

222a. Plasma Physics: Lecture #1, What is a Plasma?

Lecture notes have mistakes - they are not intended to be a book.

- (i) Definition of a plasma:
(I do not know a good one!)
 - a) Gas that is sufficiently ionized.
 - b) Ionized gas that displays collective behavior.

(ii) Plasmas are very common in nature, - only dark matter has more mass.

EXAMPLES

a) ISM (Inter-stellar medium) $n_e = \# \text{ of electrons per } m^{-3} \sim 10^6$
 $T_e = \text{Electron temperature } \lesssim 10^4$

b) J.E.T. Magnetic fusion device, $n_e = 10^{20} m^{-3}$
 $T_e = 10 \text{ keV. } (10^8 \text{ °K.})$

(iii) Start with some very simple considerations: - $n_e = \# \text{ electrons per } m^{-3}$

⇒ $n_e^{-1/3} \sim \text{mean separation of electrons } (10^{-6-7} m \text{ in J.E.T.})$

(iv) $\frac{3}{2} k T_e = \text{mean kinetic energy of the electrons} = \frac{3}{2} m v_{\text{the}}^2$
 \nwarrow defines "temperature".

v_{the} = "thermal velocity of the electrons".

It is common to measure temperature in electron volts so that

$e T_{\text{eV}} = k T_{\text{OK}} = \frac{2}{3} \times \text{Mean Kinetic Energy of electrons}$
 \uparrow
 Boltzmann's constant

⇒ $1 \text{ eV} \approx 10^4 K = 1.6 \times 10^{-19} \text{ Joules.}$
 \uparrow
 Kelvin

(v) Ions: Density = n_i Temperature = T_i Charge = Ze .

In nature plasmas are usually neutral (on average) so that $n_e \approx n_i Z$. Often the plasma is hydrogen - $Z=1$.

(vi) The Classical Plasma: Formally it is not hard to write down the equations we are solving.

For the i^{th} particle:- Position = $\underline{x}_i(t)$

$$\text{velocity} = \underline{v}_i(t) = \frac{d\underline{x}_i}{dt} \quad (1)$$

$$(2) \frac{d\underline{p}_i}{dt} = q_i \left\{ \underline{E}(\underline{x}_i, t) + \underline{v}_i \times \underline{B}(\underline{x}_i, t) \right\} \quad \begin{matrix} \text{EQUATION OF} \\ \text{MOTION.} \end{matrix}$$

\Leftrightarrow LORENTZ FORCE

$$\text{NON RELATIVISTICALLY } \underline{p}_i = m_i \underline{v}_i$$

To find the fields we solve Maxwell's equations with

$$\rho(\underline{r}, t) = \text{charge density} = \sum_i q_i \delta(\underline{r} - \underline{x}_i(t))$$

$\delta(\underline{r} - \underline{x}_i(t))$ DIRAC DELTA FUNCTION - ZERO EVERYWHERE BUT ON PARTICLE, $\int \delta(r_i) d^3r = 1$

$$\underline{j}(\underline{r}, t) = \text{current density} = \sum_i q_i \underline{v}_i \delta(\underline{r} - \underline{x}_i(t))$$

AND MAXWELL'S EQUATIONS ARE :-

$$(3) \nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0}$$

$$(4) \nabla \cdot \underline{B} = 0$$

$$(5) \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \nabla \times \underline{B} - \mu_0 \underline{j}$$

$$(6) \frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E}$$

(vii) Solving Eqns. ① - ⑥ for the 10^{23} plus particles and for \underline{E} and \underline{B} would be IMPOSSIBLE. Particle simulation does it for $< 10^9$ particles. Schematically one does:

use ① to update \underline{x}_i of each particle $\underline{x}_i(t+\Delta t) = \int_{t+\Delta t}^t \underline{v}_i dt$.

use ② to update \underline{v}_i of each particle $\underline{v}_i(t+\Delta t) = \int_t^{t+\Delta t} \underline{q}_i(\underline{E} + \underline{v}_i \times \underline{B}) dt$

RECONSTRUCT \underline{P} AND \underline{J} (APPROXIMATELY)

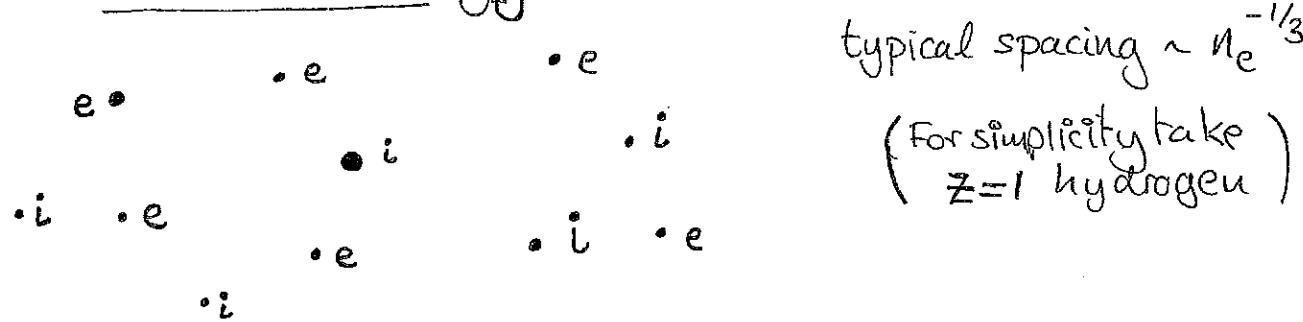
use ③ - ⑥ to solve for \underline{E} and \underline{B} .

EACH Timestep.

(viii) All this is formal and of little use since it is too complex for practical calculation. Things are much simpler in practice because the plasma is "weakly coupled" - as we shall see.

ss _____ ss

(ix) Electrostatic Energy:



typical local interaction energy $\sim \frac{e^2}{4\pi\epsilon_0 r} \sim \frac{e^2}{4\pi\epsilon_0 n_e^{1/3}}$
of closest particle.

BUT IF THE PARTICLES ARE DISTRIBUTED RANDOMLY THIS COULD BE EITHER i-e OR e-e OR i-i INTERACTION - SO ON AVERAGE NO INTERACTION ENERGY. HOWEVER THE PARTICLES ARE NOT DISTRIBUTED RANDOMLY SINCE ELECTRONS ARE ATTRACTED TO IONS AND REPULLED FROM OTHER ELECTRONS.

(x) Distribution of charge around a test charge in thermal equilibrium.

From Boltzmann's law in thermal equilibrium.

Assuming a "smoothed out" distribution of charges.

$$\left\{ \begin{array}{l} n_e = n_0 e^{-\frac{\epsilon_e}{kT}} \\ \quad \quad \quad = n_0 e^{\frac{e\phi}{kT}} \\ n_i = n_0 e^{-\frac{\epsilon_i}{kT}} \\ \quad \quad \quad = n_0 e^{-\frac{e\phi}{kT}} \end{array} \right.$$

ϵ_e = electron potential energy.
 $= -e\phi$

ϕ = electrostatic potential.

ϵ_i = ion potential energy
 $= e\phi$ (hydrogen)

Poisson's Equation.

$$(xi) \nabla^2 \phi = -\frac{f}{\epsilon_0} = -\frac{n_0 e}{\epsilon_0} \left\{ e^{-\frac{e\phi}{kT}} - e^{\frac{e\phi}{kT}} \right\} - \frac{Q \delta(r)}{\epsilon_0}$$

test charge.

- If $\frac{e\phi}{kT} \ll 1$ we can expand the exponentials

→ show this later (a posteriori)

- Since the source in this equation - the delta function is spherically symmetric we have to expect a spherically symmetric solution so - $\phi = \phi(r)$ and thus

$$\nabla^2 \phi = \frac{1}{r} \frac{d^2(r\phi)}{dr^2} = \left(2 \frac{n_0 e^2}{\epsilon_0 k T} \right) \phi - Q \delta(r)$$

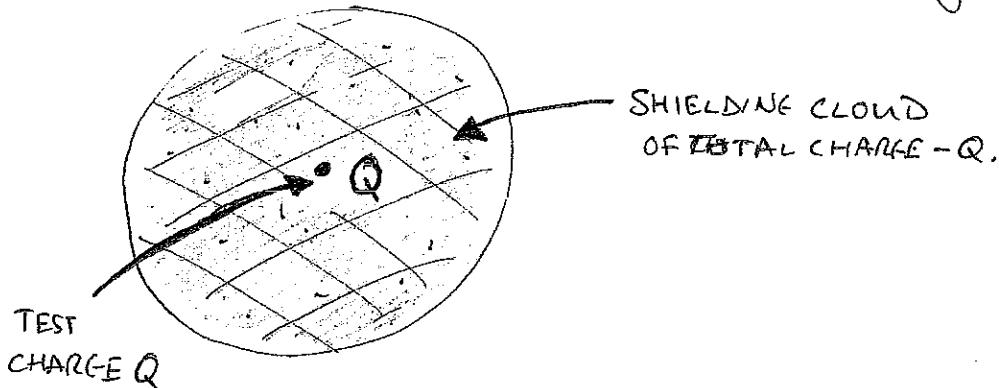
this easily solved to give:-

$$\phi(r) = \frac{Q}{4\pi\epsilon_0 r} e^{-\frac{\sqrt{2}r}{\lambda_D}}$$

$$\lambda_D = \sqrt{\frac{\epsilon_0 k T}{n_0 e^2}}$$

DEBYE LENGTH.

Shielded Coulomb field:- "Dressed Charge"



SO THE CHARGES ARE NOT RANDOM; THERE ARE MORE ELECTRONS THAN IONS NEAR EACH ION. WE CAN THINK OF EACH PARTICLE AS A TEST PARTICLE SURROUNDED BY A SHIELDING CLOUD.

(xi) Plasma Parameter

$$g = \frac{4\pi}{3} \lambda_D^3 n_e \equiv \text{Plasma Parameter} \equiv \# \text{ of electrons in a Debye sphere.}$$

$g \gg 1$ • Ideal plasma "weakly coupled" THIS COURSE FROM NOW ON,

$g \ll 1$ • "Strongly coupled" plasma rare and not treated by me.

(xii) In the weak coupling limit the perturbation caused by each particle on the others is small.

$$\frac{e\phi}{kT_e} \sim \frac{e^2}{4\pi\epsilon_0 k \lambda_D} \quad \text{For typical electron/ion in cloud.}$$

$$\sim \left(\frac{n_e e^2}{\epsilon_0 k T} \right) \frac{1}{n_e \lambda_D} \frac{1}{4\pi} \sim \frac{1}{4\pi \lambda_D^3 n} \sim \frac{1}{3g} \ll 1$$

(xiii) Another way to look at this is:

$$\frac{3}{2} kT \sim \text{Average kinetic energy of electron/ion.} = \langle K.E. \rangle$$

$$\int \rho \phi d^3r \sim \frac{e^2}{4\pi\epsilon_0 \lambda_D} \sim \text{Average potential energy of charge cloud around test ion } Q=e. = \langle P.E. \rangle$$

$$\frac{\langle K.E. \rangle}{\langle P.E. \rangle} \sim g \gg 1.$$

(compared to the K.E.)

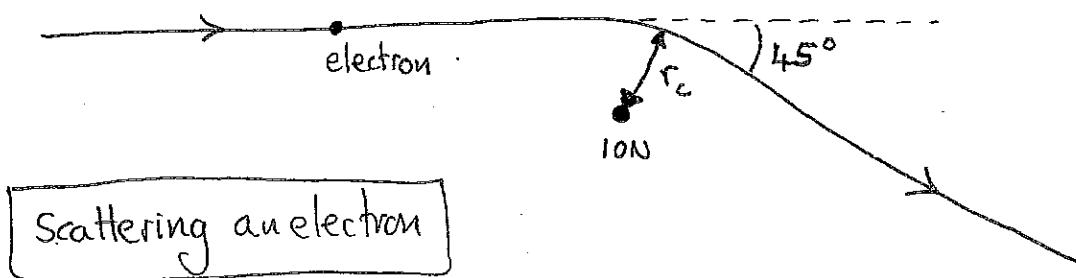
Potential energy is small, so particles in general move unperturbed by the presence of the other particles - even their own shielding cloud.

(xiv) There is "a" closest particle - the cloud is not really a smooth distribution of charge. The potential energy of this closest particle is random, larger but still not as large as the kinetic energy.

$$\frac{\text{P.E. of closest particle}}{\langle K.E. \rangle} = \frac{\frac{e^2}{4\pi\epsilon_0 n^{-1/3}}}{\frac{3kT}{2}} \sim \frac{1}{g^{2/3}} \ll 1 \quad \underline{\text{SMALL.}}$$

but no $\frac{1}{g^{2/3}} > \frac{1}{g}$ so this particle has larger P.E. than average particle.

(xv) COLLISIONS: the formal theory in the winter here we take a look at the rough scaling.



the electron scatters by about 45° when it passes close enough that K.E. = P.E.

$$\frac{1}{2}mv^2 \sim \frac{3}{2}kT \sim \frac{e^2}{4\pi\epsilon_0 T_c}$$

CROSS SECTION

$$\sigma \sim \pi r_c^2 \sim \frac{\lambda_D^2}{g^2} \ll \lambda_D^2$$

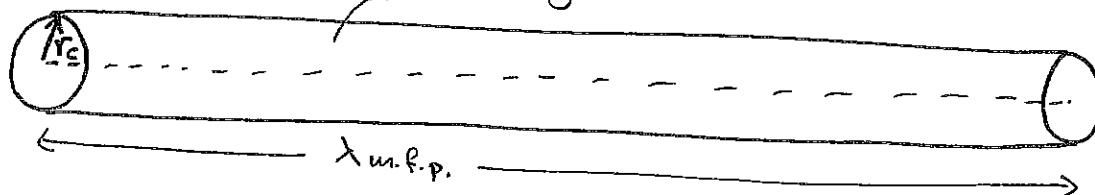
note also

$$\sigma \sim \frac{n^{-2/3}}{g^{4/3}}$$

$$r_c \ll n^{-2/3}$$

(xvi) MEAN FREE PATH - distance between collisions - $\lambda_{m.f.p.}$

n = density.



ROUGHLY ONE PARTICLE IN TUBE - THUS THERE IS ONE COLLISION AS ELECTRON GOES THE LENGTH OF THE TUBE.

$$\pi r_c^2 \lambda_{m.f.p.} = \text{volume of tube.} \Rightarrow \text{Number of particles in tube} = n \pi r_c^2 \lambda_{m.f.p.} = 1.$$

$$\lambda_{m.f.p.} = \frac{1}{n\sigma} \approx \lambda_D g \gg \lambda_D$$

Goes a very long way before colliding.

(xvii) Example :- Magnetosphere. $n \sim 10^{10} \text{ m}^{-3}$ $T_e \sim 100 \text{ eV}$

$$\lambda_D \sim 0.74 \text{ m} \quad g \sim 1.7 \times 10^{10}$$

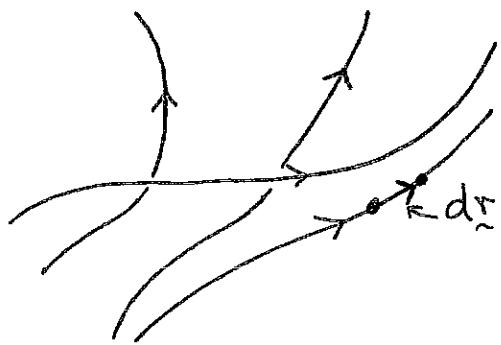
$$\lambda_{m.f.p.} \sim 10^9 \text{ m} \sim 10^6 \text{ km} \quad \text{HUGE - MUCH BIGGER THAN SIZE OF SYSTEM.}$$

222a. Lecture #2. Magnetic Fields.

Steve Cowley
2003.

(i) Many plasmas of interest are magnetized. The structure of the fields is critical to plasma behavior. They can be very complicated.

(ii) Field lines: $\underline{B} = \underline{B}(\underline{r}, t)$ We freeze field at some instant and follow field lines



dr - little step
in direction
of field.

$$|dr| = dl \quad |\underline{B}| = B$$

$$\frac{dr}{dl} = \frac{\underline{B}}{B} = \text{unit vector in direction of field} = \underline{b}$$

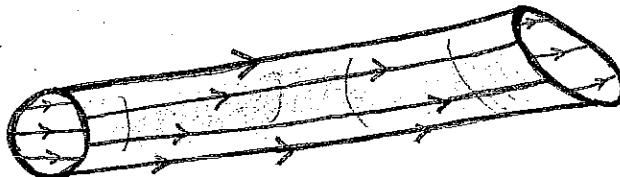
EQUATION DEFINES "FIELD LINE".

note this gives $\frac{dx}{dy} = \frac{B_x}{B_y} \quad \frac{dz}{dx} = \frac{B_z}{B_x} \quad \frac{dy}{dz} = \frac{B_y}{B_z}$

what about $B=0$?

Charged particles tend to follow field lines ~~will see~~ little helices.

(iii) Flux Tubes: A flux tube is a tube whose sides are everywhere parallel to the field.



From $\nabla \cdot \underline{B} = 0$ $\oint \underline{B} \cdot d\underline{s} = 0$ so magnetic flux along tube is constant.

Usually we take narrow flux tubes.

