on plasma physics.

(ii) Today we will talk about light waves in an inhomogeneous plasma - we will treat the ions as stationary and the electrons as cold. There will be some repetition of the case.

(iii)  $\overset{\text{Fluid}}{\text{Electron}}$ , Equation of motion (Linearized)

$$m_e n_0 \frac{d \underline{v}_e}{dt} = -e \underline{H} \underline{E}$$

$$n_0 = n_0(\underline{r})$$

we take a single frequency so that all quantities go like  $e^{-i\omega t}$ .

$$(iv) \quad \underline{J} = -e n_0 \underline{v}_e = \frac{i \omega_p^2(\underline{r}) \underline{E}}{4\pi \omega} \quad \omega_p^2 = \frac{4\pi n_0(\underline{r}) e^2}{m_e}$$

From Faraday's Law:

$$\nabla \times \underline{E} = \frac{i\omega}{c} \underline{B}$$

and the Ampere/Maxwell equation

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} - \frac{i\omega}{c} \underline{E}$$

we combine to obtain [eliminate  $\underline{B}$  &  $\underline{J}$ ]

$$(v) \quad \boxed{\nabla^2 \underline{E} - \nabla(\nabla \cdot \underline{E}) + \frac{\omega^2}{c^2} \epsilon(\underline{r}) \underline{E} = 0} \quad (1)$$

$$(vi) \text{ With } E(z) = 1 - \frac{\omega_{pe}^2(z)}{\omega^2}$$

a similar equation for  $\underline{B}$  is easy to obtain

$$\nabla^2 \underline{B} + \frac{\omega^2}{c^2} \epsilon \underline{B} + \frac{\nabla \epsilon}{\epsilon} \times (\nabla \times \underline{B}) = 0$$

(vii) When  $\nabla \epsilon = 0$  (THE HOMOGENEOUS CASE) we can take

solutions of the form,  $\underline{E} = \underline{E}_0 e^{ikz - i\omega t}$

$\Rightarrow ik \cdot \underline{E} = 0 \quad (\nabla \underline{E} = 0)$  or  
 $\omega^2 = \omega_{pe}^2 + k^2 c^2$   
 LIGHT WAVE  $\omega > \omega_{pe}$

$\underline{E} = 0 \quad \omega = \omega_{pe}$   
 and  $ik \times \underline{E}_0 = 0 \quad \nabla \times \underline{E} = 0$   
 PLASMA WAVE

(viii) Plane Stratification - Normal Incidence

$$n_0 = n_0(z) \quad \text{Plane stratification}$$

$$\underline{E} = \hat{\underline{E}}(z) e^{-i\omega t} \quad \text{Normal incidence - propagates in } z \text{ direction}$$

From ① we find

either,  $\frac{d^2 \hat{\underline{E}}}{dz^2} + \frac{\omega^2}{c^2} \epsilon(z) \hat{\underline{E}} = 0 \quad \text{with} \quad \hat{\underline{z}} \cdot \hat{\underline{E}} = 0$   
Light wave

or  $\epsilon \underline{E}_z = 0 \quad \text{Plasma wave at } \omega^2 = \omega_{pe}^2(z)$   
 only one height.

(ix) We concentrate on the light wave, and take  $\hat{E} = E \hat{x}$  without loss of generality.

$$\boxed{\frac{d^2 E}{dz^2} + \frac{\omega^2}{c^2} \epsilon(z) E = 0} \quad (2)$$

(x) W.K.B. theory Suppose  $\epsilon(z)$  varies on a scale

$L$  (i.e.  $\frac{d\epsilon}{dz} \sim \frac{\epsilon}{L}$ ) and further suppose the typical wavelength  $\lambda = \frac{c}{\omega \sqrt{\epsilon}}$  is short compared to  $L$  so that

$$\boxed{\frac{c}{L \omega \sqrt{\epsilon}} \ll 1}$$

we look for a solution for which it locally looks like the homogeneous solution.

$$(xi) \quad \hat{E}(z) = E_0(z) \exp i \int k(z') dz' \quad (3)$$

where  $(k_L)$  is large and  $\frac{dE_0}{dz} \sim \mathcal{O}\left(\frac{E_0}{L}\right)$  and  $\frac{dk}{dz} \sim \mathcal{O}\left(\frac{k}{L}\right)$

THE AMPLITUDE AND WAVELENGTH VARY ON THE SCALE OF THE DENSITY BUT THE WAVELENGTH IS SHORT,

Note  $\lambda(z) = \frac{2\pi}{k(z)}$

OUR FORM OF THE SOLUTION IS GUessed WE NOW SEE IF WE CAN SOLVE THE EQUATION (2) WITH IT.

(xi) Substituting ③ into ② we get:

$$\frac{d^2 E_0}{dz^2} + 2ik \frac{dE_0}{dz} - k^2 E_0 + i \frac{dk}{dz} E_0 + \frac{\omega^2}{c^2} \epsilon(z) E_0 = 0$$

<u>ORDERING:</u>	$\frac{\partial(E_0)}{L^2}$	$\frac{\partial(kE_0)}{L}$	$\frac{\partial(k^2 E_0)}{L}$	$\frac{\partial(kE_0)}{L}$	$\frac{\omega^2}{c^2} \epsilon E_0$
	SMALLEST	NEXT LARGEST	"A LARGEST" TERM	NEXT LARGEST	"OTHER LARGEST" TERM

Since  $kL \gg 1$  we Balance the largest terms first to find

$$④ - \boxed{k^2 = \frac{\omega^2 \epsilon(z)}{c^2}}$$

LOCAL DISPERSION RELATION.  
SAME AS HOMOGENEOUS CASE.  
= A CHOICE REALLY.

Now we expand  $E_0(z)$  as a power series in  $\frac{1}{kL} \ll 1$

$$E_0 = E_0^\circ + E_0' + E_0'' \dots$$

$$\boxed{E_0' \approx \frac{E_0^\circ}{kL}}$$

Now substitute in and equate powers of  $\frac{1}{kL}$

$$\mathcal{O}\left(\frac{kE_0^\circ}{L}\right) : 2ik \frac{dE_0^\circ}{dz} + i \frac{dk}{dz} E_0^\circ = 0 \quad -⑤$$

$$\mathcal{O}\left(\frac{E_0^\circ}{L^2}\right) : \frac{d^2 E_0^\circ}{dz^2} + 2ik \frac{dE_0'}{dz} + i \frac{dk}{dz} E_0' = 0$$

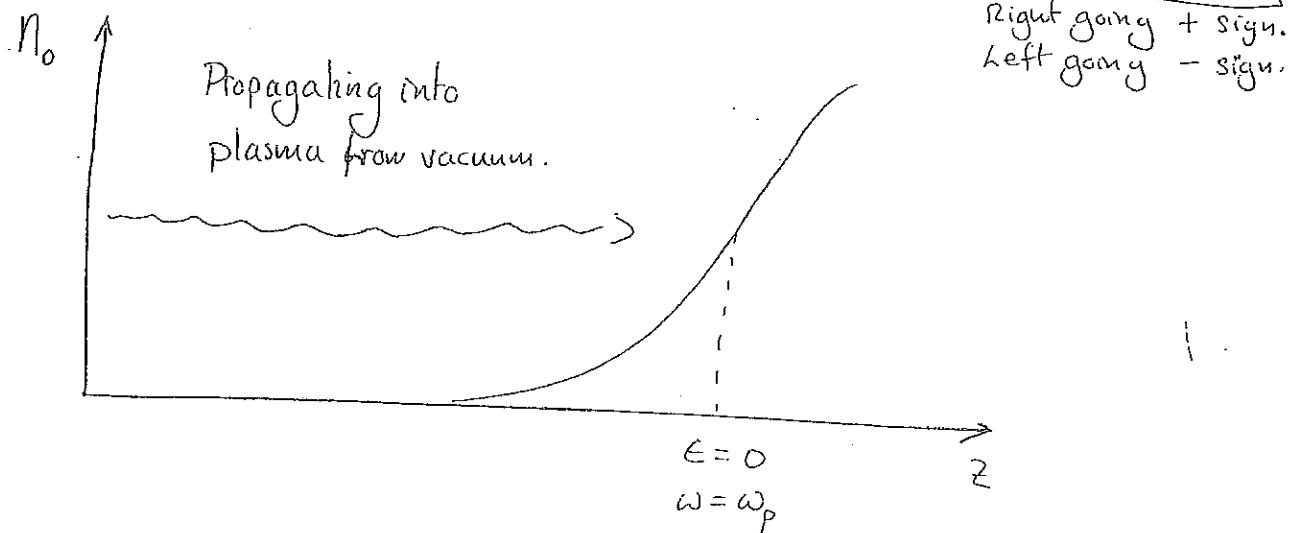
etc.

We solve ⑤

$$\frac{d}{dz} \ln E_0^0 = -\frac{1}{2} \frac{d \ln |k|}{dz} = -\frac{1}{2} \frac{d \ln \epsilon}{dz}$$

$$\Rightarrow E_0^0 = \boxed{\frac{\bar{E}}{\epsilon^{1/4}}} \quad k = \pm \frac{\omega}{c} \sqrt{\epsilon}$$

$$E_0 = \frac{\bar{E}}{(\epsilon(z))^{1/4}} \exp \left[ \frac{i\omega}{c} \int_{z'}^z \sqrt{\epsilon(z')} dz' \right] + \text{H.O.T.}$$



NOTE

ENERGY FLUX in waves. =  $\frac{V_g/E_0}{8\pi} I^2$

$V_g = \text{group velocity}$

=  $\frac{\partial \omega}{\partial k} = c \sqrt{\epsilon}$

=  $\frac{c \bar{E}^2}{8\pi} = \text{constant. (As might be expected)}$

Wave slows down as it approaches the critical layer where  $\epsilon = 0$  and  $\omega = \omega_{pe}$ . To preserve the flux

the amplitude swells (like  $\frac{1}{\epsilon^{1/4}}$ ) HOWEVER NEAR  $\epsilon = 0$

the expansion breaks down because  $kL \rightarrow 0$ .



222b. lecture #12: Propagation of light in an inhomogeneous plasma Part II.

(i) Last time we derived the WKB solution for propagation of light waves into a plasma with  $k$  along  $\nabla n$ :  $n = n(z)$ ,  $k = k_z(z)$

$$E = \frac{\bar{E} \hat{x}}{(\epsilon(z))^{\frac{1}{4}}} \exp \left\{ \pm \frac{i\omega}{c} \int_{z'}^z \sqrt{\epsilon(z')} dz' - i\omega t \right\} + \text{H.O.T.} \quad (1)$$

$$\epsilon = 1 - \frac{\omega_p^2(z)}{\omega^2}$$

So the local  $k_z$  vector =  $\pm \frac{\omega}{c} \sqrt{\epsilon(z)}$  and  $k_z^2 c^2 = \omega^2 - \omega_p^2(z)$   
 $\pm$  solution waves going in  $\pm$ ve  $z$  direction

(ii) Wave group velocity =  $v_g = c/\epsilon$  goes to zero at the  $\omega = \omega_p$  surface and amplitude gets large.