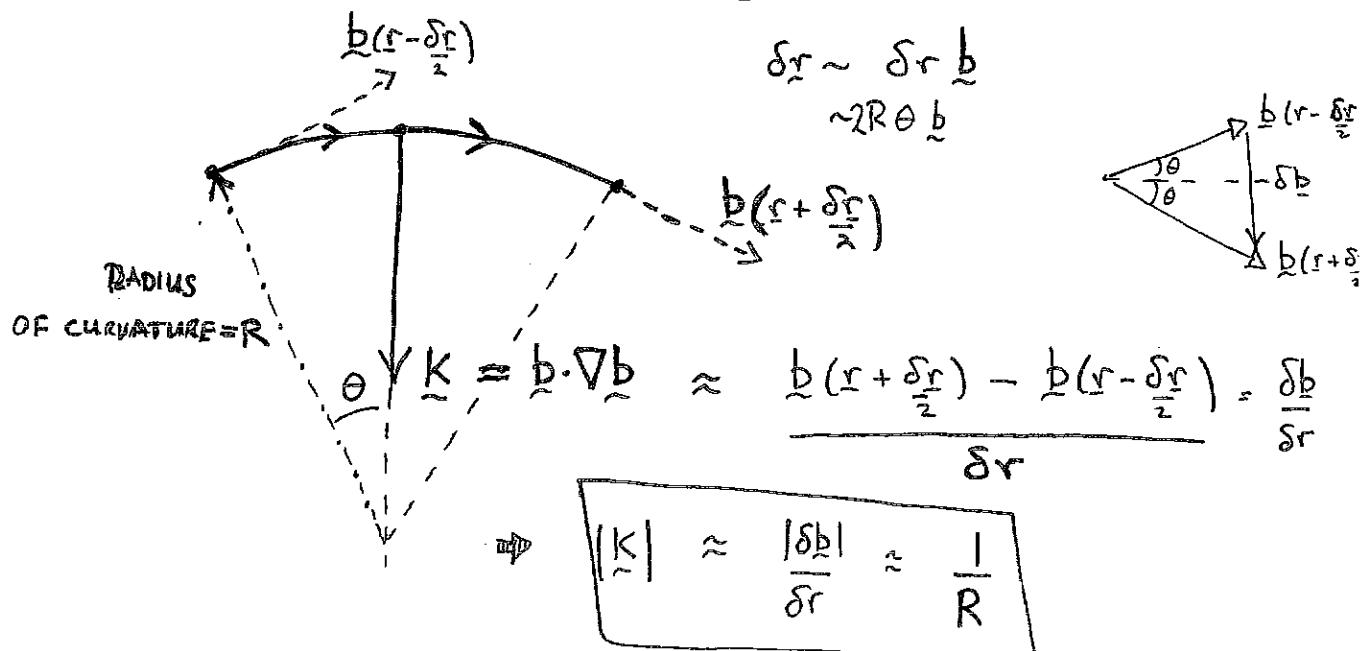
(iv) Current: For a stationary field ($\frac{\partial}{\partial t} = 0$) the magnetic field has

$$\nabla \times \underline{B} = \mu_0 \underline{J}$$

(V) CURVATURE: We define the curvature of a field line as

$$\underline{b} \cdot \nabla \underline{b} = K = \text{CURVATURE} \quad \hat{\underline{b}} = \frac{\underline{B}}{|\underline{B}|} = \text{unit vector along } \underline{B}.$$

= "How much \underline{b} varies along itself."

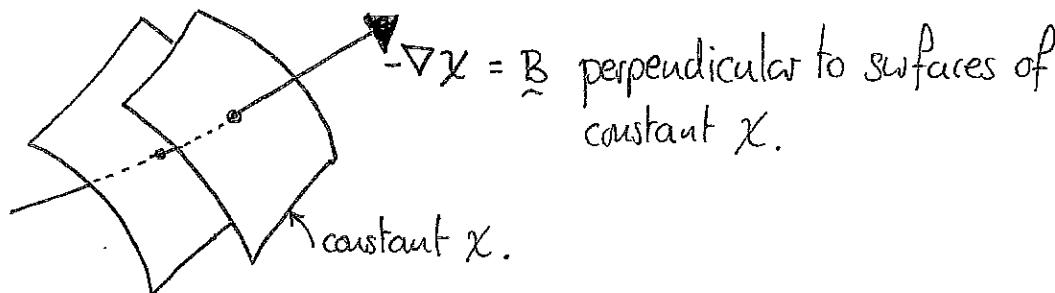


(VI) VACUUM FIELDS: Fields can often be written in terms of scalar "potentials." The best known - but not very often useful - is the potential for a vacuum field with $\frac{\partial}{\partial t} = 0$.

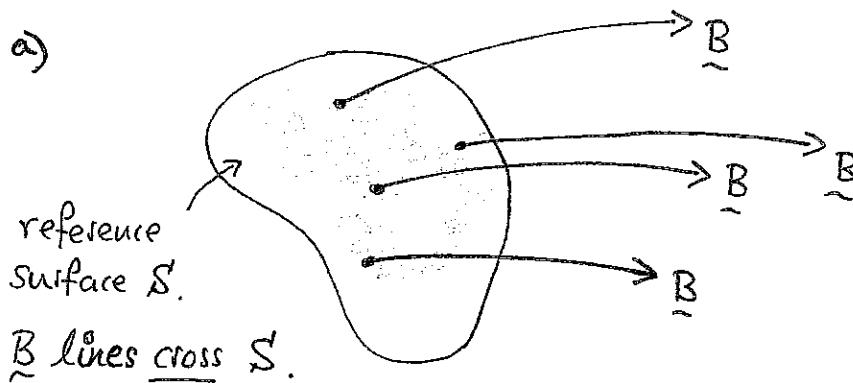
$$\Rightarrow \nabla \times \underline{B} = \mu_0 \underline{J} = 0 \text{ NOCURRENT.}$$

$$\underline{B} = -\nabla \chi$$

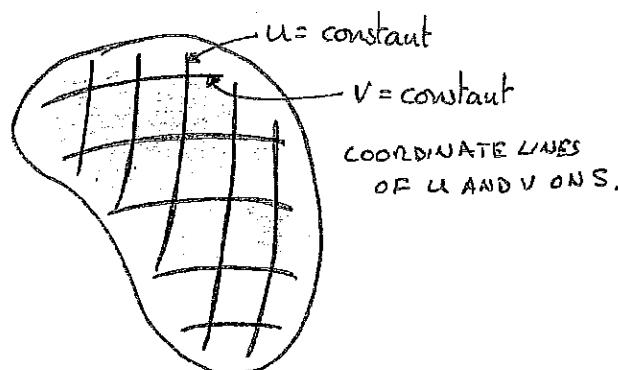
$$\text{Then } \nabla \cdot \underline{B} = 0 \quad \nabla^2 \chi = 0 \quad \text{Laplace's Equation.}$$



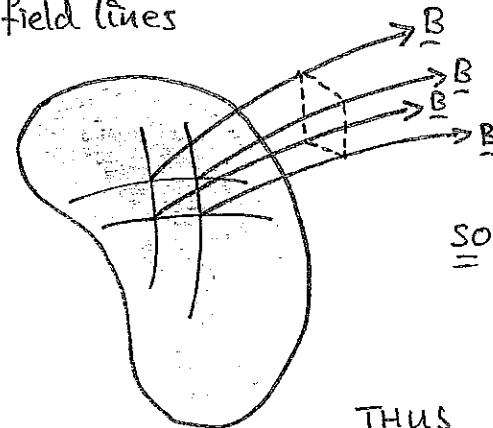
(VI) Glebsch Potentials: General Field. Here we show how to construct them



b) Construct any coordinate system on S i.e. label each point on S with two numbers u and v .



c) CONTINUE u and v off S by making them constant along field lines



$$\underline{B} \cdot \nabla u = 0 \\ \text{AND } \underline{B} \cdot \nabla v = 0$$

so \underline{B} is perpendicular to both ∇u and ∇v

THUS

$$\underline{B} = a(r) \nabla u \times \nabla v$$

d) Now use the condition:-

$$\nabla \cdot \underline{B} = 0 \quad \nabla \cdot (a \nabla u \times \nabla v) = \nabla u \times \nabla v \cdot \nabla a = 0$$

$\Rightarrow \underline{B} \cdot \nabla a = 0 \Rightarrow "a"$ constant along field lines

$$\Rightarrow a = a(u, v)$$

② Now construct the Clebsch coordinates α and β .

$$\alpha = \int^u a(u,v) du' \quad \beta = v$$

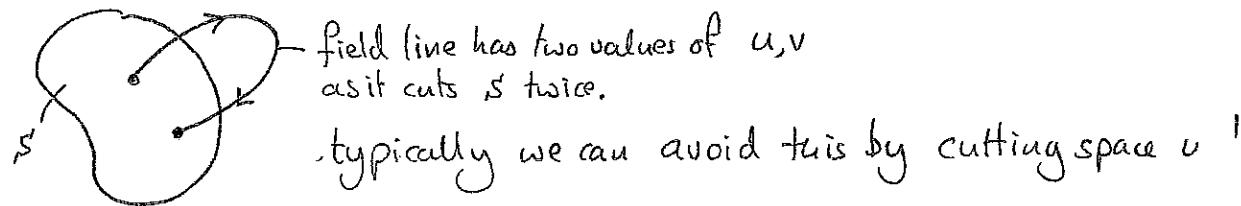
$$\nabla \alpha = a \nabla u + \nabla v \left[\int^u \frac{\partial a(u';v)}{\partial v} du' \right]$$

$\Rightarrow \boxed{B = \nabla \alpha \times \nabla \beta}$

CLEBSCH REPRESENTATION

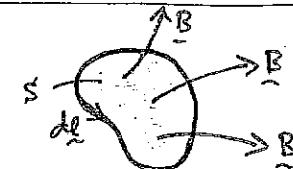
Note the Clebsch representation guarantee that $\nabla \cdot B = 0$ but $\nabla \times B$ can be anything - clearly it applies to time dependant fields etc.

ONE CAVEAT. the coordinates α and β can become multi-valued.



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Magnetic Flux in Clebsch representation.



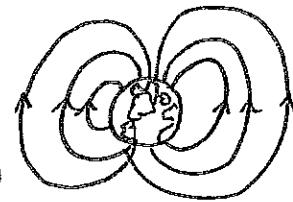
$$\begin{aligned} \int_S \underline{B} \cdot d\underline{A} &= \oint_L \underline{B} \cdot d\underline{l} = \text{Flux through surface } S = \int \nabla \alpha \times \nabla \beta \cdot d\underline{A} = \int \nabla \times (\alpha \nabla \beta) \cdot d\underline{A} \\ &= \oint_L \alpha \nabla \beta \cdot d\underline{l} \quad (\text{using Stoke's theorem.}) \quad (\text{Loop } L \text{ around } S) \\ &= \oint \alpha d\beta \end{aligned}$$

Let's do an example:

(vii) Dipole Magnetic Field: A good model for earth's field for ionosphere and lower magnetosphere.

$$\chi = \chi_0 \frac{\cos \theta}{r^2}$$

Dipole field. (Vacuum pot.)



Note: $\nabla \chi = 0$

NO TOROIDAL FIELD i.e. field in $\hat{\phi}$ direction.
POLOIDAL FIELD

$$\underline{B} = \frac{\partial \chi}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial \chi}{\partial \theta} \hat{\theta} = -\chi_0 \left\{ \frac{2 \cos \theta}{r^3} \hat{r} + \frac{\sin \theta}{r^3} \hat{\theta} \right\}$$

FIND THE CLEBSCH REPRESENTATION.

We choose one of the coordinates to be the angle ϕ (east-west angle) as we know $\underline{B} \cdot \nabla \phi = 0$.

$$\underline{B} = \nabla \psi \times \nabla \phi$$

$$\nabla \phi = \frac{\hat{\phi}}{r \sin \theta} = \frac{\hat{\phi}}{R}$$

unit vector
in eastward
direction.

$$B_r = \hat{r} \cdot \nabla \psi \times \nabla \phi = \frac{\hat{\phi} \cdot \nabla \psi}{r \sin \theta}$$

$$B_r = \frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial \theta} = -\chi_0 \frac{2 \cos \theta}{r^3}$$

$$B_\theta = \hat{\theta} \cdot \nabla \psi \times \nabla \phi = -\frac{\hat{z} \cdot \nabla \psi}{r \sin \theta} = -\frac{1}{r^2 \sin \theta} \frac{\partial \psi}{\partial r} = -\chi_0 \frac{\sin \theta}{r^3}$$

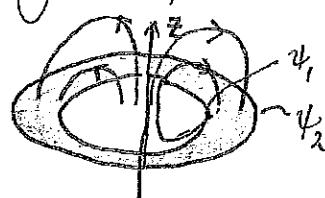
INTEGRATING WE GET

$$\psi = -\chi_0 \frac{\sin^2 \theta}{r} + \text{constant that we ignore}$$

Equation for field line is $\psi = \text{constant}$ i.e.

$$r = -\frac{\chi_0 \sin^2 \theta}{\psi}$$

Magnetic flux through equatorial strip = $\int_0^{2\pi} (\psi_1 - \psi_2) d\phi = 2\pi (\psi_1 - \psi_2)$



- "poloidal flux" = $2\pi \Delta \psi$

Plasma Physics, 222: Homework 2.

Lecturer: Steve Cowley

Question 1. Sheared Magnetic Field.

To model more complicated plasmas simple configurations have been studied extensively. One such configuration is the *sheared slab* here we look at this simple model as an example of the things we discussed in class.

- (i) The *sheared slab* field is:

$$\mathbf{B} = B_0(z + \frac{x}{l_s}\mathbf{y}), \quad (1)$$

where \mathbf{z} and \mathbf{y} are unit vectors in the z and y direction respectively and B_0 is a constant. Show that $\nabla \cdot \mathbf{B} = 0$ and calculate the current density. Are the field lines curved?

- (ii) Draw the field lines – do your best this is hard. You might try drawing the field lines on different x planes separately.

- (iii) Find Clebsch potentials α and β so that

$$\mathbf{B} = \nabla\alpha \times \nabla\beta. \quad (2)$$

Hint. Start by finding a direction that is always perpendicular to the field, this will determine one of the potentials – say β .

- (iv) A flux tube has a circular cross section $x^2 + y^2 = 1$ in the plane $z = 0$. What shape is the cross section in the plane $z = l_s$. Draw the tube.

222a. Lecture # 3. Particle Motion in Constant Fields.

(i) Start with the simplest case - we use vector notation because it helps later. See Boyd's Panderson Chapter 2. for cartesian derivation

(ii) CONSTANT \underline{B} FIELD - NON RELATIVISTIC.

$$m \frac{d\underline{v}}{dt} = q(\underline{v} \times \underline{B}) \quad \text{--- (1)}$$

$$\underline{B} = B \underline{b}$$

\underline{b} = constant unit vector, B = constant magnitude of \underline{B} .

(iii) Conservation of energy - $\underline{v} \cdot \text{--- (1)}$

$$m \frac{d\underline{v} \cdot \underline{v}}{dt} = q \underline{v} \cdot (\underline{v} \times \underline{B}) = 0 \Rightarrow \frac{d}{dt} \left[\frac{1}{2} m v^2 \right] = 0$$

$\frac{1}{2} m v^2$ = Kinetic Energy is conserved.

(iv) Parallel equation.

$$m \frac{d\underline{v} \cdot \underline{b}}{dt} = q \underline{b} \cdot (\underline{v} \times \underline{B}) = 0 \Rightarrow \frac{d(v \cdot b)}{dt} = 0$$

$$\underline{v} \cdot \underline{b} = v_{\parallel} = \text{constant.}$$

$$\underline{v} = v_{\parallel} \underline{b} + \underline{v}_{\perp} \quad \text{and} \quad \underline{v}_{\perp} \cdot \underline{b} = 0$$

(v) Note $v^2 = v_{\parallel}^2 + \underline{v}_{\perp} \cdot \underline{v}_{\perp} = \text{constant.} \Rightarrow v_{\perp} v_{\perp} = v_{\perp}^2 = \text{constant.}$

(vi) Thus we write $\underline{v} = v_{\perp} [\cos \theta \underline{e}_1 + \sin \theta \underline{e}_2] + v_{\parallel} \underline{b}$
 \nwarrow constant.

where \underline{e}_1 , \underline{e}_2 and \underline{b} are mutually orthogonal $\underline{e}_1 \cdot \underline{e}_2 = \underline{e}_1 \cdot \underline{b} = \underline{e}_2 \cdot \underline{b} = 0$
and $\underline{e}_1 \times \underline{e}_2 = \underline{b}$ LOCAL COORDINATE VECTORS.

$$(vii) m \frac{d\underline{v}_\perp}{dt} = m \underline{v}_\perp \left[-\sin\phi \frac{d\phi}{dt} \underline{e}_1 + \cos\phi \frac{d\phi}{dt} \underline{e}_2 \right] = q \left[\underline{v}_\perp \times \underline{B} \right]$$

$$= \underline{v}_\perp q \underline{B} \left[\begin{matrix} \cos\phi (\underline{e}_1 \times \underline{b}) \\ \parallel \\ -\underline{e}_2 \end{matrix} + \sin\phi (\underline{e}_2 \times \underline{b}) \right] \begin{matrix} \parallel \\ \underline{e}_1 \end{matrix}$$

equating these two expressions yields:-

$$\boxed{\frac{d\phi}{dt} = -\frac{qB}{m}} \rightarrow \boxed{\phi = -\Omega t + \phi_0} \quad \phi = \text{Gyro-angle.}$$

$$\Omega = \pm \frac{qB}{m}$$

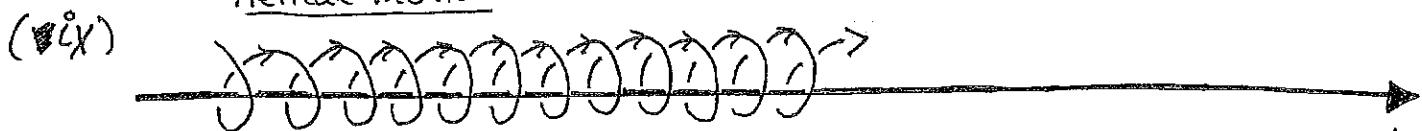
"cyclotron frequency" "Larmor frequency."
"gyro-frequency."

$$(viii) \underline{v} = \frac{d\underline{r}}{dt} = \underline{v}_\perp \left[\cos(\phi_0 - \Omega t) \underline{e}_1 + \sin(\phi_0 - \Omega t) \underline{e}_2 \right] + \underline{v}_\parallel \underline{b}$$

Integrate to find $\underline{r}(t)$

$$\underline{r}(t) = \underline{r}_0 + \frac{\underline{v}_\perp}{\Omega} \left[-\sin(\phi_0 - \Omega t) \underline{e}_1 + \cos(\phi_0 - \Omega t) \underline{e}_2 \right] + \underline{v}_\parallel t \underline{b}$$

Helical motion.



$$\text{Radius of the helix} = \frac{\underline{v}_\perp}{\Omega} = \frac{\text{LARMOR RADIUS.}}{\text{CYCLOTRON RADIUS.}}$$

Time to go around the field line once = $\frac{2\pi}{\Omega}$ Gyro-period.
Cyclotron-period.
Larmor-period.