AROUND  $\epsilon(z) = 0$  ( $\omega = \omega_p$ ) THE WKB SOLUTION/APPROXIMATION BREAKS DOWN AS THE AMPLITUDE VARIES AS FAST AS THE PHASE.

(iii) BUT, We can find the solution near  $\epsilon = 0$  by another approximation, we can then patch the WKB & local solution together this is called asymptotic matching.

LOCAL/INNER SOLUTION

def:-  $\omega_p = \omega$  at  $z = z_0$ , this is called the TURNING POINT.

we can TAYLOR EXPAND ABOUT  $z = z_0$ . THE DENSITY

$$n(z) \approx n(z_0) + n(z_0) \frac{(z-z_0)}{L} + \dots \quad \text{GOOD FOR } z-z_0 \ll L$$

$$\epsilon \approx \frac{z-z_0}{L}$$

note: we set  $\left( \frac{\partial n}{\partial z} \right)_{z_0} = \frac{n(z_0)}{L}$

defining  $L$  = density scale length.

(iv) Using this approximation for  $n(z)$  we can write the wave equation:-

$$\frac{d^2 E_x}{dz^2} + \frac{\omega^2}{c^2} \left( \frac{z - z_0}{L} \right) E_x \approx 0$$

We make this into a standard form by rescaling  $z$ , let

$$S = + \left( \frac{\omega^2}{c^2 L} \right)^{1/3} (z - z_0)$$

Note new scale length

$$L_{\text{Airy}} = \left( \frac{c^2 L}{\omega^2} \right)^{1/3} \ll L$$

$$\boxed{\frac{d^2 E_x}{ds^2} - S E_x \approx 0} \quad (2)$$

Airy's Equation

Pg: 446. Abramowitz & Stegun  
Table of Mathematical Functions.  
(see Xerox)

Solutions are

$$E_x = a \text{Ai}(s) + b \text{Bi}(s)$$

but for  $s \gg 1$

$$\text{Bi}(s) \sim \frac{1}{\sqrt{\pi}} \frac{1}{s^{1/4}} e^{\frac{2}{3}s^{3/2}}$$

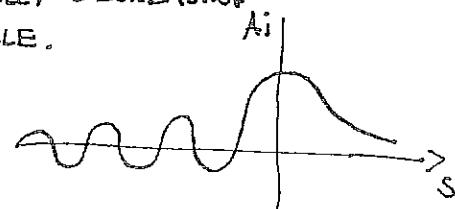
EXPONENTIALLY INCREASING  
UNACCEPTABLE

$$\text{Ai}(s) \sim \frac{1}{2} \frac{1}{\sqrt{\pi}} \frac{1}{s^{1/4}} e^{-\frac{2}{3}s^{3/2}}$$

EXPONENTIALLY DECREASING  
ACCEPTABLE.

So  $b=0$  and

$$\boxed{E_x = a \text{Ai}(s)} \quad (3)$$



NOTE THIS SOLUTION IS VALID WHERE  $|z - z_0| \ll L$  or in terms of  $s$

$$\boxed{|s| \ll \left( \frac{\omega^2 L^2}{c^2} \right)^{1/3} \gg 1}$$

## 10.4. Airy Functions

## Definitions and Elementary Properties

## Differential Equation

10.4.1

$$w'' - zw = 0$$

Pairs of linearly independent solutions are

$$\text{Ai}(z), \text{Bi}(z),$$

$$\text{Ai}(z), \text{Ai}(ze^{2\pi i/3}),$$

$$\text{Ai}(z), \text{Ai}(ze^{-2\pi i/3}).$$

## Ascending Series

10.4.2  $\text{Ai}(z) = c_1 f(z) - c_2 g(z)$

10.4.3  $\text{Bi}(z) = \sqrt{3} [c_1 f(z) + c_2 g(z)]$

$$f(z) = 1 + \frac{1}{3!} z^3 + \frac{1 \cdot 4}{6!} z^6 + \frac{1 \cdot 4 \cdot 7}{9!} z^9 + \dots$$

$$= \sum_0^{\infty} 3^k \left(\frac{1}{3}\right)_k \frac{z^{3k}}{(3k)!}$$

$$g(z) = z + \frac{2}{4!} z^4 + \frac{2 \cdot 5}{7!} z^7 + \frac{2 \cdot 5 \cdot 8}{10!} z^{10} + \dots$$

$$= \sum_0^{\infty} 3^k \left(\frac{2}{3}\right)_k \frac{z^{3k+1}}{(3k+1)!}$$

$$\left(\alpha + \frac{1}{3}\right)_0 = 1$$

$$3^k \left(\alpha + \frac{1}{3}\right)_k = (3\alpha + 1)(3\alpha + 4) \dots (3\alpha + 3k - 2)$$

(α arbitrary; k=1, 2, 3, ...)

(See 6.1.22.)

10.4.4

$$c_1 = \text{Ai}(0) = \text{Bi}(0)/\sqrt{3} = 3^{-2/3}/\Gamma(2/3)$$

$$= .35502 80538 87817$$

10.4.5

$$c_2 = -\text{Ai}'(0) = \text{Bi}'(0)/\sqrt{3} = 3^{-1/3}/\Gamma(1/3)$$

$$= .25881 94037 92807$$

## Relations Between Solutions

10.4.6  $\text{Bi}(z) = e^{\pi i/6} \text{Ai}(ze^{2\pi i/3}) + e^{-\pi i/6} \text{Ai}(ze^{-2\pi i/3})$

10.4.7

$$\text{Ai}(z) + e^{2\pi i/3} \text{Ai}(ze^{2\pi i/3}) + e^{-2\pi i/3} \text{Ai}(ze^{-2\pi i/3}) = 0$$

10.4.8

$$\text{Bi}(z) + e^{2\pi i/3} \text{Bi}(ze^{2\pi i/3}) + e^{-2\pi i/3} \text{Bi}(ze^{-2\pi i/3}) = 0$$

10.4.9  $\text{Ai}(ze^{\pm 2\pi i/3}) = \frac{1}{2} e^{\pm \pi i/3} [\text{Ai}(z) \mp i \text{Bi}(z)]$

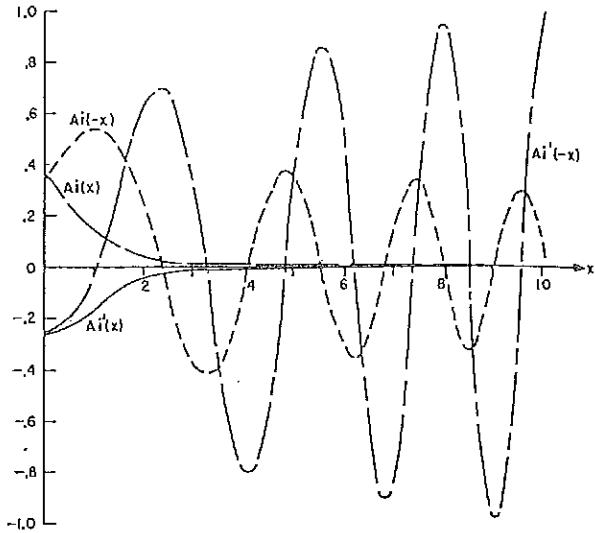
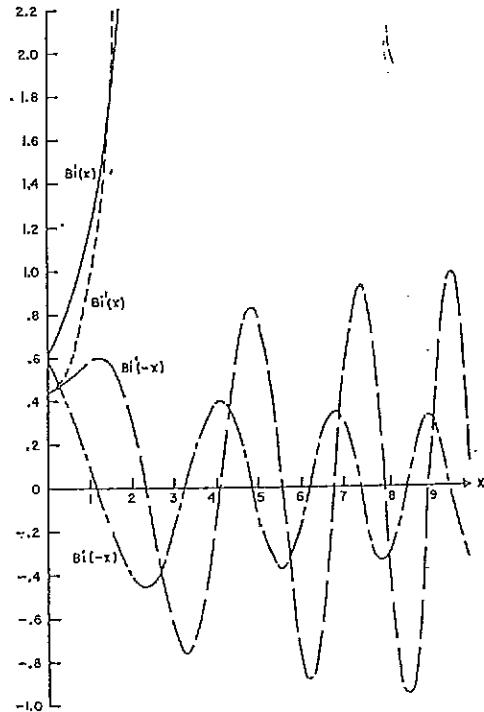
## Wronskians

10.4.10  $W\{\text{Ai}(z), \text{Bi}(z)\} = \pi^{-1}$

10.4.11  $W\{\text{Ai}(z), \text{Ai}(ze^{2\pi i/3})\} = \frac{1}{2} \pi^{-1} e^{-\pi i/6}$

10.4.12  $W\{\text{Ai}(z), \text{Ai}(ze^{-2\pi i/3})\} = \frac{1}{2} \pi^{-1} e^{\pi i/6}$

10.4.13  $W\{\text{Ai}(ze^{2\pi i/3}), \text{Ai}(ze^{-2\pi i/3})\} = \frac{1}{2} i \pi^{-1}$

FIGURE 10.6.  $\text{Ai}(\pm x), \text{Ai}'(\pm x)$ .FIGURE 10.7.  $\text{Bi}(\pm x), \text{Bi}'(\pm x)$ .

Differential Equations for  $\text{Gi}(z)$ ,  $\text{Hi}(z)$ 

10.4.55  $w'' - zw = -\pi^{-1}$

$w(0) = \frac{1}{3} \text{Bi}(0) = \frac{1}{\sqrt{3}} \text{Ai}(0) = .204975542478$

$w'(0) = \frac{1}{3} \text{Bi}'(0) = -\frac{1}{\sqrt{3}} \text{Ai}'(0) = .149429452449$

$w(z) = \text{Gi}(z)$

10.4.56  $w'' - zw = \pi^{-1}$

$w(0) = \frac{2}{3} \text{Bi}(0) = \frac{2}{\sqrt{3}} \text{Ai}(0) = .409951084956$

$w'(0) = \frac{2}{3} \text{Bi}'(0) = -\frac{2}{\sqrt{3}} \text{Ai}'(0) = .298858904898$

$w(z) = \text{Hi}(z)$

Differential Equation for Products of Airy Functions

10.4.57  $w''' - 4zw' - 2w = 0$

Linearly independent solutions are  $\text{Ai}^2(z)$ ,  $\text{Ai}(z)\text{Bi}(z)$ ,  $\text{Bi}^2(z)$ .