Motion along field = $v_\parallel t$ - constant velocity.

(X) For particles moving with the thermal velocity $v_{th} = \sqrt{\frac{kT}{m}}$

$$v_{\perp}, v_{\parallel} \sim \mathcal{O}(v_{th}) \propto \frac{1}{\sqrt{m}}$$

$$\text{LARMOR RADIUS } r = \frac{v_{th}}{\omega} = \sqrt{\frac{kT}{m}} \frac{m}{qB} \propto \sqrt{m}$$

(xi) For comparison consider electrons and protons in thermal equilibrium

$$T_e = T_i$$

$$\text{THERMAL VELOCITIES:- } v_{the} = \sqrt{\frac{m_p}{m_e}} v_{thi} \simeq 40 v_{thi}$$

$$\text{LARMOR RADII:- } r_e = \sqrt{\frac{m_e}{m_p}} r_i \simeq \frac{1}{40} r_i$$

$$\text{CYCLOTRON FREQUENCY:- } \Omega_e = \frac{m_p}{m_e} \Omega_i \simeq 2000 \Omega_i$$

(xii) Motion in Constant B and E fields.

$$m \frac{d\vec{v}}{dt} = q \{ \vec{E} + \vec{v} \times \vec{B} \}$$

$$\vec{B} = B \hat{b}$$

$$\vec{E} = E_{\parallel} \hat{b} + \vec{E}_{\perp}$$

$$\vec{v}(t) = v_{\parallel}(t) \hat{b} + \vec{v}_{\perp}$$

Parallel motion.

$$m \frac{dV_{\parallel}}{dt} = q E_{\parallel}$$



$$V_{\parallel} = \frac{q E_{\parallel} t}{m} + V_{\parallel}(0)$$

Acceleration in parallel direction.

Perpendicular motion.

$$m \frac{dV_{\perp}}{dt} = q \left\{ \vec{E}_{\perp} + \vec{v}_{\perp} \times \vec{B} \right\}$$

constant.

To solve this we write $\vec{v}_{\perp} = \vec{v}_{\perp}(t) + \vec{v}_{EXB}$

then the equation becomes:-

$$m \frac{d\vec{c}_\perp}{dt} = q \left\{ \vec{E}_\perp + (\vec{c}_\perp + \vec{v}_{ExB}) \times \vec{B} \right\}$$

We choose \vec{v}_{ExB} so that

$$(xiii) \quad \vec{E}_\perp + \vec{v}_{ExB} \times \vec{B} = 0 \quad \rightarrow \quad \vec{v}_{ExB} = \frac{\vec{E} \times \vec{B}}{B^2}$$

CALLED
"E cross B
velocity"
or
"ExB drift"

THEN, the equation for \vec{c}_\perp is identical to the one for \vec{v}_\perp with no E field.

$$m \frac{d\vec{c}_\perp}{dt} = q \left\{ \vec{c}_\perp \times \vec{B} \right\}$$

Thus we can write down the solution.

$$|\vec{c}_\perp| = c_\perp = \text{constant.}$$

$$\boxed{\vec{v} = \left[\frac{qE_{||}t}{m} + v_{||}(0) \right] \hat{b} + \vec{v}_{ExB} + c_\perp \left\{ \cos(\phi_0 - \omega t) \hat{e}_1 + \sin(\phi_0 - \omega t) \hat{e}_2 \right\}}$$

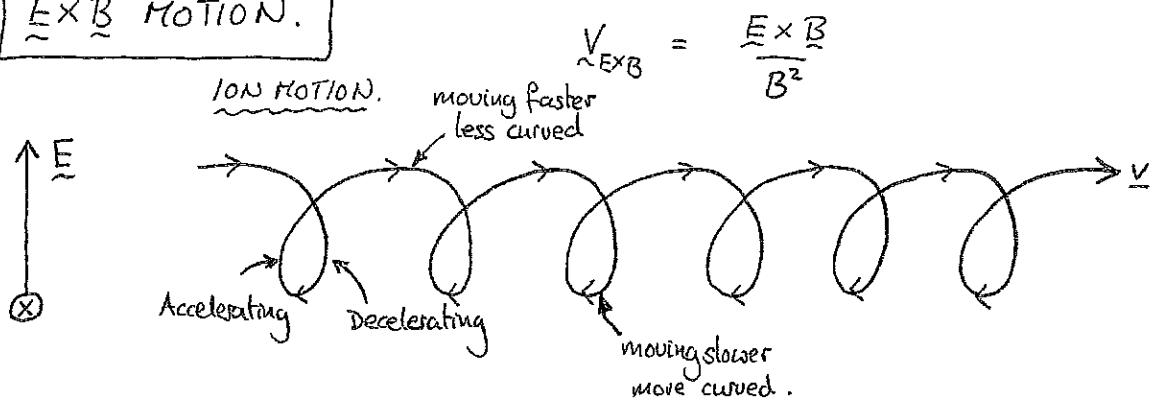
(xiv) And of course to get $\vec{r}(t)$ we integrate.

$$\vec{r}(t) = \vec{r}(0) + \left[\frac{qE_{||}}{m} \frac{t^2}{2} + v_{||}(0) \right] \hat{b} + \vec{v}_{ExB} t + \frac{c_1}{\omega} \left\{ -\sin(\phi_0 - \omega t) \hat{e}_1 + \cos(\phi_0 - \omega t) \hat{e}_2 \right\}$$

(xv) The constant ExB motion needs some explanation - the particle is being pushed in the direction of E but moves perpendicular to E !

First thing to realize is that when particle is moving faster

$\underline{E} \times \underline{B}$ MOTION.



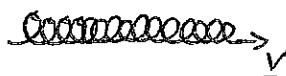
\underline{B} into page.

ION MOTION.

$$\underline{v}_{EXB} = \frac{\underline{E} \times \underline{B}}{B^2}$$

ELECTRON MOTION.

$$\underline{v}_{EXB} = \frac{\underline{E} \times \underline{B}}{B^2}$$



NOTE: $\underline{E} \times \underline{B}$ motion is independent of charge and mass. Therefore the same for electrons and ions.. it thus produces no current in a charge neutral plasma.

(XV) Time Dependant Electric Fields - Polarization.

We now go one stage more complicated we let $\underline{E} = \underline{E}_0 t$

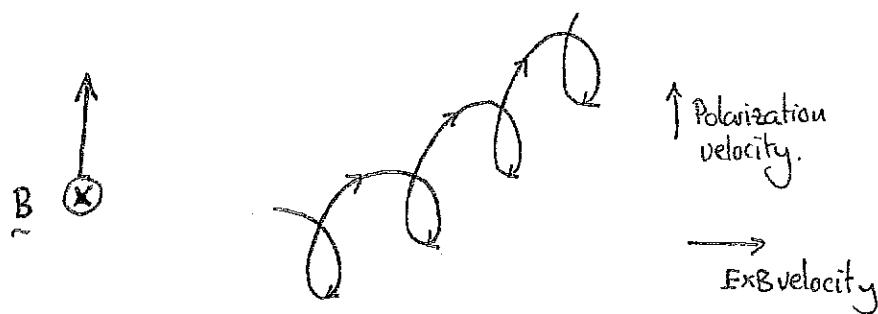
and $\underline{E}_0 \cdot \underline{B} = 0$

$$m \frac{d\underline{v}}{dt} = q \left\{ \underline{E}_0 t + \underline{v} \times \underline{B} \right\}$$

Solution proceeds in the same way - it's just one step more.

$$\underline{v} = \left(\frac{\underline{E}_0 \times \underline{B}}{B^2} \right) t + \frac{m}{q} \frac{\underline{E}_0}{B^2} + v_{||} \underline{b} + c_{\perp} \left\{ \cos(\phi_0 - \omega t) \underline{e}_1 + \sin(\phi_0 - \omega t) \underline{e}_2 \right\}$$

EXB DRIFT "POLARIZATION DRIFT"



222a. Lecture #4: Particle Motion in Inhomogeneous Fields I.

Good treatment of this in Hazeltine & Waelbroeck Chapter 2.

(i) We are not going to do everything at once - TODAY

$$\underline{E} = 0 \text{ so } \frac{\partial \underline{B}}{\partial t} = 0 \quad \underline{B} = \underline{B}(t) \quad \Omega = \frac{q}{m} \underline{B}(t)$$

$\underline{B} = \underline{B}(x)$ } Arbitrary variation of
 \underline{B} in space.

$$\frac{d\underline{v}}{dt} = \Omega(t) \underline{v} \times \underline{B}(t)$$

Note

$$\frac{1}{2} m v^2 = \text{constant in this case} = E = \text{K.E.}$$

(ii) Two length-scales: $\rho = \text{gyroradius/larmor radius} \approx v/\omega$

$$L = \text{distance over which } \underline{B} \text{ varies} \approx \left(\frac{\|\nabla \underline{B}\|}{\|\underline{B}\|} \right)^{-1}$$

Two time-scales: $\tau_s = \frac{1}{\omega} = \text{Gyro period}$

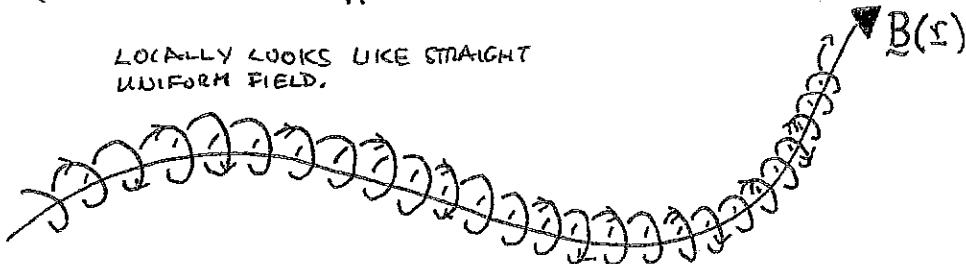
$$\tau_L = \frac{L}{v} = \text{time for particle to "see } \underline{B} \text{ change."}$$

(iii) FUNDAMENTAL SMALL PARAMETER FOR GUIDING CENTER APPROXIMATION

$$\epsilon = \frac{\tau_s}{\tau_L} = \frac{v}{\omega L} = \frac{r}{L} \ll 1$$

(iv) With this approximation \underline{B} field varies very little over one gyroperiod

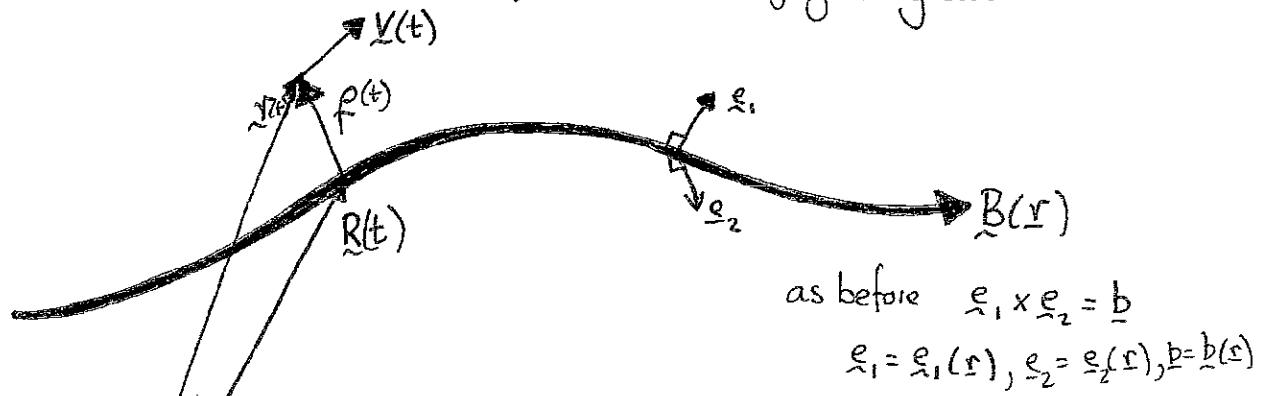
LOCALLY LOOKS LIKE STRAIGHT
 UNIFORM FIELD.



- SMALL VARIATION OF \underline{B}
 GIVES TWO EFFECTS
- a) SMALL WOBBLE ON EACH GYRATION - ELIPTICAL DISTORTION.
 b) ACCUMULATION OF SMALL CHANGES OVER MANY GYRATIONS.
 GIVES FORCE ALONG \underline{B} & DRIFT ACROSS \underline{B} .

(v) GUIDING CENTER MOTION

Aim:- To describe motion as a) Rapid gyration around \underline{B} plus
b) smooth motion of guiding center -



$\underline{r}(t)$ = position of particle.

$\underline{R}(t)$ = guiding center position. (smoothly varying)

$f(t)$ = Larmor radius.

= $\frac{\underline{b} \times \underline{v}}{\omega}$ in a straight field.

$f \sim \mathcal{O}(\epsilon)$ means roughly of size ϵ .

SO WE SET:

$$\underline{r}(t) = \underline{R}\left(\frac{t}{T_L}\right) + \frac{\underline{b} \times \underline{v}}{\omega} + \epsilon \Delta \underline{r}(rt) \quad (1)$$

GLIDING CENTER
POSITION SMOOTHLY
VARYING

CIRCULAR
MOTION VARIES
ON FAST TIMESCALE

Wobble on
FAST TIMESCALE

SPLITTING ONE VECTOR INTO THREE IS OBVIOUSLY NOT UNIQUE BUT

WE DETERMINE THE SPLITTING BY TWO CONDITIONS (1) THAT \underline{R} VARY SMOOTHLY/SLOWLY
(2) THAT $\Delta \underline{r}$ REMAIN SMALL.

WE ALSO WRITE :

$$\underline{r}(t) = \frac{d\underline{R}}{dt} + \underline{v}_\perp (\cos \theta \underline{e}_1 + \sin \theta \underline{e}_2) + \Delta \underline{v}(rt) \quad (2)$$

SLOWLY VARYING
"MEAN VELOCITY"

FLUCTUATING PERPENDICULAR
VELOCITY

Wobble on
VELOCITY
MUST REMAIN
SMALL.

$\Delta \underline{v} \sim \mathcal{O}(\epsilon v)$

note. $\frac{d\Delta r(r)}{dt} = \frac{dr}{dt} \cdot \frac{dr}{dr} = \underline{v} \cdot \nabla r$

(vi) We substitute into the equations of motion:

$$\textcircled{1} \rightarrow \frac{d\mathbf{r}}{dt} = \mathbf{v} \rightarrow$$

$$\frac{d\mathbf{R}}{dt} = \mathbf{v}_{||}\underline{\mathbf{b}} - \mathbf{v} \cdot \nabla \left(\frac{\mathbf{b}}{\omega} \right) \times \mathbf{v} - \frac{d(\epsilon \Delta \phi)}{dt}$$

$\mathcal{O}(V)$

$\mathcal{O}(V)$

$\mathcal{O}(\epsilon V)$

$\mathcal{O}(\epsilon V)$

TO LOWEST ORDER
IN ϵ .

$$\frac{d\mathbf{R}^{(0)}}{dt} = \mathbf{v}_{||}\underline{\mathbf{b}}$$

Motion along magnetic
field lines

$$\textcircled{1} \& \textcircled{2} \text{ into } \frac{d\mathbf{v}}{dt} = \omega (\mathbf{v} \times \underline{\mathbf{b}}) \rightarrow \text{ORDER WRITTEN OVER TERM}$$

$$\begin{aligned} & \frac{d^2\mathbf{R}}{dt^2} + \frac{d\mathbf{v}_1}{dt} (\cos \theta \underline{\mathbf{e}}_1 + \sin \theta \underline{\mathbf{e}}_2) + \mathbf{v}_1 \frac{d\theta}{dt} (-\sin \theta \underline{\mathbf{e}}_1 + \cos \theta \underline{\mathbf{e}}_2) + \mathbf{v}_1 \left(\cos \theta \frac{d\underline{\mathbf{e}}_1}{dt} + \sin \theta \frac{d\underline{\mathbf{e}}_2}{dt} \right) \\ & + \frac{d\Delta \mathbf{v}}{dt} = \omega \left\{ \frac{d\mathbf{R}}{dt} \times \underline{\mathbf{b}} + \mathbf{v}_1 \cos \theta (\underline{\mathbf{e}}_1 \times \underline{\mathbf{b}}) + \mathbf{v}_1 \sin \theta (\underline{\mathbf{e}}_2 \times \underline{\mathbf{b}}) + \Delta \mathbf{v} \times \underline{\mathbf{b}} \right\} \end{aligned}$$

VANISHES
TO LOWEST
ORDER

LOOKS LIKE A MESS BUT WE ONLY WANT LOWEST ORDER.

$$\frac{d\theta}{dt} = -\omega (\pm +)$$

$$\theta = \theta_0 - \int \omega dt'$$

LIKE THE CONSTANT
FIELD.

$$\mathbf{v}^{(0)} = \mathbf{v}_{||}(t)\underline{\mathbf{b}} + \mathbf{v}_1(t) \left\{ \cos(\theta_0 - \int \omega dt') \underline{\mathbf{e}}_1 + \sin(\theta_0 - \int \omega dt') \underline{\mathbf{e}}_2 \right\}$$

③

WE HAVE NOT YET CONCLUDED THE LOWEST ORDER SOLUTION

BECAUSE $\mathbf{v}_{||}(t)$ & $\mathbf{v}_1(t)$ ARE UNKNOWN, THEY VARY SLOWLY OF COURSE.

(V19) DETERMINING THE EVOLUTION OF $V_{||}(t)$

From $\frac{d\mathbf{v}}{dt} = \mathbf{s}(\mathbf{v} \times \mathbf{b})$ we have $\frac{d\mathbf{v}}{dt} \cdot \mathbf{b} = 0$

$$\Rightarrow \frac{d(\mathbf{v} \cdot \mathbf{b})}{dt} = \mathbf{v} \cdot \left(\frac{d\mathbf{b}}{dt} \right) = \mathbf{v} \cdot \nabla \mathbf{b} \cdot \mathbf{v}$$

$$\boxed{\frac{dV_{||}}{dt} + \frac{d(\mathbf{v} \cdot \mathbf{b})}{dt} = \mathbf{v} \cdot \nabla \mathbf{b} \cdot \mathbf{v}} \quad \mathcal{O}\left(\frac{V}{c_L}\right) \text{ only need lowest order } v \text{ in this term.}$$

this term varies
on the fast timescale
and must average to zero on
long timescale.

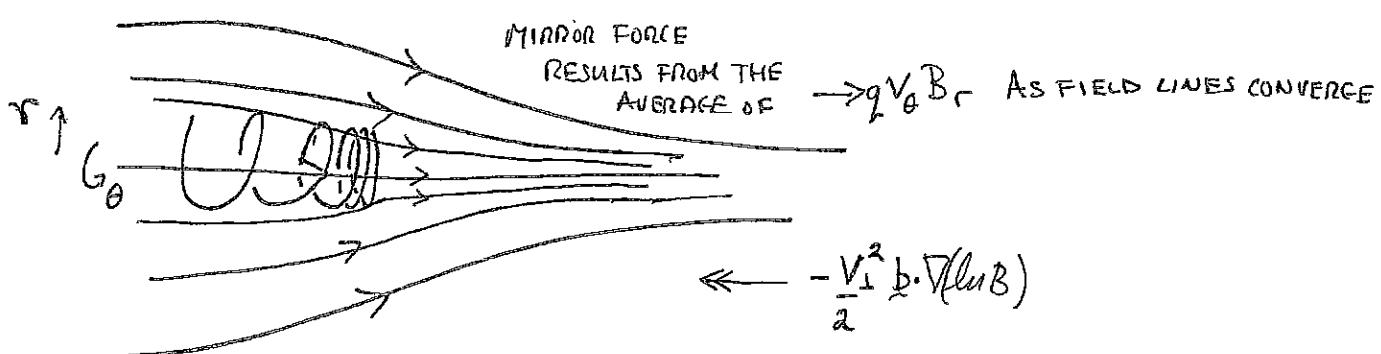
AVERAGE THIS EQUATION

OVER SHORTTIMESCALE TO REMOVE $d\mathbf{v}$ TERM.

$$\frac{dV_{||}}{dt} = \mathbf{s} \int_0^{\frac{1}{2}} dt \left[\mathbf{v}^{(0)} \nabla \mathbf{b} \cdot \mathbf{v}^{(0)} \right] = V_{||}^2 \underbrace{\mathbf{b} \cdot \nabla \mathbf{b} \cdot \mathbf{b}}_{\text{ZERO}} + \frac{V_{||}^2}{2} \left\{ \mathbf{e}_1 \cdot \nabla \mathbf{b} \cdot \mathbf{e}_1 + \mathbf{e}_2 \cdot \nabla \mathbf{b} \cdot \mathbf{e}_2 \right\}$$

$$\boxed{\frac{dV_{||}}{dt} = \frac{V_{||}^2}{2} \mathbf{B} \cdot \nabla \left(\frac{1}{B} \right) = -\frac{V_{||}^2}{2} \mathbf{b} \cdot \nabla (\ln B)} \quad \begin{aligned} &\text{MIRROR FORCE/ACCELERATION.} \\ &\text{PUSHES AWAY FROM STRONG FIELD.} \end{aligned}$$

$$\nabla \mathbf{b} = \nabla \frac{\mathbf{B}}{B}$$



(viii) ADIABATIC INVARIANT

Note first that:

$$\frac{d}{dt} \frac{V_{||}^2}{2} = V_{||} \frac{dV_{||}}{dt} = -\frac{1}{2} V_{||}^2 V_{||} \frac{\underline{b} \cdot \nabla B}{B} = -\frac{V_{||}^2}{2B} \frac{dR}{dt} \frac{dB}{dR}$$

$$= -\frac{V_{||}^2}{2B} \frac{dB}{dt}$$

But $V_{||}^2 = V^2 - V_{\perp}^2 = \frac{2E}{m} - V_{\perp}^2$ and recall that $\frac{dE}{dt} = \text{constant}$.

$$-\frac{d}{dt} \left(\frac{V_{||}^2}{2} \right) = -\frac{V_{||}^2}{2B} \frac{dB}{dt} \Rightarrow \frac{d}{dt} \left(\frac{V_{||}^2}{B} \right) = 0$$

$\mu = \frac{1}{2} m \frac{V_{ }^2}{B}$	1ST ADIABATIC INVARIANT - IT IS APPROXIMATELY CONSTANT ON THE LONG TIME SCALE	SOMETIMES CALLED "MAGNETIC MOMENT"
--	---	---------------------------------------

(ix) More generally μ is an approximate constant (to $\mathcal{O}(E)$ corrections) in a slowly varying E and B field.(x) Solving for $V_{||}$ we get -

$$V_{||} = \pm \sqrt{\frac{2}{m} (E - \mu B)}$$

The only thing that varies here is $B(r)$

And:

$$f = \left(\frac{2\mu B}{m} \right)^{1/2} \frac{1}{\Omega} \left(-\cos \theta(t) \underline{e}_1 + \sin \theta(t) \underline{e}_2 \right)$$

(xi) NOW WE HAVE ALL THE LOWEST ORDER INFORMATION

TO FIND THE ORBIT:

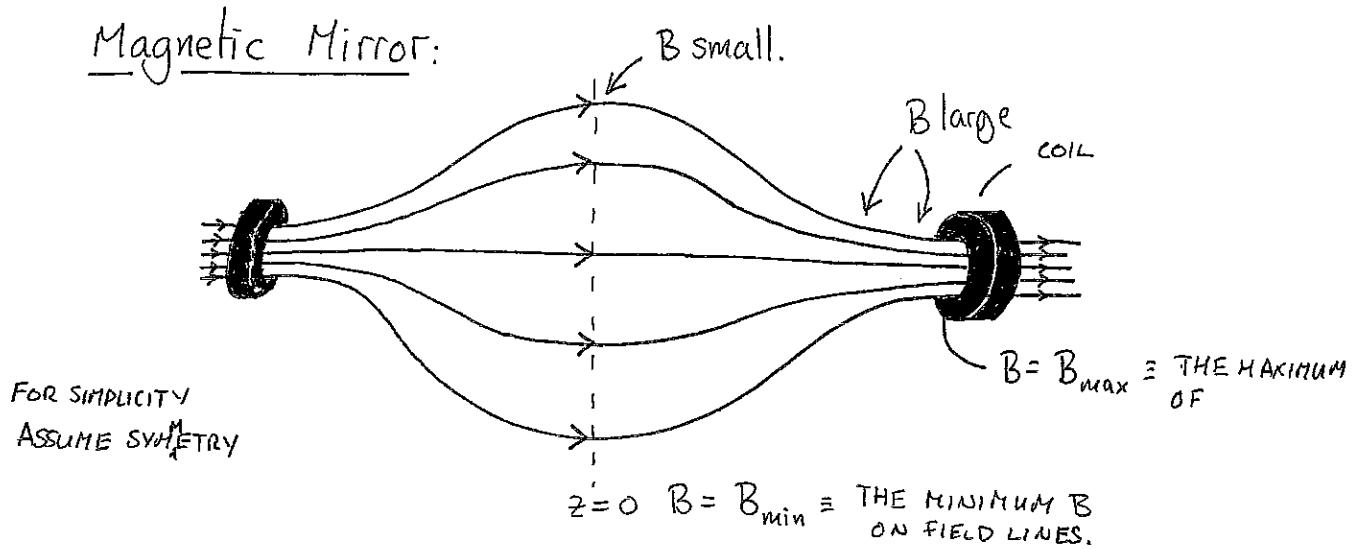
$$\underline{r}(t) = \underline{R}(t) d + f(t)$$

 $E = \text{CONSTANT}$, $\mu = \text{CONSTANT}$.

$$\frac{dR}{dt} = V_{||} \underline{b}, \quad V_{||} = \pm \sqrt{\frac{2}{m} (E - \mu B)}$$

$$f(t) = \left(\frac{2\mu B}{m} \right)^{1/2} \frac{1}{\Omega} \left(-\cos \theta(t) \underline{e}_1 + \sin \theta(t) \underline{e}_2 \right)$$

Magnetic Mirror:



Suppose we consider a particle with Energy ϵ and Magnetic Moment / 1st Adiabatic Invariant μ .

TRAPPED PARTICLE

$$\text{At } z=0 \quad V_{||} = V_{||0} = \sqrt{\frac{2}{m}(\epsilon - \mu B_{\min})}$$

$\mu B_{\min} < \epsilon$
OBVIOUSLY.
WE TAKE $\mu B_{\max} > \epsilon$

As particle moves to the right $V_{||}$ decreases because B increases

$$V_{||} = \sqrt{\frac{2}{m}(\epsilon - \mu B)}$$

$$\text{At } \epsilon = \mu B(z_0)$$

$$V_{||} = 0 \quad V_{\perp} = V = \sqrt{\frac{2\epsilon}{m}}$$

"BOUNCE POINT" PARTICLE STOPS INSTANTANEOUSLY
BUT MIRROR FORCE STILL PUSHING PARTICLE TO LEFT
THEREFORE IT GOES BACK TOWARDS $z=0$

$$V_{||} = -\sqrt{\frac{2}{m}(\epsilon - \mu B)}$$

CLEARLY MOTION IS OSCILLATION BETWEEN BOUNCE POINTS, at $\pm z_0$

UNTRAPPED PARTICLES

If $\epsilon > \mu B_{\max}$ PARTICLES LEAVE

MIRROR - NO BOUNCE POINTS.

$$\epsilon > \mu B_{\max} \text{ means at } z=0 \quad V^2 > V_{\perp}^2 \frac{B_{\max}}{B_{\min}} \text{ or } \frac{V_{\perp}}{V} < \left(\frac{B_{\min}}{B_{\max}} \right)^{1/2}$$

MIRROR RATIO,

222a. Lecture #5. Particle Motion in Inhomogeneous Fields. II.

REVIEW OF LAST TIME,

$$\rightarrow \underline{B} \quad \epsilon \sim \frac{L}{R} \ll 1$$

DEFINES
VARIABLES.

GUIDING
CENTER
POSITION

Larmor
radius r_L

WOBBLE.

$$\underline{x}(t) = \underline{R}\left(\frac{t}{\omega_L}\right) + \frac{\underline{b} \times \underline{v}}{\omega_L} + \epsilon \Delta \underline{f}(\omega t)$$

$$\underline{v}(t) = \frac{d\underline{R}}{dt} + v_{\perp} (\cos \theta \underline{e}_1 + \sin \theta \underline{e}_2) + \Delta \underline{v}(\omega t)$$

we derived

$$\frac{d\underline{R}}{dt} = v_{\parallel} \underline{b} - \underline{v} \cdot \nabla \left(\frac{\epsilon}{m} \right) \times \underline{v} - \frac{d(\epsilon \Delta f)}{dt}$$

EXACT

The we derived the lowest order motion. - the Guiding centre equation

$$\frac{d\underline{R}^{(0)}}{dt} = v_{\parallel} \underline{b}$$

- Motion along field lines.

$$\theta = \theta_0 - \int_{t'}^t \omega(t') dt' \quad - \text{Fast motion around field lines.}$$

slow evolution of v_{\parallel} & v_{\perp} is calculated from either

$$m \frac{dv_{\parallel}}{dt} = - m v_{\perp}^2 \frac{\underline{b} \cdot \nabla B}{2B} = - \mu \nabla B \quad \mu = \frac{1}{2} m \frac{v_{\perp}^2}{B}$$

or from

$$\mu = \frac{1}{2} m \frac{v_{\perp}^2}{B} = \text{approximately constant even on long timescale.}$$

(1st Adiabatic Invariant.)

$$\epsilon = \frac{1}{2} m v^2 = \text{exact constant.}$$

$$v_{\parallel} = \pm \sqrt{\frac{2}{m} (\epsilon - \mu B)}$$

(ii) Perpendicular Drifts.

We found guiding center moves along field lines to lowest order. Now we compute the drift across the field.

Since $\frac{d}{dt}(\epsilon_{dp})$ averaged over fast timescale must average to zero we have:

$$\frac{dR}{dt} = V_{||}\underline{b} - \alpha \int_0^{\frac{1}{\Omega}} dt' (\underline{v} \cdot \nabla \left(\frac{\underline{b}}{\omega} \right) \times \underline{v})$$

we use that $\int_0^{\frac{1}{\Omega}} \underline{v} \underline{v} dt' = V_{||}^2 \underline{b} \underline{b} + \frac{V_{\perp}^2}{2} (\underline{e}_1 \underline{e}_1 + \underline{e}_2 \underline{e}_2)$

to get,

$$\boxed{\frac{dR}{dt} = V_{||}\underline{b} + \frac{V_{||}^2}{2\Omega} (\underline{b} \times \underline{b} \cdot \nabla \underline{b}) + \frac{V_{\perp}^2}{2\Omega} \left(\underline{b} \times \frac{\nabla B}{B} \right) \dots}$$

CURVATURE
DRIFT

∇B Drift.

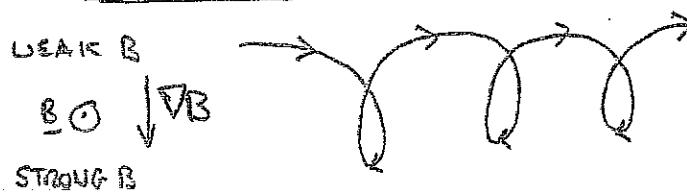
$$+ \frac{V_{\perp}^2}{2\Omega} (\underline{b} \cdot \nabla \times \underline{b}) \underline{b}$$

CONNECTION TO PARALLEL MOTION.

(iii)

Physical Picture

∇B Drift.



$$\boxed{\text{CURVATURE DRIFT}} \equiv \frac{\text{CENTRIFUGAL FORCE} \times B}{(like E \times B)}$$

CENTRIFUGAL FORCE
TO GO AROUND CORNER

$$= -mV_{||}^2 \underline{b} \cdot \nabla \underline{b}$$

$$q B^2$$

$$\nabla \underline{b}$$

3 (iv) Magnetic Fields in a Tokamak's Guiding Center Motion.