Wronskian for Products of Airy Functions

10.4.58  $W\{\text{Ai}^2(z), \text{Ai}(z)\text{Bi}(z), \text{Bi}^2(z)\} = 2\pi^{-3}$

Asymptotic Expansions for  $|z|$  Large

$c_0 = 1, c_k = \frac{\Gamma(3k + \frac{1}{2})}{54^k k! \Gamma(k + \frac{1}{2})} = \frac{(2k+1)(2k+3)\dots(6k-1)}{216^k k!}$

$d_0 = 1, d_k = -\frac{6k+1}{6k-1} c_k \quad (k=1, 2, 3, \dots)$

$$\boxed{\zeta = \frac{2}{3} z^{3/2}}$$

10.4.59

$\text{Ai}(z) \sim \frac{1}{2} \pi^{-1/2} z^{-1/4} e^{-\zeta} \sum_0^\infty (-1)^k c_k \zeta^{-k} \quad (|\arg z| < \pi)$

10.4.60

$\text{Ai}(-z) \sim \pi^{-1/2} z^{-1/4} \left[ \sin\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k c_{2k} \zeta^{-2k} \right. \\ \left. - \cos\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k c_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$

10.4.61

$\text{Ai}'(z) \sim -\frac{1}{2} \pi^{-1/2} z^{1/4} e^{-\zeta} \sum_0^\infty (-1)^k d_k \zeta^{-k} \\ (|\arg z| < \pi)$

10.4.62

$\text{Ai}'(-z) \sim -\pi^{-\frac{1}{2}} z^{\frac{1}{2}} \left[ \cos\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k d_{2k} \zeta^{-2k} \right. \\ \left. + \sin\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k d_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$

10.4.63

$\text{Bi}(z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{2}} e^{\zeta} \sum_0^\infty c_k \zeta^{-k} \quad (|\arg z| < \frac{1}{3}\pi)$

10.4.64

$\text{Bi}(-z) \sim \pi^{-\frac{1}{2}} z^{-\frac{1}{2}} \left[ \cos\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k c_{2k} \zeta^{-2k} \right. \\ \left. + \sin\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k c_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$

10.4.65

$\text{Bi}(ze^{\pm\pi i/3})$

$\sim \sqrt{2/\pi} e^{\mp\pi i/6} z^{-\frac{1}{2}} \left[ \sin\left(\zeta + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_0^\infty (-1)^k c_{2k} \zeta^{-2k} \right. \\ \left. - \cos\left(\zeta + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_0^\infty (-1)^k c_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$

10.4.66

$* \text{Bi}'(z) \sim \pi^{-\frac{1}{2}} z^{\frac{1}{2}} \sum_0^\infty d_k \zeta^{-k} \quad (|\arg z| < \frac{1}{3}\pi)$

10.4.67

$\text{Bi}'(-z) \sim \pi^{-\frac{1}{2}} z^{\frac{1}{2}} \left[ \sin\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k d_{2k} \zeta^{-2k} \right. \\ \left. - \cos\left(\zeta + \frac{\pi}{4}\right) \sum_0^\infty (-1)^k d_{2k+1} \zeta^{-2k-1} \right]$

10.4.68

$\text{Bi}'(ze^{\pm\pi i/3})$

$\sim \sqrt{2/\pi} e^{\mp\pi i/6} z^{\frac{1}{2}} \left[ \cos\left(\zeta + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_0^\infty (-1)^k d_{2k} \zeta^{-2k} \right. \\ \left. + \sin\left(\zeta + \frac{\pi}{4} \mp \frac{i}{2} \ln 2\right) \sum_0^\infty (-1)^k d_{2k+1} \zeta^{-2k-1} \right] \\ (|\arg z| < \frac{2}{3}\pi)$

Modulus and Phase

10.4.69

$\text{Ai}(-x) = M(x) \cos \theta(x), \text{Bi}(-x) = M(x) \sin \theta(x)$

$M(x) = \sqrt{[\text{Ai}^2(-x) + \text{Bi}^2(-x)]},$

$\theta(x) = \arctan [\text{Bi}(-x)/\text{Ai}(-x)]$

See page II.

(v) Both Solutions - the WKB solution & the local/inner solution - are valid where

$$\bullet \quad L_{\text{Airy}} \ll z_0 - z \ll L \quad (4)$$

This is called the MATCHING OR OVERLAP REGION.

We can make the solutions equal in this region by picking "a". This is the process called matching.

#### LOCAL/INNER SOLUTION IN THE MATCHING REGION:

For  $s$  large and negative ( $z_0 - z \gg L_{\text{Airy}}$ ) we have:

$$\approx a \text{Ai}(s) \approx \frac{a}{\sqrt{\pi}} \frac{1}{(-s)^{1/4}} \sin \left( \frac{2}{3} (-s)^{3/2} + \frac{\pi}{4} \right) \dots \dots \quad (5)$$

$$-s = (z_0 - z) \left( \frac{\omega^2}{c^2 L} \right)^{1/3}$$

WKB solution in the matching region:  $\epsilon \sim (z_0 - z)/L$  (need both +z-waves)

$$E_x \approx \frac{L^{1/4} e^{-i\omega t}}{(z_0 - z)^{1/4}} \left\{ E_+ \exp \left\{ -i \frac{2}{3} \frac{\omega}{c L^{1/2}} (z_0 - z)^{3/2} + i\phi \right\} + E_- \exp \left\{ +i \frac{2}{3} \frac{\omega}{c L^{1/2}} (z_0 - z)^{3/2} - i\phi \right\} \right\} \quad (6)$$

where we have written:-

$$\int_z^{\infty} dz' \sqrt{E(z')} \approx \int_{z_0}^{\infty} dz' \sqrt{E} + \phi \approx \frac{1}{L^{1/2}} \frac{2}{3} (z_0 - z)^{3/2} + \phi$$

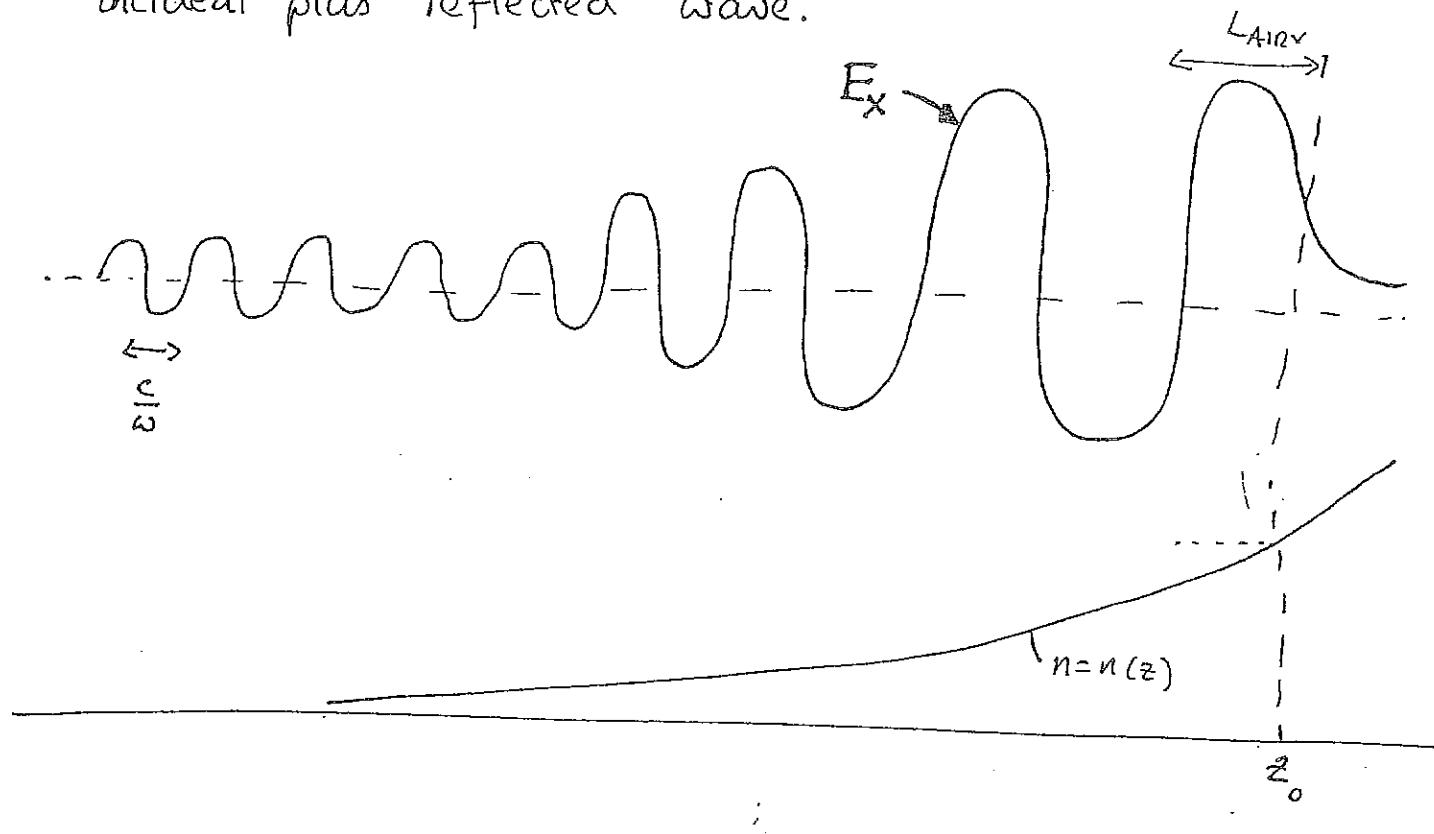
THE SOLUTIONS (5) AND (6) ARE EQUAL - i.e. MATCH IN THE OVERLAP REGION IF :-

$$E_- = -E_+, \quad \phi = \frac{\pi}{4} \quad \text{and} \quad a = -2i\sqrt{\pi} \left( \frac{L}{L_{\text{Airy}}} \right)^{1/4} E_+$$

Thus the WKB solution is

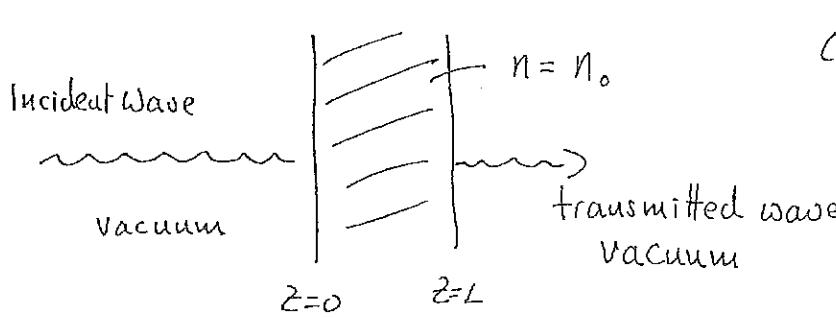
$$E_x = \frac{E_+ e^{-i\omega t}}{(\epsilon(z))^{1/4}} \sin \left\{ \frac{\omega}{c} \int_z^{z_0} dz' (\epsilon(z'))^{1/2} + \frac{\pi}{4} \right\}$$