TOROIDAL FIELD (comes from bent Solenoid)

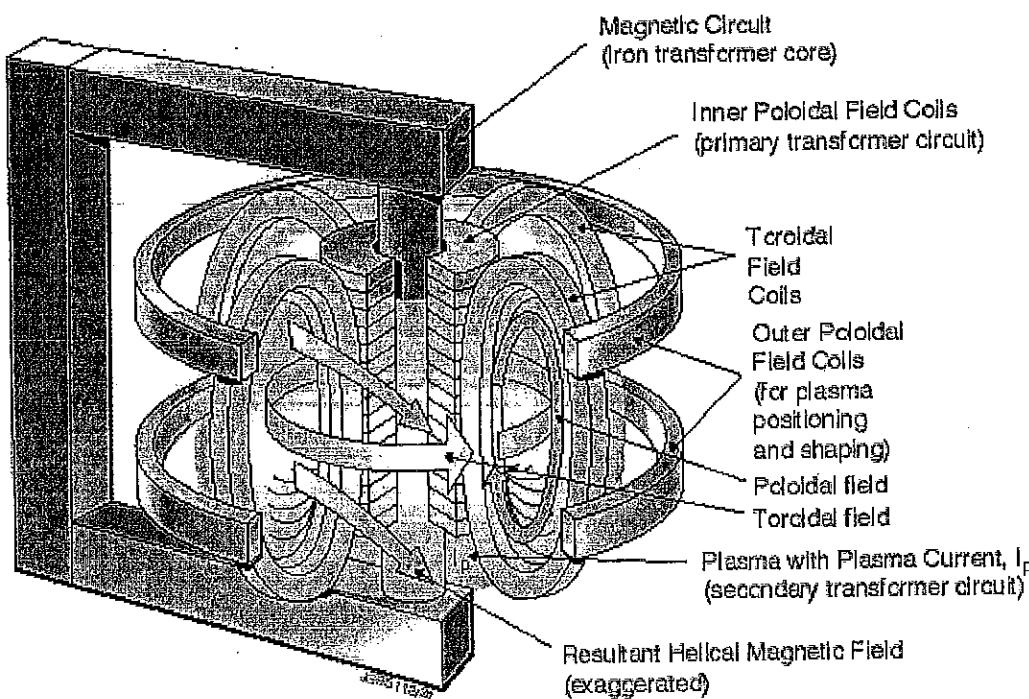


Figure 1:
Magnetic
Field
Configuration

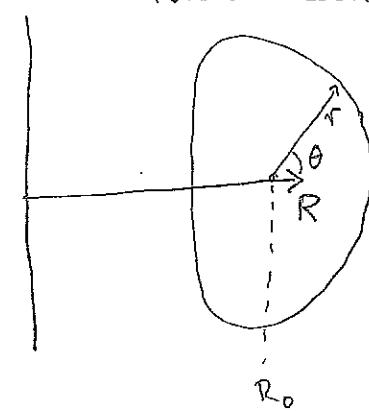
ϕ = Toroidal Angle.

TOROIDAL MAGNETIC FIELD DUE ALMOST ENTIRELY TO
CURRENT IN COILS. THEREFORE

$$B_\phi = \frac{\mu_0 I}{2\pi R}$$

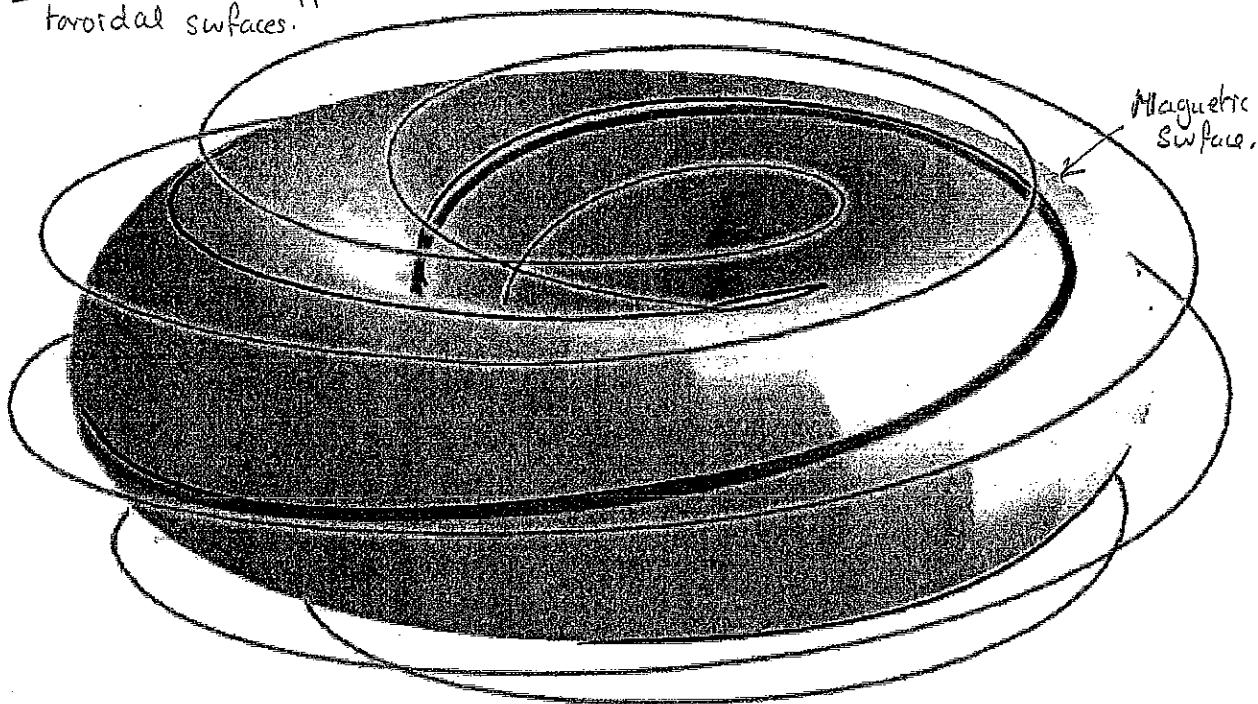
I = total current in toroidal
field coils flowing through
the hole in the donut.

Toroidal Coords.



$$R = R_0 + r \cos\theta$$

B Helices wrapped around toroidal surfaces.



POLOIDAL FIELD Comes from plasma current

flowing in toroidal direction $B_p \sim \frac{1}{R} B_T$

$$\Rightarrow |B| \sim \frac{\mu_0 I}{2\pi R} \quad \text{and} \quad \underline{b} \cdot \nabla \underline{b} \sim - \frac{\nabla R}{R}$$

Take circular magnetic surfaces. Then

$$\underline{B} \sim \frac{\mu_0 I}{2\pi R} \underline{e}_\phi + B_p(r) \underline{e}_\theta$$

PARALLEL MOTION,

$$\frac{dR}{dt} = \sqrt{(\epsilon - MB)^2 \frac{m}{M}} \underline{b} - \frac{1}{R} \left(\frac{V_{||}^2}{\mu} + \frac{V_\perp^2}{2\mu} \right) \underline{b} \times \nabla R$$

Look at motion in poloidal (r, θ) plane

PARALLEL MOTION.

$$r \frac{d\theta}{dt} = \left(\sqrt{\epsilon - \mu B_{min} \frac{R+r}{R}} \right) \frac{B_p}{B} - \left(V_{||}^2 + \frac{V_{\perp}^2}{2} \right) \frac{\cos\theta}{R\omega}$$

$$\frac{dr}{dt} = - \left(V_{||}^2 + \frac{V_{\perp}^2}{2} \right) \frac{\sin\theta}{R\omega} \quad R = R_0 + r \cos\theta.$$

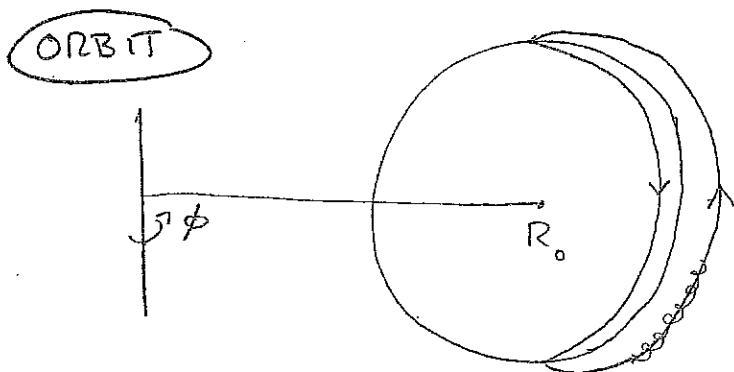
$$B_{min} = \frac{\mu_0 I}{2\pi(R_0+r)}$$

Field at $\theta = 0$ minimum value of field.

$$B_{max} = \frac{\mu_0 I}{2\pi(R_0-r)} \quad \text{field at } \theta = \pi \text{ maximum value of field.}$$

TRAPPED PARTICLE

$$\epsilon < \mu B_0 \left(\frac{R_0+r}{R_0-r} \right), \quad \frac{R_0+r}{R_0-r} < \frac{V_{\perp}^2}{V^2}$$



Banana Orbit, $\frac{V_{||}}{V} \gg \sqrt{\frac{2\pi}{R_0+r}}$
Projected on this plane.
(Guiding Center motion)

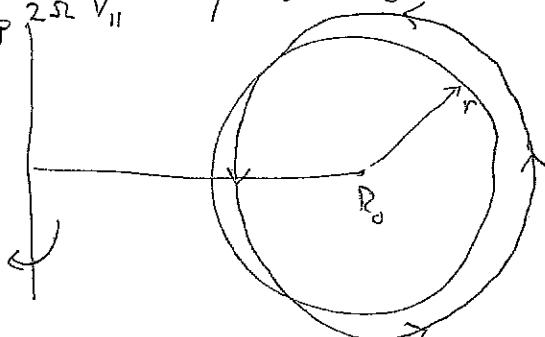
Banana width. $\sim \Delta r \sim \frac{V_{\perp}^2}{2} \frac{1}{R\omega} \Delta t$ — bounce time.

Bounce time $\Delta t \sim \frac{r}{V_{||}} \frac{B}{B_p}$

$$\Delta r \sim \left(\frac{r}{R} \right) \frac{B}{B_p} \frac{V_{\perp}^2}{2\omega} \frac{1}{V_{||}} \sim 10 \rho \text{ typically.}$$

PASSING PARTICLES

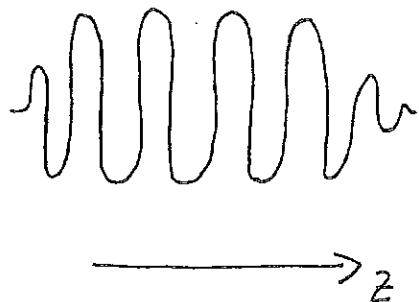
$$\frac{V_{||}}{V} > \left(\frac{2r}{R_0+r} \right)$$



222a. Lecture #6. Motion of an Electron in an Electromagnetic Wave.

- (i) In the last 20 years the power of lasers has risen by something like a factor of 10^6 . Modern lasers ~~cannot~~ accelerate electrons to relativistic energies. To understand laser plasma interaction it is very helpful to understand the relativistic motion of an electron in a plane wave.

(ii) PLANE ELECTROMAGNETIC (EM) WAVE



electron at rest. ($v=0$)

we study the motion of the electron in the prescribed wave field.

(iii) First a Recap of Electromagnetic Waves

We take a wave polarized in the "x" direction.

$$\underline{E} = E(t, z) \hat{x}$$

$$\underline{B} = B(t, z)$$

MAXWELL'S EQUATIONS IN A VACUUM:- $\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E} \rightarrow$

$$\therefore \underline{B} = B \hat{y}$$

$$\text{and } \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \nabla \times \underline{B} \rightarrow$$

$$\frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = -\frac{\partial \underline{B}}{\partial z}$$

①

$$\frac{\partial B}{\partial t} = -\frac{\partial E}{\partial z}$$

COMBINING GIVES THE WAVE EQUATION.

$$\frac{1}{c^2} \frac{\partial^2 E}{\partial t^2} = \frac{\partial^2 E}{\partial z^2} \Rightarrow \boxed{E = E(t - \frac{z}{c})} \text{ AND FROM } ① B = \frac{E}{c} (t - \frac{z}{c})$$

WE SHALL USE THE VECTOR POTENTIAL (NO SCALAR POTENTIAL)

$$\underline{E} = - \frac{\partial \underline{A}}{\partial t} \Rightarrow \underline{B} = \nabla \times \underline{A} \quad \underline{A} = A(t - \frac{z}{c}) \hat{x} \quad E = - \frac{\partial A}{\partial t}$$

$$B = -\frac{1}{c} \frac{\partial A}{\partial t}$$

(iv) We shall take a general pulse but also specialize to

$$\underline{E} = E_0 \sin(\omega t - kz), \quad \omega = kc$$

in places. Note for a pulse E_0 must actually depend on t too.

(v) RELATIVISTIC EQUATIONS OF MOTION FOR ELECTRON.

$$\textcircled{3} \quad \frac{d\underline{P}}{dt} = -e(\underline{E} + \underline{v} \times \underline{B})$$

$$\underline{P} = m\gamma \underline{v}$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

the equation for the energy can be derived from this.

$$\textcircled{4} \quad \frac{dE}{dt} = -e \underline{v} \cdot \underline{E}$$

$$E = m\gamma c^2 \quad \text{also } E = \sqrt{m^2 c^4 + p^2 c^2}$$

(vi) NON-RELATIVISTIC MOTION:

$$\frac{v}{c} \ll 1 \quad \gamma = 1.$$

$$|\underline{v} \times \underline{B}| \sim \frac{vE}{c} \ll |E|$$

$$\Rightarrow m \frac{d\underline{v}}{dt} = -e \underline{E} = +e \frac{\partial \underline{A}}{\partial t} \hat{x}$$

using $\underline{v} = 0$ for $t \rightarrow -\infty$

$$V_y = V_z = 0 \quad y = y_0, \quad z = z_0. \quad \text{NO MOTION IN } z \text{ DIRECTION.}$$

$$A = A(t - \frac{z_0}{c}) \quad \text{therefore from } \textcircled{5}$$

$$V_x = +\frac{eA}{m}$$

and integrating

$$x - x_0 = \frac{e}{mc} \int_{t'}^t A dt'$$

In a sinusoidal wave $E_0 = \omega A_0$

$$\frac{V_x}{c} = \frac{eA_0}{mc} \cos(\omega t - kx_0) \quad \text{Electron jiggles in } \hat{x} \text{ direction only.}$$

We define the oscillation velocity v_{osc} by

$$\boxed{\frac{v_{\text{osc}}}{c} = \frac{eA_0}{mc}}$$

(vii) RELATIVISTIC MOTION.

$$\phi = t - \frac{x}{c} = \text{PHASE OF ELECTRON} \quad \frac{d\phi}{dt} = 1 - \frac{v_x}{c}$$

$$A = A(\phi) \quad \text{so} \quad \frac{dA}{dt} = \frac{dA}{d\phi} \quad \text{and} \quad \frac{dA}{dt} = \frac{d\phi}{dt} \frac{dA}{d\phi}$$

(viii) \hat{y} component of (3): $\frac{dp_y}{dt} = 0 \Rightarrow \boxed{v_y = 0} \quad (6)$

(ix) \hat{x} component of (3): $\frac{dp_x}{dt} = \frac{d(m\gamma v_x)}{dt} = -e(E_x - v_z B_y) = +e \frac{dA}{dt} \left(1 - \frac{v_z}{c}\right)$
INTEGRATING. $= +e \frac{dA}{d\phi} \frac{d\phi}{dt} = +e \frac{dA}{dt} \quad (7)$

$$\boxed{p_x = m\gamma v_x = +eA}$$

(x) \hat{z} component of (3): $\frac{dp_z}{dt} = \frac{d(m\gamma v_z)}{dt} = -e v_x B_y = -\frac{e}{c} v_x \frac{dA}{d\phi} \quad (8)$

(xi) From (4) : $\frac{d(m\gamma c^2)}{dt} = -e v_x E_x = -e v_x \frac{dA}{d\phi} \quad (9)$

COMPARING THESE LAST 2 EQUATIONS WE HAVE.

(xii) $\frac{d}{dt} m\gamma c \left(1 - \frac{v_z}{c}\right) = 0 \Rightarrow \gamma \left(1 - \frac{v_z}{c}\right) = \text{constant} = 1 \text{ from initial conditions.}$

$$\Rightarrow \boxed{\frac{d\phi}{dt} = \frac{1}{\gamma}} \quad (10)$$

WE (10)

$$(xiii) \text{ From (8) and (10) we have: } \frac{d(m\gamma V_z)}{dt} = \frac{e}{c} \gamma V_x \frac{1}{\gamma} \frac{dA}{d\phi} = \frac{e^2}{mc} A \frac{dA}{d\phi}$$

INTEGRATING WE GET

USE (7)

$$\boxed{\gamma V_z = \frac{e^2}{mc^2} \frac{A^2}{2}} \quad (11)$$

Similarly from (9) we get

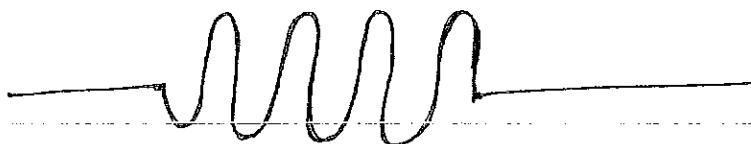
$$\boxed{\gamma = \frac{e^2}{mc^2} \frac{A^2}{2} + 1} \quad (12)$$

$$(xiv) \quad \underline{\text{FROM (7)}} \quad x - x_0 = \int V_x dt = \int \frac{eA}{m\gamma} dt = \int \frac{eA}{m} d\phi$$

FROM (11)

$$z - z_0 = \int_{-\infty}^{\phi} \frac{1}{2} \frac{e^2 A^2}{mc^2} d\phi'$$

$$(xv) \text{ Take a pulse with } A = A_0 \cos \omega \phi \text{ for } -\phi_0 < \phi < \phi_0$$



During pulse ($\phi < \phi_0$)

$$z - z_0 = \frac{e^2 A_0^2}{2mc^2} \int_{-\phi_0}^{\phi} \cos^2 \omega \phi' d\phi' =$$

$$x - x_0 = \frac{eA}{m\omega} [\sin \omega \phi + \sin \omega \phi_0]$$

$$z - z_0 = \frac{V_{osc}^2}{4c} \left\{ \left(t - \frac{z}{c} \right) + \phi_0 + \frac{\sin 2\omega \phi + \sin 2\omega \phi_0}{2\omega} \right\}$$

NOTICE THAT z HAS AN OSCILLATORY PART AND A MEAN DRIFT (FOR $\Delta t \gg \frac{1}{\omega}$)

$$V_{z\text{mean}} = \frac{Dz}{Dt} = \frac{a^2/4}{1+a^2/4} c$$

where $a = \frac{V_{osc}}{c} = \frac{eA_0}{mc}$.