STANDING WAVES - actually easier to think of as an incident plus reflected wave.



$$\omega = \omega_p$$

Homework - Calculate the  $\epsilon$  structure of the waves and the transmission through a plasma slab as below



Consider frequencies

$$(i) \omega > \omega_p$$

$$(ii) \omega < \omega_p$$

$$\omega_p^2 = \frac{4\pi n_0 e^2}{m_e}$$

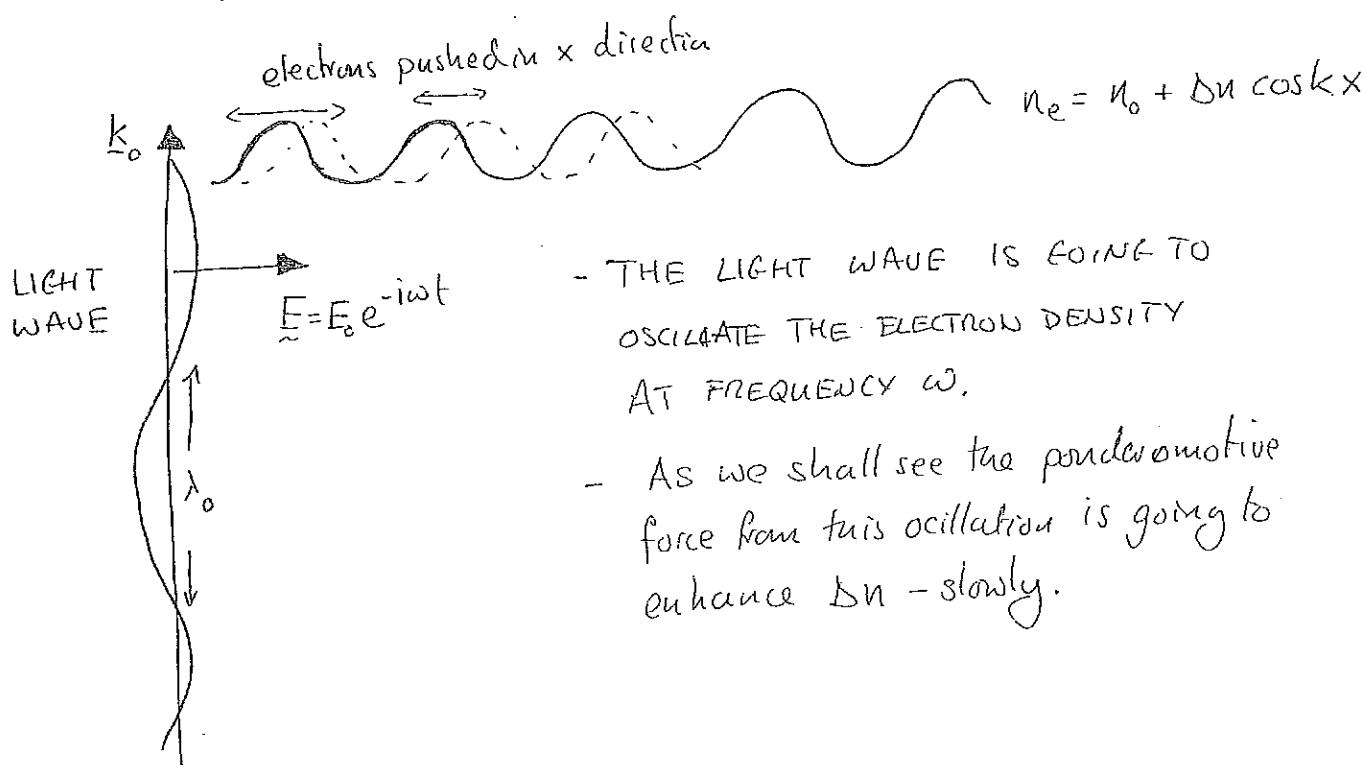
222b. Lecture #14Parametric Instability.

- (i) In today's lecture we examine a simple parametric instability - the "oscillating two stream instability".
- (ii) Consider a light wave propagating into a plasma - in the  $z$  direction - the plasma has a small density modulation in the  $x$  direction

$$n_e = n_0 + \Delta n(t) \cos kx$$

- $\Delta n(t)$  is slowly varying in time (compared to the frequency of the light wave).

- (iii) The incident light wave has a frequency  $\omega$  of order  $\omega_p = (4\pi n e^2/m_e)^{1/2}$  and a wavelength  $\lambda_0 \approx \frac{2\pi}{k_0} \approx \frac{c}{\omega}$   
we will assume  $k\lambda_0 \gg 1$  so the light wave looks like
- $$\tilde{E} = E_0 \hat{x} e^{-i\omega t} \quad - \text{it is polarised in the } x \text{ direction}$$



(iv) **FAST TIMESCALE**

- On this timescale the ions remain stationary.

Electron equations: Assume very cold electrons for simplicity.

electro momentum,

$$\frac{\partial \hat{V}_{ex}}{\partial t} = -\frac{e}{m_e} (E_0 e^{-i\omega t} + \hat{E}) - \nu_{ei} \hat{V}_{ex} \quad (1)$$

Drag on the ions  
by collisions (a model)

electron density,

$$\frac{\partial \hat{n}_e}{\partial t} + n_e \frac{\partial \hat{V}_{ex}}{\partial x} + \hat{V}_{ex} \frac{\partial n_e}{\partial x} = 0 \quad (2)$$

Poisson's Equation

$$\frac{\partial \hat{E}}{\partial x} = -4\pi e \hat{n}_e \quad (3)$$

$\hat{E}$  = electrostatic field  
created by electron charge.

with  $\frac{\Delta n}{n_0} \ll 1$  we linearize. Combining (1), (2) and (3),

$$\frac{\partial}{\partial x} \left\{ \frac{\partial^2 \hat{E}}{\partial t^2} + \omega_p^2 \hat{E} + \nu_{ei} \frac{\partial \hat{E}}{\partial t} \right\} = \omega_p^2 \frac{\Delta n}{n} k \sin kx E_0 e^{-i\omega t}$$

$$\hat{E} = \frac{\omega_p^2}{\omega^2 - \omega_p^2 + i\nu_{ei}\omega} \frac{\Delta n}{n_0} \cos kx E_0 e^{-i\omega t} = \hat{E} e^{-i\omega t}$$

Resonant response close to  $\omega = \omega_p$  gives large  $\hat{E}$  even for small  $\Delta n/n_0$ . This is the plasma wave / Langmuir wave (if we keep the thermal corrections which we don't here).

$$\hat{V}_{ex} \approx \frac{i}{\omega + i\nu_{ei}} \frac{e}{m_e} [E_0 e^{-i\omega t} + \hat{E}]$$

SLOW TIMESCALE

$$m_e \frac{d\vec{V}_e}{dt} = -e\vec{E} - m_e \hat{\vec{V}_e} \cdot \nabla \tilde{\vec{V}_e}$$

↑  
Slowly varying  
 $\vec{E}$

electron momentum  
equation.

Ponderomotive Force  
on electrons =  $F_p$

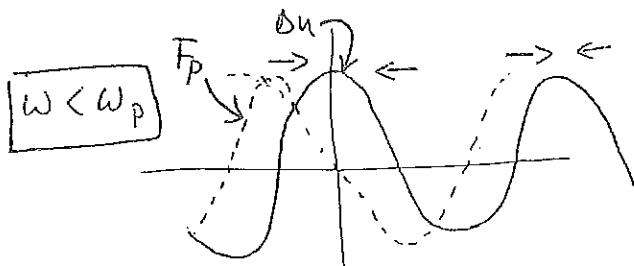
- on the slow timescale the electrons can be treated as having no inertia since  $\frac{d}{dt}$  is small.

$$\Rightarrow e\vec{E} = -m_e \hat{\vec{V}_e} \cdot \nabla \tilde{\vec{V}_e} = \frac{1}{4} \frac{e^2}{m_e \omega^2} \nabla (E_0 + \tilde{E})^2$$

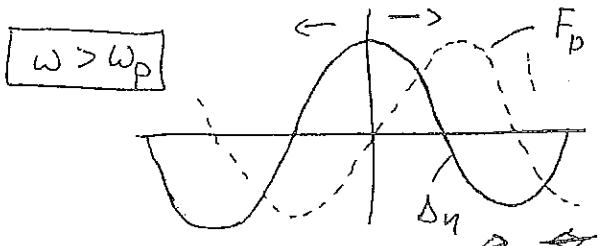
Ions:

$$= \frac{1}{4} \frac{e^2}{m_e \omega^2} \frac{\omega_p^2}{\omega^2 - \omega_p^2} \left( \frac{\Delta n}{n_0} \right) E_0^2 k \sin kx + \mathcal{I} \left( \frac{\delta n^2}{n^2} \right)$$

(4)



$\omega < \omega_p$   
 $F_p$  Pushes towards density maximum



$\omega > \omega_p$   
 $F_p$  Pushes away from density maxima.

- Ponderomotive force pushes electrons - balanced by  $\vec{E}$  somewhat.

-  $\vec{E}$  pulls ions. so now we must calculate the ions response.

## Ion Response

Ion Momentum :  $m_i \frac{\partial V_i}{\partial t} = e \bar{E}$  (5)

$$\frac{\partial n_i}{\partial t} \approx n_0 \frac{\partial V_i}{\partial x} \quad (6)$$

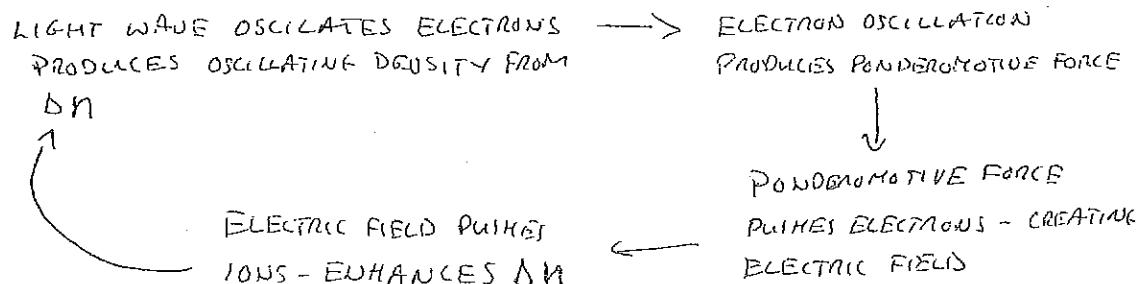
On this slow timescale  $n_i \sim n_0 + \Delta n \cos kx \sim n_e$   
 that is the plasma is neutral. Combining (4), (5) and (6) we  
 get

$$\frac{\partial^2 \Delta n}{\partial t^2} = -\frac{1}{4} \frac{k^2 e^2 n_0}{m_i m_e \omega^2} \frac{\omega_p^2}{\omega^2 - \omega_p^2} \frac{\Delta n}{n_0} E_o^2$$

$$\Delta n = \overline{\Delta n} e^{\gamma t}$$

$$\boxed{\gamma^2 = -\frac{1}{4} \frac{k^2 e^2}{m_i m_e \omega^2} \frac{\omega_p^2}{\omega^2 - \omega_p^2} E_o^2} \simeq k^2 v_e^2 \frac{m_e}{m_i}$$

SLOWLY GROWING FOR  $\omega \ll \omega_p$



222b. Lecture # 15Waves in Inhomogeneous Plasmas - Ray Tracing. 3D.

(i) Ray tracing is a technique used to solve for the wave fields in many physical situations - of particular interest here is the propagation of radio waves in plasmas - ray tracing is also used to analyze the propagation of seismic waves in earth and the sun (Helioseismology), the bending of light by gravity in distant galaxy clusters etc.

(ii) Let us take a particularly simple plasma model - the cold unmagnetized plasma. Thus in the usual way we can obtain

$$-c^2 \nabla \times (\nabla \times \underline{E}) = \omega_p^2 \underline{E} + \frac{\partial^2 \underline{E}}{\partial t^2} \quad (1) \quad \omega_p^2 = \omega_p^2(r, t)$$

where we shall be interested in waves of frequency such that

$$\frac{\omega}{c} \gg \frac{\nabla \omega_p^2}{\omega_p^2} = \frac{1}{L} \quad \text{i.e. wavelength is shorter than the typical scale length.}$$

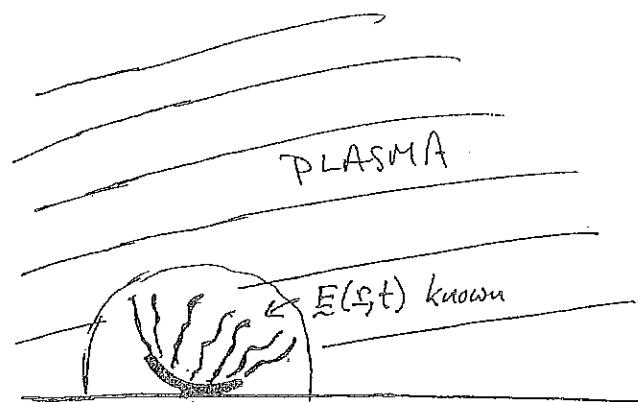
$$\omega \gg \frac{1}{\omega_p^2} \frac{\partial \omega_p^2}{\partial t} = \frac{1}{\tau} \quad \text{i.e. the wave oscillates fast compared to the changes in the plasma.}$$

THIS IS (AS IN THE 1D CASE) THE WKB LIMIT

(iii) The typical situation is when we know the field near some antenna as waves traveling into the plasma - we presume around the antenna is a non plasma region i.e. vacuum or air.

- Typically we can write the far field solution to the antenna as:-

$$\underline{E} \approx \bar{\underline{E}}(r) e^{ikr - i\omega t} \quad k = \frac{\omega}{c}$$



where  $\bar{\underline{E}}$  is known and the "local"  $k = k \hat{r}$

(iv) We write  $\underline{E}$  as almost a plane wave:-

$$\underline{E} = \bar{\underline{E}}(x, t) e^{is(x, t)} \quad (2)$$

where we assume  $s$ , the phase, is fast varying compared to  $\bar{\underline{E}}$  specifically

$$\frac{\partial s}{\partial t} \gg \frac{1}{|\bar{\underline{E}}|} \frac{\partial |\bar{\underline{E}}|}{\partial t} \sim \frac{1}{c}$$

$$\nabla s \gg \frac{1}{|\bar{\underline{E}}|} \frac{\nabla |\bar{\underline{E}}|}{\partial t} \sim \frac{1}{L}$$

we shall  
treat  $\frac{\omega L}{c} = \bar{\underline{E}}$   
and  $\omega t \sim \partial(\bar{\underline{E}})$

THE LOCAL  $\underline{k}$  VECTOR IS DEFINED AS:-

$$\underline{k} = \nabla s \quad \underline{k} = \underline{k}(x, t)$$

THE LOCAL FREQUENCY  $\omega$  IS DEFINED AS:-

$$\omega(x, t) = -\frac{\partial s}{\partial t}$$

note:  $\frac{\partial \underline{k}}{\partial t} = -\nabla \omega$

by definition

Substituting (2) into (1).

$$(v) \quad \underline{k} \times (\underline{k} \times \bar{\underline{E}}) - i \underline{k} \times (\nabla \times \bar{\underline{E}}) - i \nabla \times (\underline{k} \times \bar{\underline{E}}) - \nabla \times (\nabla \times \bar{\underline{E}}) \\ = \left( \frac{\omega_p - \omega^2}{c^2} \right) \bar{\underline{E}} - \frac{i\omega}{c^2} \frac{\partial \bar{\underline{E}}}{\partial t} - \frac{i}{c^2} \frac{\partial (\omega \bar{\underline{E}})}{\partial t} + \frac{1}{c^2} \frac{\partial^2 \bar{\underline{E}}}{\partial t^2}$$

Now treating  $\nabla \sim \mathcal{J}(\epsilon \underline{k})$  and  $\frac{\partial}{\partial t} \sim \mathcal{J}(\epsilon \omega)$  (see above) we expand

$$\bar{\underline{E}} = \bar{\underline{E}}_0 + \bar{\underline{E}}_1 + \bar{\underline{E}}_2 \dots \quad |\bar{\underline{E}}_1| \sim \mathcal{J}(\epsilon |\bar{\underline{E}}_0|) \quad \text{etc.}$$

(vi) LOWEST ORDER:

$$\underline{k} \times (\underline{k} \times \bar{\underline{E}}_0) = \frac{\omega_p^2 - \omega^2}{c^2} \bar{\underline{E}}_0 \Rightarrow \omega^2 = \omega_p^2 + k^2 c^2$$

$\underline{k} \cdot \bar{\underline{E}}_0 = 0 \quad (\text{except at } \omega = \omega_p)$

Local homogeneous dispersion relation must be satisfied at all  $x$  and  $t$ .

we can consider either  $\omega = \omega(r, k, t)$  but remember  $k = k(r, t)$  so:

$$\omega^2(r, t) = \omega_p^2(r, t) + k^2(r, t) c^2 \quad (3)$$

as it stands this appears impossible to use since we know  
 $\omega_p^2$  and there seems to be two unknowns ( $\omega$  &  $k$ ) and only  
one equation. BUT:  $\omega$  and  $k$  are related they are both derived from  
 $s(r, t)$  one function.

(v) Evolution of  $k$ : consider,  $\frac{\partial k}{\partial t} = -\nabla \omega = -\left(\frac{\partial \omega}{\partial r}\right)_{k,t} - \nabla k \cdot \left(\frac{\partial \omega}{\partial k}\right)_{r,t}$

now note  $\nabla k = \nabla(\nabla s)$  is a symmetric tensor so  $\nabla k \cdot \frac{\partial \omega}{\partial k} = \frac{\partial \omega}{\partial k} \cdot \nabla k$

THUS:-

$$\frac{\partial k}{\partial t} + \left(\frac{\partial \omega}{\partial k}\right)_{r,t} \cdot \nabla k = -\left(\frac{\partial \omega}{\partial r}\right)_{k,t}$$

If we define a moving point by  $\frac{dr}{dt} = \frac{\partial \omega}{\partial k} = V_g$  = GROUP VELOCITY

$$\frac{\partial}{\partial t} + V_g \cdot \nabla \equiv \frac{d}{dt} \equiv \text{CONVECTIVE DERIVATIVE MOVING WITH  
VELOCITY } V_g \text{ THE GROUP VELOCITY OF WAVES.}$$

$$\frac{dk}{dt} = -\left(\frac{\partial \omega}{\partial r}\right)_{k,t} \quad (4) \text{ CHANGE OF } k \text{ ALONG "RAY"}$$

$$\frac{dr}{dt} = \left(\frac{\partial \omega}{\partial k}\right)_{r,t} \quad (5) \text{ CHANGE IN } r \text{ ALONG "RAY"}$$

$$\frac{d\omega}{dt} = \left(\frac{\partial \omega}{\partial t}\right)_{k,r} + \frac{dk}{dt} \cdot \frac{\partial \omega}{\partial k} + \frac{dr}{dt} \cdot \frac{\partial \omega}{\partial r} \Rightarrow$$

CANCEL

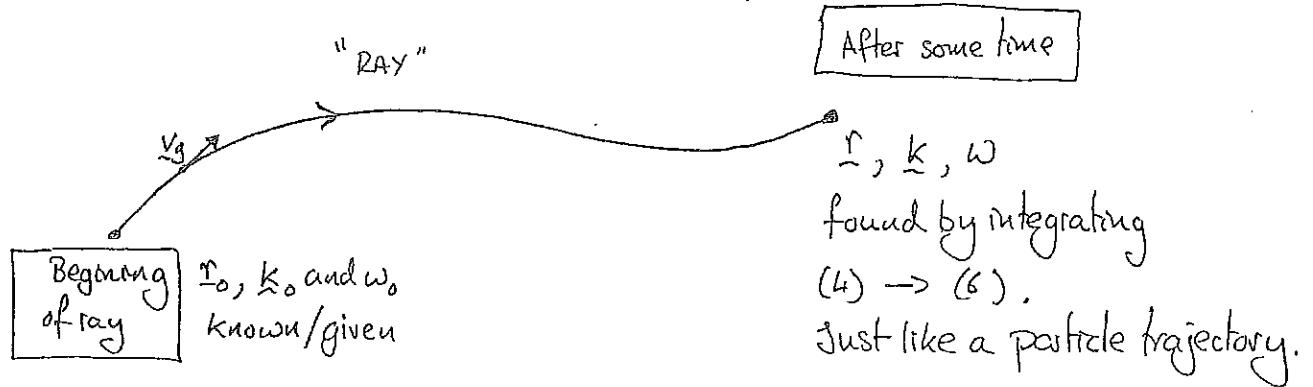
$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} \quad (6)$$

(vi) The ray equations (4)  $\rightarrow$  (6) are analogous to Hamilton's equations

$$\frac{dr}{dt} = \frac{\partial H}{\partial p}, \quad \frac{dp}{dt} = -\frac{\partial H}{\partial r}, \quad \frac{dH}{dt} = \frac{\partial H}{\partial t}$$

with  $H \Rightarrow \omega, \underline{r} \rightarrow \underline{s}, \underline{p} \rightarrow \underline{k}$ .