(xvi) After we define a moving coordinate. $z' = z - v_{\text{ZMEAN}} t$

$$z' = \frac{v_{\text{ZMEAN}}}{c} \frac{\sin 2(\omega' t - kz')}{2k} \quad x = \frac{v_{\text{osc}}}{c} \frac{\sin(\omega' t - kz')}{k}$$

$$\text{where } \omega' = \omega(1 - \frac{v_{\text{ZMEAN}}}{c}) = \omega - k v_{\text{ZMEAN}}$$

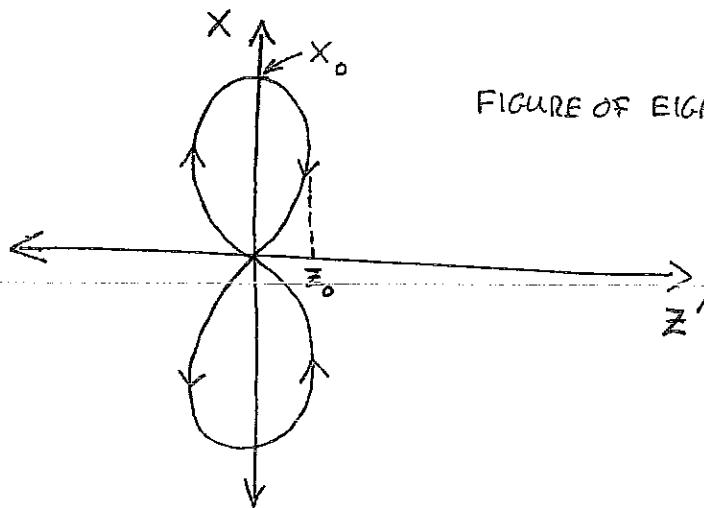
IN THIS TRANSFORMED VARIABLE (NOT A LORENTZ TRANSFORM) ELECTRON OSCILLATES WITH PERIOD $\frac{2\pi}{\omega'}$ IN X DIRECTION AND WITH PERIOD $\frac{\pi}{\omega'}$ IN Z DIRECTION.

$$z' = \pm 2 z_0 \left(\frac{x}{x_0} \right) \sqrt{1 - \left(\frac{x}{x_0} \right)^2}$$

$$z_0 = \frac{v_{\text{ZMEAN}}}{2kc}$$

$$x_0 = \frac{v_{\text{osc}}}{kc}$$

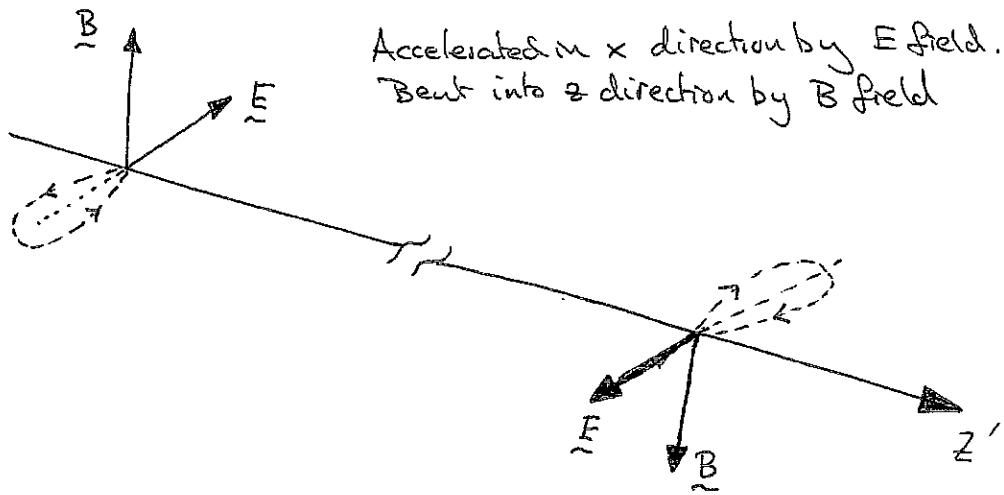
FIGURE OF EIGHT MOTION.



MOTION IN Z (NOT) Z IS ALWAYS IN +VE DIRECTION. $v_z > 0$.

(xvii) Several people have proposed direct acceleration of electrons by lasers. It doesn't work because as soon as the electron leaves the pulse v_z is once again zero - see Eq. (11). No net momentum given to electron.

6

(xviii) Motion of the Electron: Physically.

$$(XIX) \text{ Note } \bar{\gamma} = \frac{e^2}{m^2 c^2} \frac{\bar{A}^2}{2} + 1 = \frac{e^2 A_0^2}{4 m^2 c^2} + 1 = \frac{a^2}{4} + 1$$

$$\text{Also note for } \frac{V_{osc}}{c} \gg 1 \quad \frac{V_x}{c} = \frac{V_{osc}}{c}$$

state today.

22a. Lecture #7.

Kinetic Description of a Plasma

(i) To solve the equations of motion for every particle is intractable - and usually impossible as our knowledge of the initial conditions is limited. We need a STATISTICAL DESCRIPTION.

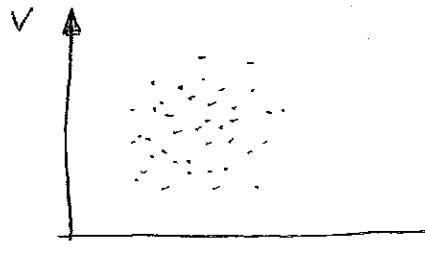
(ii) CONFIGURATION SPACE / PHASE SPACE

IDENTIFYING $\underline{r}, \underline{v}$ for each particle gives position in

CONFIGURATION SPACE $\equiv (\underline{r}, \underline{v})$ SIX DIMENSIONAL.

IDENTIFYING $\underline{r}, \underline{p}$ for each particle gives position in

PHASE SPACE $\equiv (\underline{r}, \underline{p})$ SIX DIMENSIONAL.



• EACH PARTICLE IS A DOT IN THIS SPACE.

• IN ALMOST ALL SITUATIONS THE NUMBER OF PARTICLES IS HUGE

(iii) KLIMONTOVICH DISTRIBUTION:

Let $\underline{r}_i(t)$ and $\underline{v}_i(t)$ be the position and velocity of the i^{th} particle.

$$\left. \begin{array}{l} \text{DENSITY OF} \\ \text{PARTICLES IN} \\ \text{CONFIGURATION SPACE} \end{array} \right\} F = \sum_{i=1}^N \delta(\underline{r} - \underline{r}_i(t)) \delta(\underline{v} - \underline{v}_i(t))$$

$$\frac{d\underline{v}_i}{dt} = \underline{a}_i = \text{acceleration}$$

$$\frac{d\underline{r}_i}{dt} = \underline{v}_i$$

$$\int d^3\underline{r} d^3\underline{v} F = \text{NUMBER OF PARTICLES IN A VOLUME OF CONFIGURATION SPACE} \rightarrow ,$$

2

$$\int \underline{F} d^3\underline{v} = \sum_{i=1}^N \delta(\underline{r} - \underline{r}_i(t))$$

DENSITY OF PARTICLES
IN REAL SPACE.

(iv) This is not much progress since we have just restated the problem of following all the particles.

(v) \underline{F} is constant along the particle orbits [since it is ∞ where there are particles and zero elsewhere] so FORMALLY:-

$$\boxed{\frac{d\underline{F}}{dt} = 0}$$

where $\frac{d}{dt}$ is the derivative along the particle orbits

$$\frac{d\underline{F}}{dt} = \frac{\partial \underline{F}}{\partial t} + \frac{d\underline{v}}{dt} \cdot \frac{\partial \underline{F}}{\partial \underline{v}} + \frac{d\underline{v}}{dt} \cdot \frac{\partial \underline{F}}{\partial \underline{v}} = 0$$

You can verify this by direct substitution.

$$\boxed{\frac{d\underline{F}}{dt} = \frac{\partial \underline{F}}{\partial t} + \underline{v} \cdot \nabla \underline{F} + \underline{a} \cdot \frac{\partial \underline{F}}{\partial \underline{v}} = 0}$$

KLIMOVICH EQUATION.
ONE FOR EACH SPECIES.

(vi) For MAXWELL'S EQUATIONS

$$\epsilon_0 \nabla \cdot \underline{E} = \rho(\underline{r}, t) = \text{CHARGE DENSITY} = \sum_j q_j \int F_j(\underline{v}, \underline{v}, t) d^3\underline{v}$$

SUM OVER SPECIES j .

$$\frac{1}{c^2} \frac{\partial \underline{E}}{\partial t} = \nabla \times \underline{B} - \mu_0 \underline{J} \quad \& \quad \boxed{\underline{J} = \text{CURRENT DENSITY} = \sum_j q_j \int \underline{v} F_j d^3\underline{v}}$$

$$\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E}$$

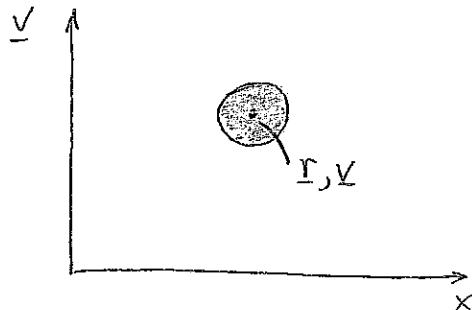
$$\nabla \cdot \underline{B} = 0$$

(vii) COARSE GRAINED AVERAGE.

We create a smooth distribution by averaging F over a small volume of configuration space. We want the average to satisfy 2 conditions:

a) The averaging volume contains many particles.

b) The average \bar{f} does not change much over the averaging volume.



DEFINE THE
SMOOTHED DISTRIBUTION BY:

$$f(\underline{r}, \underline{v}, t) = \int d^3R d^3V K(\underline{r}, \underline{v}) F(\underline{r}-\underline{R}, \underline{v}-\underline{V}, t)$$

WE TAKE A "SMOOTHING KERNEL"

$$K(R, V)$$

THE ACTUAL FUNCTION IS UNIMPORTANT
BUT A GOOD EXAMPLE IS THE GAUSSIAN:

$$\frac{1}{R_0^3 V_0^3 \pi^3} e^{-\frac{R^2}{R_0^2} - \frac{V^2}{V_0^2}}$$

HENCE LOCALIZE
SMOOTHING AND
BE NORMALIZED TO
ONE.

(viipp) { CONDITION a) IS ROUGHLY $R_0^3 V_0^3 f \gg 1$
CONDITION b) IS ROUGHLY $R_0 |\nabla f| \ll 1$ AND $V_0 \left| \frac{\partial f}{\partial v} \right| \ll 1$.

(ix) Another approach is to F averaged over an ensemble but I am not going to do that here.

(x) We will then split the Electric field into a smoothed part \bar{E} and a fluctuating part \hat{E} so that $E = \bar{E} + \hat{E}$.

(xi) The smooth fields are then just the fields generated by the smoothed distribution. I will pretend there is only one species for convenience.

$$\epsilon_0 \nabla \cdot \bar{E} = \bar{\rho} = \int d^3v q f(r, v, t), \quad \nabla \cdot \bar{B} = 0$$

$$\frac{1}{c^2} \frac{\partial \bar{E}}{\partial t} = \nabla \times \bar{B} - \mu_0 \underbrace{\int d^3v q v f(r, v, t)}_{J}, \quad \frac{\partial \bar{B}}{\partial t} = - \nabla \times \bar{E}$$

AVERAGING THE KLMONTOVICH EQUATION YIELDS THE "ONE PARTICLE EQUATION"

$$\boxed{\frac{\partial f}{\partial t} + v \cdot \nabla f + \frac{q}{m} (\bar{E} + v \times \bar{B}) \cdot \frac{\partial f}{\partial v} = \frac{q}{m} \left\langle (\bar{E} + v \times \bar{B}) \cdot \frac{\partial F}{\partial v} \right\rangle \stackrel{\text{collisions}}{=} \left(\frac{\partial f}{\partial t} \right)}$$

$$\nabla \cdot \hat{E} = q \int d^3v (F - f) \quad \begin{array}{l} \text{FIELD DUE TO THE} \\ \text{NON-SMOOTHED PART OF} \\ \text{DISTRIBUTION, "PARTICLE} \\ \text{DISCRETESS".} \end{array}$$

$$\frac{1}{c^2} \frac{\partial \hat{E}}{\partial t} = \nabla \times \hat{B} - \mu_0 \int d^3v q v \{ F - f \}$$

(xii) If we drop the fluctuating fields, we get the VLAOV EQUATION.

(xiii) The correlation of particles is all in the fluctuating parts - f doesn't know there are particles!

State today.

(xiv) Moments:-

$$\text{SMOOTHED DENSITY.} = \int d^3v f(\underline{r}, \underline{v}, t) = n(\underline{r}, t)$$

very like a fluid density. It is a simpler function than f as it depends only on \underline{r}, t - not $\underline{v}, \underline{r}, t$.

$$\text{SMOOTHED VELOCITY.} = \frac{1}{n} \int d^3v \underline{v} f(\underline{r}, \underline{v}, t) = \underline{V}(\underline{r}, t)$$

$$\begin{aligned} \text{KINETIC ENERGY} &= \int d^3v \frac{1}{2} m v^2 f(\underline{r}, \underline{v}, t) = U(\underline{r}, t) \\ &= \frac{3}{2} n(\underline{r}, t) k T(\underline{r}, t) \end{aligned}$$

etc.

(xv) Of course the most important distribution is the MAXWELL-BOLTZMANN DISTRIBUTION for a classical plasma in THERMAL EQUILIBRIUM.

$$f_{\text{MB}}(\underline{v}) = n \left(\frac{m}{2\pi k T} \right)^{3/2} e^{-\frac{1}{2} \frac{m(\underline{v} - \underline{V})^2}{k T}}$$

Here n = DENSITY T = TEMPERATURE \underline{V} = MEAN "FLUID" VELOCITY

We usually call this the MAXWELLIAN.

222a. Lecture #8: Fluid Equations.

(i) Revision of last time.

a) KLIMONTOVICH DISTRIBUTION \equiv

$$F(\underline{r}, t) = \sum_{i=1}^N \delta(\underline{r} - \underline{r}_i(t)) \delta(\underline{v} - \underline{v}_i(t))$$

b) KLIMONTOVICH EQUATION \Rightarrow

$$\frac{\partial F}{\partial t} + \underline{v} \cdot \nabla F + \frac{q}{m} (\underline{E} + \underline{v} \times \underline{B}) \cdot \frac{\partial F}{\partial \underline{v}} = 0$$

c) COARSE GRAINED / SMOOTHED DISTRIBUTION

$$f(\underline{r}, \underline{v}, t) = \int d^3 \underline{R} d^3 \underline{V} G(\underline{R}, \underline{V}) F(\underline{r} - \underline{R}, \underline{v} - \underline{V}, t)$$

"Smoothed particle distribution in configuration space." continuous smooth function for appropriate G .

$$= \langle F \rangle$$

d) Evolution of smoothed distribution:

$$\textcircled{1} - \boxed{\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{q}{m} (\bar{\underline{E}} + \underline{v} \times \bar{\underline{B}}) \cdot \frac{\partial f}{\partial \underline{v}} = \left\langle \frac{q}{m} (\hat{\underline{E}} + \underline{v} \times \hat{\underline{B}}) \cdot \frac{\partial F}{\partial \underline{v}} \right\rangle}$$

with

$$\begin{aligned} \epsilon_0 \nabla \cdot \bar{\underline{E}} &= \bar{\rho} = \int q f d^3 v & \nabla \cdot \bar{\underline{B}} &= 0 \\ \frac{1}{c^2} \frac{\partial \bar{\underline{E}}}{\partial t} &= \nabla \times \bar{\underline{B}} - M_0 \int q \underline{v} f d^3 v & \frac{\partial \bar{\underline{B}}}{\partial t} &= -\nabla \times \bar{\underline{E}} \end{aligned} \quad \left. \begin{array}{l} \text{SMOOTHED FIELDS DRIVEN} \\ \text{BY SMOOTHED CURRENT AND} \\ \text{CHARGE.} \end{array} \right\}$$

$\underline{E} = \bar{\underline{E}} + \hat{\underline{E}}$, $\underline{B} = \bar{\underline{B}} + \hat{\underline{B}}$ the fluctuating fields obey Maxwell's equations driven by $(F - f)$ the fluctuating charge and current.