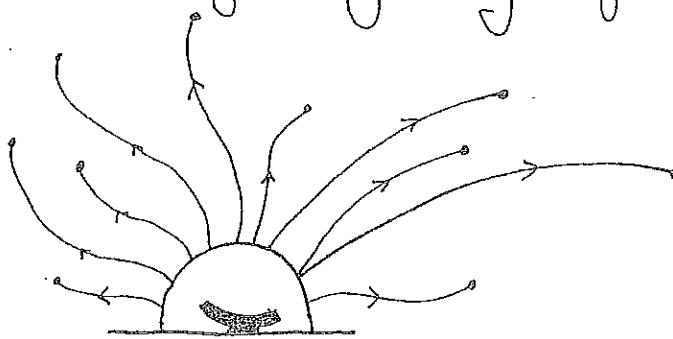
(vii) Once  $\omega = \omega(\underline{k}, \underline{s}, t)$  is known we have the ray equations in explicit form — here we do of course.



(viii) We integrate the ray equations along rays — starting usually on a hemisphere around the antenna. We find  $S(\underline{r}, t)$  by integrating along the ray.

$$\frac{ds}{dt} = \frac{\partial S}{\partial t} + v_g \cdot \nabla S = -\omega(\underline{s}, t) + v_g \cdot \underline{k}(\underline{s}, t)$$

This gives  $S$  at one point — a moving point. We can do better by starting many rays at many times all over the hemisphere.



- by computing (on a computer) lots of rays we can get  $S(\underline{s}, t)$  by interpolation between the points. PROBLEMS ARISE AT CAUSTICS WHERE RAYS INTERSECT.
- Easy to do this on a computer because it splits task into many subtasks.

(ix) Computing the Amplitude:- We now know  $S$  but usually we would like to know the amplitude - we go back to point (v) and go to next order in the expansion:

$$\underline{k} \times (\underline{k} \times \underline{E}_1) + \left( \frac{\omega^2 - \omega_p^2}{c^2} \right) \underline{E}_1 = i \underline{k} \times (\nabla \times \underline{E}_0) + i \nabla \times (\underline{k} \times \underline{E}_0) \\ - i \frac{\omega}{c^2} \frac{\partial \underline{E}_0}{\partial t} - i \frac{\partial}{c^2 \partial t} \omega \underline{E}_0$$

noting that  $\underline{E}_0^* \cdot [\underline{k} \times (\underline{k} \times \underline{E}_1) + \frac{\omega^2 - \omega_p^2}{c^2} \underline{E}_1] = 0$  we annihilate  $\underline{E}_1$  by dotting with  $\underline{E}_0^*$  and we add the resulting equation to the complex conjugate of itself to obtain

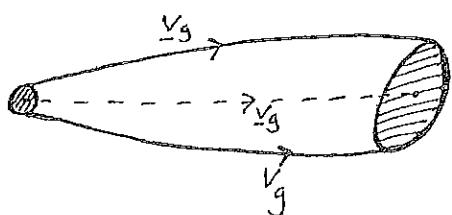
$$\boxed{\frac{\partial}{\partial t} \mathcal{E} + c^2 \nabla \cdot [\underline{k} \cdot \underline{E}] = 0}$$

CONTINUITY EQUATION  
FOR WAVE ENERGY.

$$\mathcal{E} = \frac{\omega |\underline{E}_0|^2}{8\pi}$$

$$K_g = \left( \frac{\partial \omega}{\partial k} \right)_{S,t} = \frac{c^2 k}{\omega} \quad \text{in this case}$$

Kind of an energy  
of the waves



- We can calculate the energy flow along a tube of rays - and thereby calculate the  $|\underline{E}_0|^2$  the amplitude

- Both phase and direction of  $\underline{E}_0$  are still unknown but they can also be found integrating along the ray.



ment given above, often by assuming that the radial width of the linear eigenmode is the radial mixing length, and the growth rate is the decorrelation time. In this paper, we argue that both the mixing length and correlation time are set by nonlinear processes. Estimates based on the linear mode structure can be vastly wrong, see Sec. VI.

The ion-temperature-gradient (ITG) mode<sup>5,6</sup> has been postulated as the major cause of thermal transport under certain plasma conditions, e.g., high-ion-temperature plasmas with sufficiently weak density profiles. For simplicity, we restrict ourselves in this paper to consideration of only this mode. This instability has the nice feature of being largely fluid in character when the stability threshold is well exceeded.

We address three questions: What drives the ITG unstable, what saturates it, and what determines the shape of an eddy? We do not attempt to calculate the saturated spectrum of the ITG or the transport arising from it. However, our results do bear directly on mixing length estimates of the transport.

In Sec. II, we present a simple, but realistic, physical model of the ITG. We introduce the  $E \times B$  nonlinearities in Sec. III and show that modes saturate by eddy turnover. Long thin eddies with  $\lambda_x \gg \lambda_y$ , where the equilibrium  $\nabla T_i$  is in the  $x$  direction and  $B$  is in the  $z$  direction, saturate at large amplitudes. These long thin eddies are similar to the "streamers" seen in the simulations by Drake *et al.*<sup>7</sup> In addition, a simple approximate model of the nonlinear dynamics is given in Sec. III. The nonlinear behavior of a single eddy is further examined in Sec. IV. In Sec. V, we introduce magnetic shear and finite Larmor radius effects. We show that the linear mode has two equivalent representations, Fourier modes, in which  $\lambda_x \sim \mathcal{O}(\rho_i)$  (Sec. V C), and twisted eddies,<sup>8,9</sup> in which the eddies are long and thin, but twist in the sheared magnetic field (Sec. V D). Section V depends on Appendices A and B. We analyze the twisted-eddy representation in a finite length box in Appendix A to show that this representation, rather than the Fourier representation, is the appropriate one, and that  $\lambda_x$  is of order  $\mathcal{O}[(\rho_i L_T)^{1/2}]$ , where  $L_T$  is the scale length of  $T_i$ . In Appendix B, we show how the twisted eddy in a torus is related to the ballooning-mode representation.<sup>10</sup> We also demonstrate that the twisted eddy gives a useful representation of the ITG in a torus.<sup>10,11</sup>

In Sec. VI, we argue that fully developed twisted eddies that have a large  $\lambda_x$  can lead to Bohm-like transport, because the mixing length could be as large as  $\sqrt{\rho_i L_n}$ . Since Bohm transport is much greater than the observed transport, this is puzzling. The problem is resolved to some extent in Sec. VII, where we show that the long thin eddies (and, therefore, twisted eddies) are vulnerable to fast growing secondary instabilities that feed on the large gradients produced by long thin eddies. This prevents them from reaching large amplitudes. The nonlinear development of the secondary instabilities is not completely clear. However, they should play a significant role in determining the mixing lengths and correlation times. The existing numerical simulations are examined for evidence of our proposed mechanisms. We argue that, if the secondary instabilities break up the twisted eddies, the characteristic eddy shape will be a twisted elliptical

eddy. In Sec. VIII, we draw the conclusion that nonlinear processes, not linear processes, determine mixing lengths and correlation times. We also discuss how experimental observations of the isotropy of the  $k$  spectrum, as a function of poloidal angle, would clarify this conclusion.

## II. SIMPLE PICTURE OF THE ITG MODE

In this section, we examine the basic dynamics that gives rise to the  $\nabla T$ -driven instability. It is helpful to have a simple physical picture of the mode for the subsequent discussion of nonlinear effects. We take  $\nabla n = 0$ ,  $B = B_0 \hat{z}$ ,  $\delta B = 0$ , and assume a fluid response for the ions. We look for instability in the range

$$k_{\parallel} v_{th,i} < \omega \ll k_{\parallel} v_{th,e}, \quad (1)$$

where  $v_{th,i} = \sqrt{2T_i/m_i}$ ,  $v_{th,e} = \sqrt{2T_e/m_e}$ , and  $k_{\parallel}$  is a typical parallel wave number. Therefore, electrons set up the "equilibrium" Boltzmann distribution along the field line

$$\delta n_e/n_0 = e\phi/T_e, \quad (2)$$

We concentrate on modes with long perpendicular wavelengths

$$k_1 \rho_i \ll 1. \quad (3)$$

The plasma is quasineutral, i.e.,  $\delta n_e \sim \delta n_i$  since  $k\lambda_D \ll 1$ , where  $\lambda_D$  is the Debye length. In this regime,  $\omega \ll \omega_{ci}$ , the ion motion perpendicular to  $B$  is given by the  $E \times B$  motion,

$$\frac{\partial \xi_1}{\partial t} = v_1 = c \frac{E \times B}{B^2} = -c \frac{\nabla \phi \times \hat{z}}{B_0}, \quad (4)$$

and  $v_1$  is, therefore, incompressible. The ion density perturbation (from the continuity equation) is entirely due to parallel compression

$$\frac{\delta n_i}{n_0} = -\nabla_{\parallel} \xi_{\parallel} = -\frac{\partial \xi_{\parallel}}{\partial z}, \quad (5)$$

where  $\xi_{\parallel}$  is the parallel displacement of the ions. The ion pressure perturbation  $\delta p_i$  is

$$\delta p_i/p_0 = -\xi_1 \cdot (\nabla p_0/p_0) - \gamma \nabla_{\parallel} \xi_{\parallel}, \quad (6)$$

where  $p_0 = n T_i$ . Here, the first term on the right-hand side of Eq. (6) is the convection of the equilibrium pressure by the  $E \times B$  motion, and the second term is the parallel compression of pressure ( $\gamma$  is the ratio of specific heats). Finally, the parallel equation of motion for the ions is

$$m_i n_0 \frac{\partial^2 \xi_{\parallel}}{\partial t^2} = -\nabla_{\parallel} \delta p_i - e n_0 \nabla_1 \phi. \quad (7)$$

When  $\nabla p_0 = 0$ , Eqs. (2)–(7) describe ion sound waves. These equations allow both standing and traveling wave solutions in the parallel ( $z$ ) direction. We consider standing-wave solutions since they are analogous to the "ballooning" type solutions that we examine in later sections. We take

$$\delta n = \delta n_0 \cos k_z z \exp(-i\omega t + ik_y y),$$

$$\delta p_i = \delta p_0 \cos k_z z \exp(-i\omega t + ik_y y), \quad (8)$$

$$\xi_{\parallel} = \xi_{\parallel 0} \sin k_z z \exp(-i\omega t + ik_y y).$$

Let us define the drift frequency

$$\omega_{pi}^* = \frac{k_y T_i c}{eB} \left( \frac{|\nabla p_0|}{p_0} \right) = \frac{k_y T_i c}{eB} \frac{1}{l_p}. \quad (9)$$

The dispersion relation for the mode is obtained from Eqs. (2)–(8):

$$\omega [\omega^2 - (k_z^2 C_s^2/2)(\Gamma + 1)] = -(k_z^2 C_s^2/2)\omega_{pi}^*, \quad (10)$$

with  $\Gamma = \gamma T_i / T_e$  and  $C_s^2 = 2T_e/m_i$ . Note that, if  $\omega_{pi}^* = 0$ , we get sound waves. Equation (10) has an unstable root if  $|\omega_{pi}^*| > 4[(\Gamma + 1)/6]^{1/2}|k_z C_s|$ . In the limit  $|\omega_{pi}^*| \gg |k_z C_s|$ , Eq. (10) becomes<sup>5</sup>

$$\omega^3 = -(k_z^2 C_s^2/2)\omega_{pi}^*. \quad (11)$$

In this limit, the mode has a simple physical description. First, note that the parallel compression term in Eq. (6) is small, so the pressure perturbation is entirely due to the convection of equilibrium pressure by the  $E \times B$  motion. Physically, this arises because the mode frequency  $\omega$  is faster than the rate of pressure relaxation along the field lines  $k_z C_s$ . Also, the electric field term in the parallel ion equation of motion is negligible compared to the pressure gradient term. Hence the equations can be written

$$\frac{\delta n_i}{n_0} = \frac{e\phi}{T_e} = -\frac{\partial \xi_z}{\partial z}, \quad (12a)$$

$$n_0 m_i \frac{\partial^2 \xi_z}{\partial t^2} = -\frac{\partial}{\partial z} \delta p_i, \quad (12b)$$

and

$$\frac{\delta p_i}{p_0} = -\xi_z \frac{\nabla p_i}{p_0} = -\frac{\omega_{pi}^*}{\omega} \frac{T_e}{T_i} \left( \frac{e\phi}{T_e} \right). \quad (12c)$$

For the unstable mode,  $\omega = |k_z^2/C_s^2\omega_{pi}^*/2|^{1/3} \times e^{i\pi/3} = \omega_0 + i\gamma$ . The various physical quantities with their relative phases are

$$\begin{aligned} \phi &= \bar{\phi}_0 e^{i\gamma t} \cos k_z z \cos(k_y y - \omega_0 t), \\ \frac{\partial \phi}{\partial t} &= |\omega| \phi_0 e^{i\gamma t} \cos k_z z \cos\left(k_y y - \omega_0 t - \frac{\pi}{6}\right), \\ \delta p &= \bar{\delta p}_0 e^{i\gamma t} \cos k_z z \cos(k_y y - \omega_0 t - \pi/3), \\ \frac{\partial \delta p}{\partial t} &= |\omega| \delta \bar{\phi}_0 e^{i\gamma t} \cos\left(k_z z \cos k_y y - \omega_0 t - \frac{\pi}{2}\right), \quad (13) \\ \xi_z &= \bar{\xi}_{z0} e^{i\gamma t} \sin k_z z \cos(k_y y - \omega_0 t), \\ \frac{\partial \xi_z}{\partial t} &= |\omega| \bar{\xi}_{z0} e^{i\gamma t} \sin k_z z \cos\left(k_y y - \omega_0 t - \frac{\pi}{6}\right), \\ \frac{\partial^2 \xi_z}{\partial t^2} &= |\omega| \bar{\xi}_{z0} e^{i\gamma t} \sin k_z z \cos\left(k_y y - \omega_0 t - \frac{\pi}{3}\right). \end{aligned}$$

Roughly, the instability proceeds as follows. Compression of the ions by parallel flow gives rise to a potential, which draws in the electrons so that  $\delta n_e = \delta n_i$ . The potential causes an  $E \times B$  drift, which brings cool ions into the compressed region and lowers the pressure. The low pressure sucks in more ions along the field line, and thereby increases the compression. This completes a kind of feedback loop that leads to instability. This expansion is a little too simple, as it ignores the relative phases of the physical quantities, which must be correct to yield instability.

In Fig. 1, we illustrate the phase relations of the various

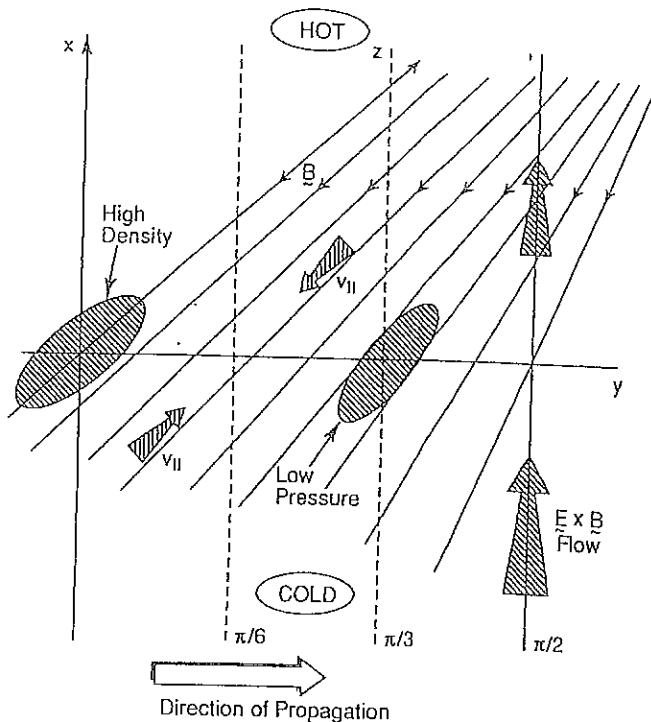


FIG. 1. Picture of the  $\nabla T_i$  instability. The ions are compressed at  $z \sim 0$ ,  $y \sim 0$ , and a potential is set up to draw in electrons to establish neutrality. The potential causes an  $E \times B$  flow with a maximum at  $z = 0, y = \pi/2$ . The  $E \times B$  flow convects cold plasma into the region  $0 < y < \pi$ . The pressure minimum that is at  $y = \pi/3$  is lowered and moved to the right by the cooling flow. The pressure minimum sucks in ions along the field lines, thereby increasing  $V_{||} v_{||}$  in the region  $-\pi/6 < y < 5\pi/6$ . Again, since the pressure minimum leads the maximum in  $V_{||} v_{||}$  by  $\pi/6$ , the maximum in  $V_{||} v_{||}$  is increasing and moving to the right. Finally,  $V_{||} v_{||}$ , which is a maximum at  $y = \pi/6$  increases with  $\delta n_i$  in the region  $-\pi/3 < y < 2\pi/3$ . Thus the density maximum increases and moves to the right.

physical quantities. The cooling  $E \times B$  flow is  $\pi/2$  out of phase with the density maximum, but since the mode is propagating to the right and growing, the pressure minimum lags the maximum of the cooling flow by  $\pi/6$ . The pressure minimum (at  $\pi/3$  relative to the density maximum) accelerates the parallel flow, but again because the model is propagating to the right, the maximum in the parallel flow (and parallel flow divergence) lags the pressure minimum by  $\pi/6$ . Finally, the rate of change of density is a maximum where the parallel flow divergence is largest ( $\pi/6$  out of phase with the density maximum). Since the density is increasing at its maximum ( $\partial \delta n_i / \partial t$  is positive at the maximum of  $\delta n_i$ ), the mode must grow. Physical pictures of the instability of varying detail are given by Rudakov and Sagdeev<sup>5</sup> and Coppi *et al.*,<sup>6</sup> and recently, for the case with a density gradient, by Dimits *et al.*<sup>12</sup>

### III. SIMPLIFIED NONLINEAR EQUATIONS

In the limit  $\omega_{pi}^* \gg k_{||} C_s$ , we can obtain simple nonlinear equations for the ITG. Note that the modes have the healthy linear growth rate given by Eq. (11). The basic expansion parameters are

## Lecture #17

222b. Ion Sound Waves and Solitons.

(i) Sound waves obey the wave equation - almost - in a plasma. So

$$\delta n \approx f(x - c_s t) + g(x + c_s t)$$

where with  $T_e \gg T_i$ :

$$c_s^2 = \frac{T_e}{m_i}$$

WAVE SHAPE PRESERVED.

for a plasma two things destroy this: DISPERSION

: NONLINEARITY

[Linear effect different frequencies travel at different speeds, spreading of packet, waves steepen to form shocks.]

the competition between these two effects can give rise to a traveling structure called the SOLITON. We examine this today.

### (ii) BASIC EQUATIONS FOR ION SOUND WAVES WITHOUT DAMPING, & $T_e \gg T_i$

$$\frac{\partial n_i}{\partial t} + \frac{\partial (n_i v)}{\partial x} = 0 \dots (1) \text{ Ion density Eqn.}$$

$$\frac{\partial p_e}{\partial x} = e n \frac{\partial \phi}{\partial x} \dots (2) \text{ Electron momentum Eqn. } (m_e \rightarrow 0)$$

$$\frac{\partial T_e}{\partial x} = 0 \dots (3) \text{ Large electron thermal conduction.}$$

$$T_i \rightarrow 0$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = - \frac{1}{n_i m_i} \frac{\partial p_i}{\partial x} - \frac{e}{m_i} \frac{\partial \phi}{\partial x} \dots (4) \text{ Ion momentum Eqn.}$$

$$\frac{\partial^2 \phi}{\partial x^2} = 4\pi e (n_e - n_i) \dots (5) \text{ Poisson Eqn.}$$

$$\frac{\partial^2 \ln(\frac{n_e}{n_0})}{\partial x^2} = \frac{n_e - n_i}{\lambda_D^2 n_0} \dots (6)$$

$$\lambda_D = \frac{T_e}{4\pi e^2 n_0}$$

Debye length

2

(iii) We make three basic approximations.

$$\text{A: } \frac{n_{ei} - n_0}{n_0} = N \sim \mathcal{O}(e) \ll 1 \quad \text{Small Amplitude.}$$

$$B: \quad \lambda_D \frac{\partial}{\partial x} \sim \mathcal{O}(\epsilon'^{1/2}) \ll 1 \quad \text{Long wavelength.}$$

c: Slow evolution in frame moving at velocity  $C_s$  - so we examine<sup>slow</sup> evolution of  $f(x-ct, t)$  function as it changes shape. Define moving coordinate:-

$$\zeta = x - c_s t$$

$$\left(\frac{\partial}{\partial t}\right)_x \rightarrow \left(\frac{\partial}{\partial t}\right)_{\tilde{x}} - c_s \left(\frac{\partial}{\partial \tilde{x}}\right)_t \text{ and } \frac{\partial}{\partial x} \rightarrow \frac{\partial}{\partial \tilde{x}}$$

$\overset{?}{J(E)}$        $\overset{?}{J(I)}$

From (1) we learn

$$N = \frac{v}{c_s} \quad \text{to lowest order}$$

and from (1) — (6) after "some" Algebra we learn

$$\left( \frac{\partial N}{\partial t} \right)_{\xi} + c_s N \frac{\partial N}{\partial \xi} + c_s \lambda_0^2 \frac{\partial^3 N}{\partial \xi^3} = 0$$

↑ ↑  
 Nonlinearity                      Dispersion

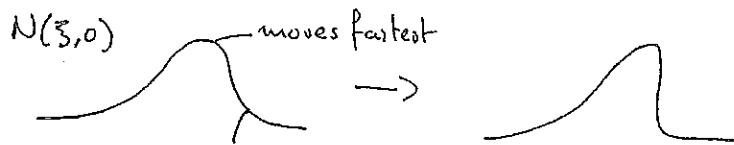
to  
 $\mathcal{J}(\epsilon)$

Korteweg - de Vries - originally derived for waves in shallow water. Gives evolution of the shape.