(ii) We generally think of the smoothed distribution as "the distribution".

The $\hat{\underline{E}}, \hat{\underline{B}}$ and $f - F$ represent particle discreteness and collisions.

(iii) **FLUID EQUATIONS** we would like to write down equations for the moments of f .

DENSITY MOMENT:-

$$n(\underline{r}, t) = \int d^3 v f(\underline{r}, \underline{v}, t)$$

(iv) TO OBTAIN AN EQUATION FOR $n(\underline{v}, t)$ WE INTEGRATE EQUATION (i) OVER V

$$\int d^3v \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \left\{ \int d^3v f \right\} = \frac{\partial n}{\partial t}$$

$$\int d^3v (\underline{v} \cdot \nabla f) = \int d^3v \nabla \cdot (\underline{v} f) = \nabla \cdot \left\{ \int d^3v \underline{v} f \right\} = \nabla \cdot (n \underline{v})$$

where

$$n \underline{v} = \int d^3v \underline{v} f$$

VELOCITY MOMENT (MEAN
PARTICLE VELOCITY LOCALLY.)

$$\begin{aligned} \int d^3v \frac{q}{m} \left\{ \underline{\underline{E}} + \underline{v} \times \underline{\underline{B}} \right\} \cdot \frac{\partial f}{\partial \underline{v}} &= \frac{q}{m} \int d^3v \frac{\partial}{\partial \underline{v}} \left[\left\{ \underline{\underline{E}} + \underline{v} \times \underline{\underline{B}} \right\} f \right] \\ &= \frac{q}{m} \oint_S d\underline{s} \cdot (\underline{\underline{E}} + \underline{v} \times \underline{\underline{B}}) f = 0 \end{aligned}$$

lim $S \rightarrow \infty$ SURFACE AT INFINITY.

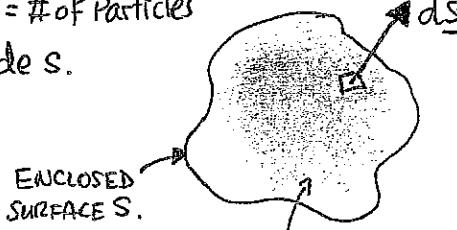
$$\int d^3v \frac{q}{m} (\underline{\underline{E}} + \underline{v} \times \underline{\underline{B}}) \cdot \frac{\partial F}{\partial \underline{v}} = 0$$

Therefore }
COLLECTIVE TERMS

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{v}) = 0$$

DENSITY OR "CONTINUITY"
EQUATION.
"CONSERVATION OF PARTICLES."

$N = \#$ of Particles
inside S .



USING GAUSS'S THEOREM

$$\frac{\partial N}{\partial t} = \frac{\partial}{\partial t} \left[\int_V d^3r n(\underline{v}, t) \right] = \int d^3v \nabla \cdot (n \underline{v})$$

$$\begin{aligned} \text{RATE OF} \\ \text{CHANGE OF \#} \\ \text{OF PARTICLES IN V.} \end{aligned} = \int_S d\underline{s} \cdot (n \underline{v}) = \text{FLUX OF PARTICLES OUT OF} \\ \text{VOLUME V THROUGH S}$$

(v) But this is not complete: THE EQUATION FOR $n(\underline{x}, t)$ INVOLVES ANOTHER MOMENT $\underline{V}(\underline{x}, t)$ - SO WE NEED AN EQUATION FOR $\underline{V}(\underline{x}, t)$!

(vi) EQUATION FOR $\underline{V}(\underline{x}, t)$ OBTAINED BY

$$\int d^3 v \underline{V} \{ \textcircled{1} \}$$

After a lot of tedious algebra we get.

$$mn \left\{ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right\} = -\nabla \cdot \underline{\underline{P}} + qn (\underline{E} + \underline{V} \times \underline{B}) + \begin{matrix} \text{DRAFT DUE} \\ \text{TO COLLISION} \end{matrix}$$

MOMENTUM EQUATION.

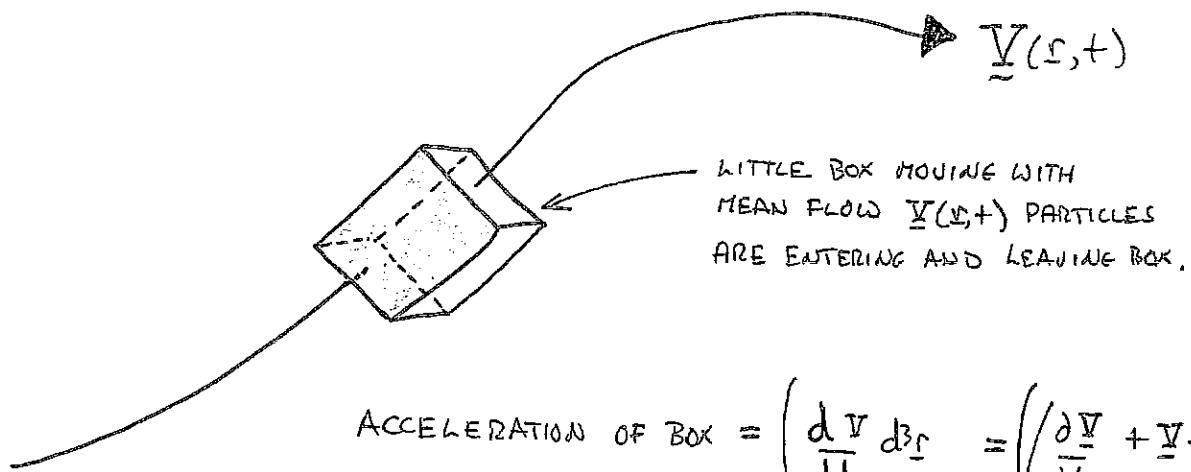
$$\underbrace{mn \left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right)}_{\frac{\partial \underline{V}}{\partial t} + \frac{d\underline{r}}{dt} \cdot \frac{\partial \underline{V}}{\partial \underline{r}}} = mn \frac{d \underline{V}}{dt} = \text{RATE OF CHANGE OF FLUID MOMENTUM IN FRAME MOVING WITH PLASMA FLOW.}$$

$$\underline{\underline{P}} = m \int d^3 v (\underline{v} - \underline{V})(\underline{v} - \underline{V}) f = \text{PRESSURE TENSOR.}$$

$$qn (\underline{E} + \underline{V} \times \underline{B}) = \text{ELECTROMAGNETIC FORCES ON PLASMA "FLUID."}$$

$$\text{DRAG} = \left\langle \int d^3 v (\underline{\underline{E}} + \underline{V} \times \underline{\underline{B}}) \underline{F} \right\rangle = \text{CAN ONLY BE BETWEEN DIFFERENT SPECIES SINCE COLLISIONS MUST CONSERVE MOMENTUM.: e-e collisions cannot change net e momentum.}$$

(vii) THIS EQUATION ISN'T CLOSED EITHER. $\underline{\underline{P}}$ IS ANOTHER NEW MOMENT



$$\text{ACCELERATION OF BOX} = \int \frac{d\underline{V}}{dt} d^3r = \left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right) d^3r$$

$$\text{EM FORCES ON PLASMA IN BOX} = \int q n (\underline{E} + \underline{V} \times \underline{B}) d^3r$$

$$\text{PRESSURE FORCES - PARTICLE CARRYING MOMENTUM } m \text{ AND OUT OF THE BOX} = \int \nabla \cdot \underline{\underline{P}} d^3r = \int \underline{\underline{P}} \cdot d\underline{s}$$

22 —————— 22

(viii) The problem is that we will get still more moments if we try and write an equation for $\underline{\underline{P}}$.

(ix) CLOSURE PROBLEM. No method to close these fluid equations in the general case — without collisions playing a dominant role.

(x) Usually we go to one more equation.

DEFINE: $P = \text{pressure} = \frac{1}{3} \text{Trace}(\underline{\underline{P}}) = \frac{1}{3} \int d^3r m(\underline{V} \cdot \underline{V})^2 f$

$$q_s = \text{Heat flux} = \int d^3r \frac{1}{2} m(\underline{V} \cdot \underline{V})^2 (\underline{V} \cdot \underline{V}) f$$

"energy flux out of moving box"

$$\left\{ k_{\text{BOULEAU}} = 1 \text{ Joules} \right.$$

$$T = \text{Temperature} = \frac{P}{n} = \frac{2}{3} \cdot \int d^3r \frac{1}{2} m(\underline{V} \cdot \underline{V})^2 f$$

$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ the identity tensor.

AND $\underline{\underline{\Pi}} = \underline{\underline{P}} - \underline{\underline{\underline{I}}}\underline{\underline{P}}$ = GENERALIZED VISCOSITY.

ENERGY EQUATION

$$\frac{3}{2} \left(\frac{\partial P}{\partial t} + \nabla \cdot (P \underline{\underline{V}}) \right) + \underline{\underline{P}} : \nabla \underline{\underline{V}} + \nabla \cdot \underline{\underline{q}} = \text{COLLISIONAL HEAT EXCHANGE WITH OTHER SPECIES.}$$

Still no equation for $\underline{\underline{P}}$.

(X) LOCAL THERMODYNAMIC EQUILIBRIUM. L.T.E.

SUPPOSE WE HAVE ENOUGH COLLISIONS THAT,

a) the distribution changes slowly on a collision time

$$\frac{\partial}{\partial t} \ll \gamma \equiv \text{collision rate.}$$

b) the distribution is almost constant over a mean free path.

$$\nabla \ll \frac{1}{\lambda_{\text{m.f.p.}}} \ll \frac{1}{\lambda_{\text{DEBYE}}}$$

THEN WE SAY THE PLASMA IS COLLISIONAL (STILL FEW COLLISIONS IN A DEBYE LENGTH). THE PLASMA IS THEN LOCALLY IN THERMODYNAMIC EQUILIBRIUM.



f must be nearly a Maxwell Boltzmann

$$f(\underline{r}, \underline{v}, t) = \left(\frac{m}{2\pi T(\underline{r}, t)} \right)^{3/2} n(\underline{r}, t) e^{-\frac{m(\underline{v} - \underline{V}(\underline{r}, t))^2}{2T(\underline{r}, t)}}$$

Then we find $\underline{P} = \underline{\underline{I}} P$ $\underline{q} = 0$ no heat flow

\uparrow
ISOTROPIC PRESSURE

$T_e = T_i$

$\underline{V}_e = \underline{V}_i$
no Drag
no Heat exchange

COLLISIONAL EQUATIONS (To lowest order)

$$\frac{\partial \underline{n}}{\partial t} + \nabla \cdot (\underline{n} \underline{V}) = 0$$

$$m \underline{n} \left(\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right) = -\nabla P + q \underline{n} (\underline{E} + \underline{v} \times \underline{B})$$

$$\frac{\partial P}{\partial t} + \nabla(P\underline{V}) + \frac{2}{3} P \nabla \cdot \underline{V} = 0$$

Adiabatic equation
"no heat flow"

"IDEAL" FLUID EQUATIONS FOR EACH SPECIES.

222a. Lecture #9.

Two Fluid - MHD.

Steve Cowley

moments are not closed.

- (i) Revision of last time.
- We took "moments" of the distribution function,
 - If collisions are large i.e. $\gamma \equiv \text{collision rate} \gg \frac{1}{\tau} \frac{\partial f}{\partial t}, \frac{v \cdot \nabla f}{f}$
- PLASMA IS LOCALLY A MAXWELLIAN - LOCAL THERMODYNAMIC EQUILIBRIUM

$$f(\mathbf{r}, \mathbf{v}, t) = n(\mathbf{r}, t) \left(\frac{m_s}{2\pi T(\mathbf{r}, t)} \right)^{3/2} \exp \left\{ - \frac{1/2 m (\mathbf{v} - \mathbf{V}(\mathbf{r}, t))^2}{T(\mathbf{r}, t)} \right\}$$

Note: Maxwellian in frame moving with FLUID VELOCITY.

- c) Equations for DENSITY $\equiv n(\mathbf{r}, t)$ PRESSURE $\equiv n(\mathbf{r}, t)T(\mathbf{r}, t)$
and VELOCITY $\equiv \mathbf{V}(\mathbf{r}, t)$

for each species (ions, electrons) can be written down in closed form. IDEAL TWO FLUID EQUATIONS:

IONS:

$$\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{V}_i) = 0$$

$$n_i m_i \left[\frac{\partial \mathbf{V}_i}{\partial t} + \mathbf{V}_i \cdot \nabla \mathbf{V}_i \right] = -\nabla p_i + q_i n_i \left[\mathbf{E} + \mathbf{V}_i \times \mathbf{B} \right] + \mathbf{F}_{ig}$$

Friction on ions from electrons

$$\frac{\partial p_i}{\partial t} + \mathbf{V}_i \cdot \nabla p_i = -\frac{5}{3} p_i \nabla \cdot \mathbf{V}_i + q_i \mathbf{V}_i \cdot \mathbf{E} + Q_{ie}$$

Heat from electrons to ions

ELECTRONS:

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{V}_e) = 0$$

$$n_e m_e \left[\frac{\partial \mathbf{V}_e}{\partial t} + \mathbf{V}_e \cdot \nabla \mathbf{V}_e \right] = -\nabla p_e - n_e e \left[\mathbf{E} + \mathbf{V}_e \times \mathbf{B} \right]$$

+ F_{ei}

Friction of electrons from ions.

$$\frac{\partial p_e}{\partial t} + \mathbf{V}_e \cdot \nabla p_e = -\frac{5}{3} p_e \nabla \cdot \mathbf{V}_e - q_i n_e \mathbf{V}_e \cdot \mathbf{E} + Q_{ei}$$

Heat from ions to electrons.

NOTE: $Q_{ie} = -Q_{ei}$ and by Newton 3rd law $F_{ie} = -F_{ei}$

The two fluid equations including the collisional corrections are given in the plasma formulary: "BRAGINSKII'S EQUATIONS".

(ii) COLLISIONAL CORRECTIONS: This involves keeping $\delta f \sim \mathcal{O}\left(\frac{1}{\gamma} \frac{\partial f}{\partial t}, \frac{1}{\gamma} \mathbf{V} \cdot \nabla f\right)$

corrections and their contribution to moment equations.

(iii) To make further progress we must assume some ORDERING ON THE size of various terms.

ORDERING (a) $\underline{V}_e, \underline{V}_i \sim \mathcal{O}(\text{ION THERMAL VELOCITY } \sim \sqrt{\frac{T}{m_i}} \equiv v_{th,i}) \ll c$

FLUID VELOCITIES CAN BE "SUPERSONIC"

^{non}
relativistic
ion velocities.

(b) TYPICAL LENGTH OF VARIATION OF $f \equiv L \quad L \sim \frac{f}{|\nabla f|}$

$$L \gg \rho_i = \frac{v_{th}}{\Omega_{ci}} \quad \Omega_{ci} = \frac{qB}{m_i} \text{ AND } L \gg \lambda_{\text{DEBYE}}$$

(c) TYPICAL TIME VARIATION OF $f \equiv \tau \quad \text{i.e. } \tau = \frac{f}{|\frac{\partial f}{\partial t}|}$

$$\tau = \frac{L}{v} \sim \frac{L}{\rho_i} \frac{1}{\Omega_{ci}} \gg \frac{1}{\Omega_{ci}}$$

(iv) Much of the simplification comes from electron ^{momentum} equation:

$$m_e n_e \left[\frac{\partial \underline{V}_e}{\partial t} + \underline{V}_e \cdot \nabla \underline{V}_e \right] = -\nabla p_e - e n_e \left[\underline{E} + \underline{V}_e \times \underline{B} \right] + \underline{F}_{ei}$$

(1)

(2)

(3)

(4)

(5)

$$\frac{(1)}{(4)} \approx \frac{m_e n_e \frac{\partial \underline{V}_e}{\partial t}}{e n_e \underline{V}_e \times \underline{B}} \approx \frac{1}{\Omega_{ce}} \frac{1}{v_e} \frac{\partial \underline{V}_e}{\partial t} \ll \frac{1}{\Omega_{ci}} \frac{1}{v_e} \frac{\partial \underline{V}_e}{\partial t} \ll 1$$

ELECTRONS ARE LIGHT SO WE MAY IGNORE INERTIA.

$$\frac{(2)}{(4)} \approx \frac{\nabla p_e}{e n_e \underline{V}_e \times \underline{B}} \approx \frac{p_e}{L} \ll 1$$

The Drag between electrons and ions must be proportional to their relative velocity i.e.

$$\underline{F}_{ei} = c_{ei} m_e n_e \nu_e (\underline{V}_i - \underline{V}_e) \quad \text{where } \nu_e \text{ is the collision rate and } c_{ei} \text{ is a number of order unity.}$$

(V) Electron Equation is then

$$\underbrace{[\underline{E} + \underline{V}_e \times \underline{B}]}_{\text{"Electric field seen by electrons."}} = \frac{\underline{F}_{ei}}{en_e} = \frac{c_{ei} m_e \underline{V}_e}{en} + \mathcal{O}\left(\frac{P_i}{L}\right)$$

↑
corrections.