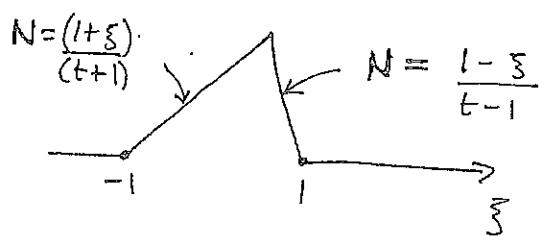
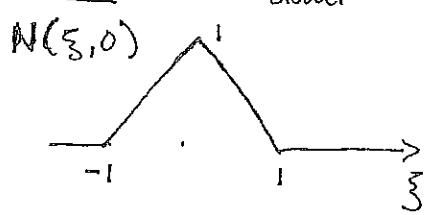
NONLINEARITY ALONE: STEEPENING.

$$\frac{\partial N}{\partial t} + N c_s \frac{\partial N}{\partial \xi} = 0 \Rightarrow$$

$N = N(x - Nt)$  implicit solution



each point of shape moves with velocity  $N$ .

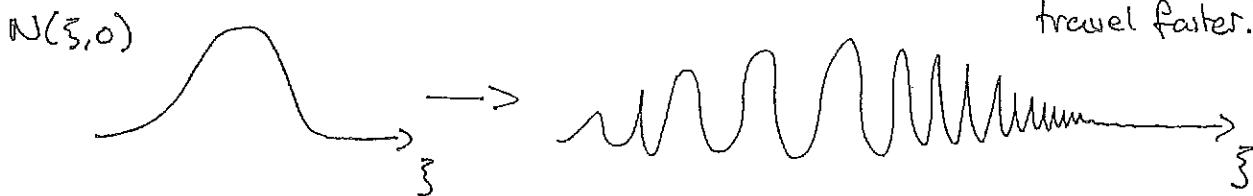
EXAMPLE

at  $t=1$   
shock forms at  $\xi=1$ .

DISPERSION ALONE: SPREADING.

$$\frac{\partial N}{\partial t} + c_s \lambda_D^2 \frac{\partial^3 N}{\partial \xi^3} = 0 \Rightarrow$$

$\omega \approx c_s \lambda_D^2 k^3$   
for waves: higher  $k$  travel faster.



To analyze full equation let's normalize to make all coefficients 1.

$$t = \tau \quad N = b y \quad \xi = c s$$

$$b \frac{\partial y}{\partial t} + \frac{b^2 c_s}{c} y \frac{\partial y}{\partial s} + \frac{b c_s \lambda_D^2}{c^3} \frac{\partial^3 y}{\partial s^3} = 0$$

$$1 = \frac{b c_s}{c} = \frac{c_s \lambda_D^2}{c^3} \quad c^3 = c_s \lambda_D^2 \Rightarrow c = (c_s \lambda_D^2)^{1/3}$$

$$b = \left( \frac{\lambda_D^2}{c_s^2} \right)^{1/3}$$

WOLMANIZED.

$$\frac{\partial y}{\partial t} + y \frac{\partial y}{\partial s} + \frac{\partial^3 y}{\partial s^3} = 0$$

### SOLITON SOLUTION:

We look for a solution of the form

$$y = y(s - u_0 t) \quad \text{where } u = \text{constant speed.}$$

i.e. a solution that does not change shape as it propagates. Let prime denote differentiation w.r.t.  $s - ut = \phi$  ↴ constant.

$$-u_0 y' + y y' + y''' = 0 \Rightarrow y'' = u_0 y - \frac{y^3}{2} + \frac{k_1}{2},$$

$$y'^2 = u_0 y^2 - \frac{y^3}{3} + k_1 y + k_2 \quad \text{another constant.}$$

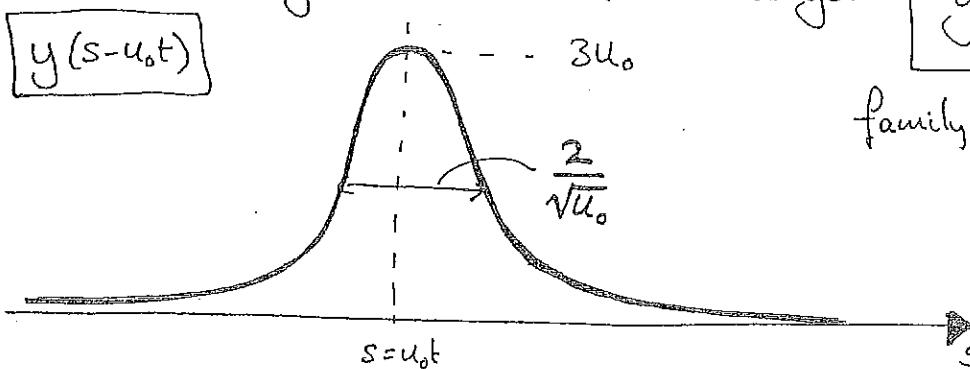
to get the SOLITON we set  $k_1 = k_2 = 0$ .

$$\int \frac{dy}{y(u_0 - \frac{y^2}{3})^{1/2}} = s - u_0 t$$

Integral can be done and we get

$$y = 3u_0 \cosh^{-2} \left[ (s - u_0 t) \frac{\sqrt{u_0}}{2} \right]$$

family of SOLITONS with different  $u_0$



### NONLINEAR SOLUTION WHERE STEEPENING BALANCES DISPERSION

A more complicated theory called "Inverse Scattering" shows that solitons can pass through each other: Gardner, Greene, Kruskal, Miura  
J. Math Phys. 1967

# Waves In Plasmas

## Levels of Approximation.

Geometry:

Plasma Model:

Linear Waves

Instability:

Numericity:

Homogeneous (Magnetized) Plasmas.

Inhomogeneous Plasmas Magnetized - often.

Closure

Vlasov / Fokker-Planck

Fluid Theory

Kinetic Theory

Fluid

Kinetic

COLD PLASMA  
"Simplest Model"

HOT PLASMA

Cold Plasma

2 Fluid MHD

2 Fluid MHD

WKB

WKB

$D(\omega, k)$   $\omega = \omega(k)$   
DISPERSION REL.

MHD WAVES  
Kinetic Dispersion Rel. WAWDAY DAMPING

Eigenmode Calculations

Wave propagation

Velocity Space Instabilities,  
2 Stream, Bump on Tail  
Mirror modes

Real Space Instabilities  
MHD Interchange/Kink.  
ITG, Drift waves

Ray tracing.

SOLITONS  
Nonlinear Waves  
BFR modes

SHOCK WAVES  
Singularities

Quasi-Linear Theory

Turbulence  
Weak/Strong.

Particle Acceleration - Cosmic Rays

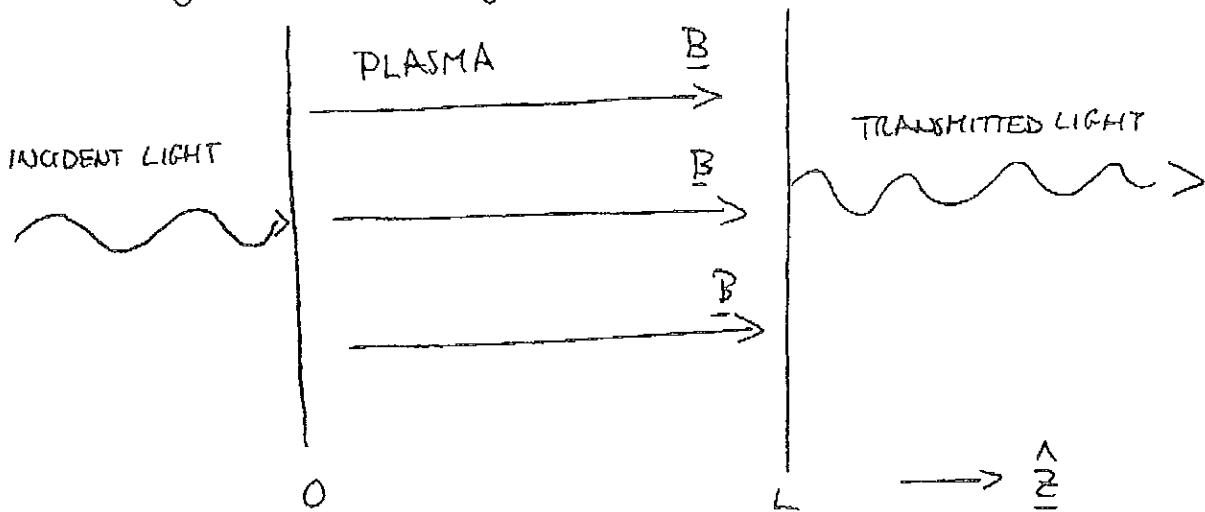
Transport



222b. Homework Faraday Effect.

Steve back for class on 2nd February.

Consider a plasma of length  $L$  (in the  $\hat{z}$  direction) and density  $n_0$  with a magnetic field  $\underline{B} = B_0 \hat{z}$ .



$$\underline{\delta E} = \underline{\delta E}_0 \hat{x} \cos(kz - \omega t) \equiv \text{INCIDENT LIGHT}$$

Let the frequency of light be high so that  $\omega > \omega_p, \omega_{ce}$  and ignore ion contributions to the dielectric tensor. CALCULATE the polarization of the transmitted light.