NOTE $E \sim \mathcal{O}(VB)$

(VI) The plasma is approximately CHARGE NEUTRAL

$$\frac{\Delta n}{n} \sim \frac{en_e - n_i q}{en_e} \sim \frac{e \nabla \cdot \underline{E}}{n_e e} \sim \frac{\epsilon_0 v B}{L n_e e} \sim \frac{\lambda_D^2 P_i}{P_i^2 L} \ll 1$$

NOTE $\left(\frac{\lambda_D}{P_i}\right)^2 \sim \left(\frac{B^2 / \mu_0}{P}\right) \left(\frac{V_i^2}{C^2}\right) = \left(\frac{\text{MAGNETIC PRESSURE}}{\text{PLASMA PRESSURE}}\right) \cdot \frac{V_i^2}{C^2} \lesssim 1$

$$n_i q = n_e e$$

(VII) IONS AND ELECTRONS MOVE AT ALMOST SAME SPEED

From ion momentum Equation we also get

$$[\underline{E} + \underline{V}_i \times \underline{B}] = - \frac{\underline{F}_{ie}}{q n_i} + \mathcal{O}\left(\frac{P_i}{L}\right)$$

To be consistent with electron equation we must have,

$$\underline{V}_i = \underline{V}_e + \mathcal{O}\left(\frac{P_i}{L}\right)$$

∴ WE CALL THE ION VELOCITY $\underline{V}_i = \underline{V}$
THE "PLASMA" VELOCITY

KEEP SMALL DIFFERENCE BETWEEN \underline{V}_i & \underline{V}_e IN ~~WITH~~ FRICTION AS COLLISIONS ARE STRONG.

(VIII) Using (V) , (V^i) and (V^{iop}) we get

OHMS LAW :-

$$\underline{E} + \underline{V} \times \underline{B} = \eta \underline{J}$$

ELECTRIC FIELD IN
PLASMA FRAME = RESISTIVITY
 X CURRENT

η = resistivity.

$$= \frac{e_i m_e \gamma_e}{e^2 n_e}$$

(ix) We need an equation for \underline{V} the plasma flow: BUT

BOTH ELECTRON AND ION EQUATIONS GIVE OHMS LAW TO DOMINANT ORDER. TO GET AN EQUATION FOR \underline{V} WE MUST ELIMINATE DOMINANT TERMS. WE CAN DO THIS BY ADDING THE MOMENTUM EQUATIONS FOR IONS AND ELECTRONS.

$$(m_e n_e + m_i n_i) \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right] = -\nabla (P_{\text{tot}}) + (q_i n_i - e n_e) \underline{E} + (q_i n_i \underline{V}_i - e n_e \underline{V}_e) \times \underline{B}$$

MASS DENSITY $\sim m_i n_i$
 TOTAL PRESSURE P
 CHARGE DENSITY STILL SMALL CAN BE NEGLECTED.
 (check this later)

MOMENTUM EQUATION.

$$P \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right] = -\nabla P + \underline{J} \times \underline{B}$$

(x) If we take electrons and ions to have the same temperature then

$$Q_{ie} = -Q_{ei} = \text{heat flow from electrons to ions} = 0.$$

as they are in local thermodynamic equilibrium.

Then adding electron and ion pressure equations we get:

PRESSURE EQUATION.

$$\frac{\partial P}{\partial t} + \underline{V} \cdot \nabla P = -\frac{5}{3} P \nabla \cdot \underline{V} + \eta \underline{J}^2$$

Ohmic heating ($I^2 R$)

(Xⁱ) In principle we think we have to know $V_i - V_e$ to find \underline{J} but we don't because we can find \underline{J} from \underline{B} in Maxwell's Equations.

$$\nabla \times \underline{B} = \mu_0 \underline{J} + \frac{1}{c^2} \frac{\partial \underline{E}}{\partial t}$$



$$\boxed{\nabla \times \underline{B} = \mu_0 \underline{J}}$$

$$\frac{V \underline{B}}{c^2 \epsilon} \sim \frac{V^2}{c^2} \frac{\underline{B}}{\epsilon} \ll \nabla \times \underline{B}$$

IGNORE DISPLACEMENT CURRENT

(X^{PP}) So GATHERING TOGETHER EQUATIONS FOR $n, V, P, \underline{E}, \underline{B}$ and \underline{J}
 MAGNETOHYDRODYNAMIC EQUATIONS - M.H.D.

CONTINUITY EQUATION

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{V}) = 0 \quad \text{EVOLVES } n.$$

FARADAYS LAW

$$\frac{\partial \underline{B}}{\partial t} = - \nabla \times \underline{E} \quad \text{EVOLVES } \underline{B}$$

OHMS LAW

$$\underline{E} + \underline{V} \times \underline{B} = \eta \underline{J} \quad \text{DETERMINES } \underline{E}$$

AMPERES LAW

$$\nabla \times \underline{B} = \mu_0 \underline{J} \quad \text{DETERMINES } \underline{J}$$

MOIMENTUM EQUATION

$$m_i n \left[\frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right] = - \nabla P + \underline{J} \times \underline{B} \quad \text{EVOLVES } \underline{V}$$

PRESSURE/ENERGY EQUATION

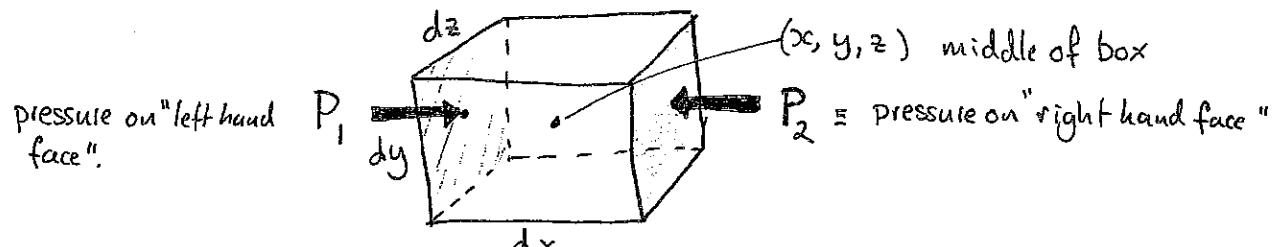
$$\frac{\partial P}{\partial t} + \underline{V} \cdot \nabla P = - \frac{5}{3} \rho \nabla \cdot \underline{V} + \eta \underline{J}^2 \quad \text{EVOLVES } P.$$

(X^{PP}) MHD is a very widely used approximation. Recall what it says:

IN THE MHD LIMIT THE PLASMA BEHAVES AS A SINGLE CONDUCTIVE FLUID WITH THE IONS AND ELECTRONS LOCALLY MAXWELLIAN WITH THE SAME FLOW, TEMPERATURE, AND WITH CHARGE NEUTRALITY.

(xiii) FORCES ON A SMALL CHUNK OF MHD PLASMA:

PRESSURE FORCE.



$$\text{PRESSURE FORCE ON LEFT HAND FACE} = P_1 dy dz$$

$$\text{" " " RIGHT HAND FACE} = P_2 dy dz$$

$$\text{NET PRESSURE FORCE IN X DIRECTION} = (P_1 - P_2) dy dz$$

$$P_1 \approx P(x - \frac{dx}{2}, y, z)$$

$$\approx p(x, y, z) - \frac{dx}{2} \frac{\partial p}{\partial x}$$

$$P_2 \approx P(x + \frac{dx}{2}, y, z)$$

$$\approx p(x, y, z) + \frac{dx}{2} \frac{\partial p}{\partial x}$$

$$\therefore \text{Net pressure force in } x \text{ direction} \approx \frac{\partial p}{\partial x} \cdot \text{volume of box.}$$

volume of box.
 $dx dy dz$.

SO PRESSURE FORCE PER UNIT VOLUME	$= -\nabla p$
--------------------------------------	---------------

<u>BODY FORCE</u> \approx NET CHARGE IN BOX $\times \frac{E}{V}$	$+ \text{CURRENT} \times \underline{B}$
---	---

<u>BODY FORCE</u> PER UNIT VOLUME	$= \underline{J} \times \underline{B}$
---	--

where $\underline{J} = \frac{\text{CURRENT}}{\text{VOLUME}} \equiv \text{CURRENT DENSITY.}$

222a. Lecture #10: MHD Part I - Flux Freezing.

(i) The equations of MHD are:

$$\text{DENSITY "CONTINUITY"} : \frac{\partial n}{\partial t} + \nabla \cdot (n \underline{v}) = 0 \quad (1)$$

$$\text{FARADAY'S LAW EVOLVES } \underline{B} : \frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E} \quad (4)$$

$$\text{OHM'S LAW DETERMINES } \underline{E} : \underline{E} + \underline{V} \times \underline{B} = \eta \underline{J} \quad (2)$$

$$\text{AMPERE'S LAW DETERMINES } \underline{J} : \nabla \times \underline{B} = \mu_0 \underline{J} \quad (5)$$

$$\text{MOMENTUM EQUATION} : m_i n \left[\frac{\partial \underline{v}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right] = -\nabla p + \underline{J} \times \underline{B} \quad (3)$$

$$\frac{\partial p}{\partial t} + \underline{V} \cdot \nabla p = -\frac{5}{3} p \nabla \cdot \underline{V} + \text{heating terms } \eta \underline{J}^2 \text{ etc.} \quad (6)$$

MHD Books. J. Freiberg. "IDEAL MHD" MIT Press.
E. Parker. "Cosmical Electrodynamics."
P. Davidson "Magnetohydrodynamics (liquid metals etc.)"

(ii) It is common to eliminate \underline{E} entirely via substituting (2) into (4).

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{V} \times \underline{B}) + \eta \nabla^2 \underline{B}$$

sometimes called the "induction equation."

- From the plasma formulary we find

$$\begin{aligned} \eta &= \left(\frac{n_e c^2}{4\pi} \right) = Z \left(\frac{ln \Lambda}{15} \right) \cdot \frac{10^7}{T_{eV}^{3/2}} \text{ cm}^2 \text{s}^{-1} \\ &= Z \left(\frac{ln \Lambda}{15} \right) \frac{10^3}{T_{eV}^{3/2}} \text{ m}^2 \text{s}^{-1} \end{aligned}$$

$$(iii) \underline{V} = 0$$

$$\frac{\partial \underline{B}}{\partial t} = \eta \nabla^2 \underline{B} \quad \text{Diffusion equation.}$$

Example solution

$$\underline{B} = \left\{ e^{-\gamma t} \sin k z \right\} \hat{x}$$

$$\gamma = \eta k^2$$

Decays away:- removes variations in \underline{B} .

2

(iv) Numbers in SOLAR CORONA.

TYPICAL LENGTH $L \sim 10^8 \text{ m}$.TEMPERATURE $\sim 100 \text{ eV}$. $\Rightarrow \eta = 1 \text{ m}^2 \text{s}^{-1}$ RESISTIVE TIME $\sim \frac{L}{\eta} \sim \frac{L^2}{\eta} \sim 10^{16} \text{ s} \sim \frac{1}{3} \text{ Billion years!}$ COLLISIONALITY $= \frac{\lambda_{m.f.p.}}{L} \sim \frac{10^3 \text{ m}}{10^8 \text{ m}} \sim 10^{-5}$ very collisional plasma.

\Rightarrow M.H.D. is good ($\lambda_{m.f.p.} \ll L$) and resistivity is small effect unless scale (L) gets very small.

(v) When resistivity can be neglected ($\eta \rightarrow 0$) we can use

IDEAL M.H.D.

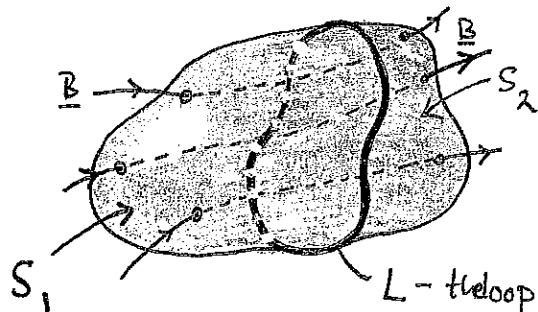
$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B})$$

this equation has special properties that are very important to understand. We spend the rest of the lecture on these properties.

(vi) MAGNETIC FLUX: $\underline{\Phi} = \int_S \underline{B} \cdot d\underline{S}$ at fixed time.

Lemma #1. Flux through any loop is independent of the choice of surface spanning the loop.

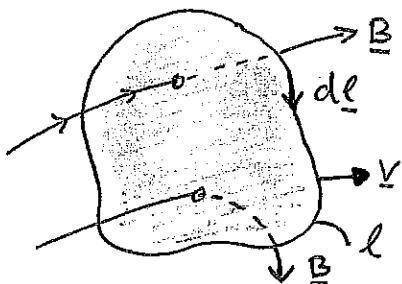
Proof. Consider two surfaces spanning Loop L , S_1 & S_2 .



$$\phi_1 = \text{Flux through } S_1 = \int_{S_1} \underline{B} \cdot d\underline{S} \quad \phi_2 = \text{Flux through } S_2 = \int_{S_2} \underline{B} \cdot d\underline{S}$$

$$\begin{aligned} \text{Flux out of volume enclosed by } S_1 \text{ & } S_2 &= \int_S \underline{B} \cdot d\underline{S} = \phi_1 - \phi_2 \\ \text{But Gauss's theorem} &= \int_V \nabla \cdot \underline{B} dV = 0 \\ \Rightarrow \phi_1 &= \phi_2 \quad Q.E.D. \end{aligned}$$

(vii) CONSIDER LOOP MOVING WITH FLOW

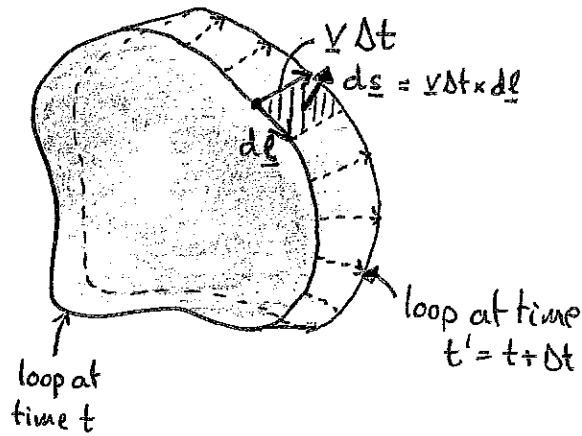


Each point on loop moves with velocity \underline{V}

THEOREM #1

FLUX FREEZING The flux through the loop ℓ moving with the flow is constant in time if $\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{V} \times \underline{B})$.

Proof: Consider loop at t and $t + \Delta t = t'$.



S = Surface spanning ℓ at time t

S' = Surface spanning ℓ at time t'

= S + ribbon connecting ℓ to ℓ'

(since we can choose it any way we like from Lemma #1.)

$$\Phi(t) = \text{FLUX THROUGH } \ell = \int_S \underline{B}(t) \cdot d\underline{s}$$

$$\Phi(t + \Delta t) = \int_{S'} \underline{B}(t + \Delta t) \cdot d\underline{s} = \int_S \underline{B}(t + \Delta t) \cdot d\underline{s} + \int_{\text{RIBBON}} \underline{B}(t + \Delta t) \cdot d\underline{s}$$

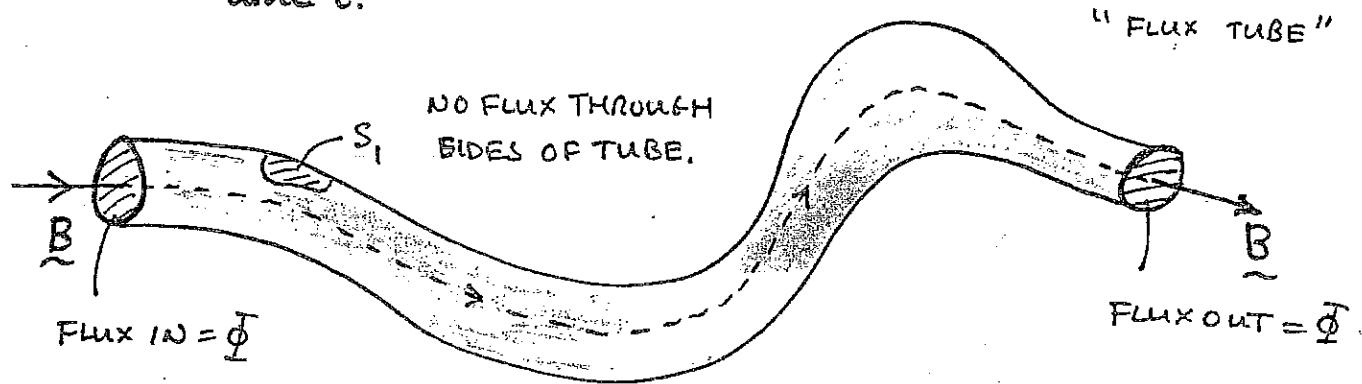
$$\Delta t \rightarrow 0 \quad \approx \int_S \underline{B}(t) \cdot d\underline{s} + \Delta t \int \frac{\partial \underline{B}}{\partial t} \cdot d\underline{s} + \Delta t \left(\int_{\ell} \underline{B} \cdot (\underline{V} \times d\underline{l}) \right) + \mathcal{O}(\Delta t^2)$$

$$\Rightarrow \frac{d\Phi}{dt} = \frac{\Phi(t + \Delta t) - \Phi(t)}{\Delta t} = \int \left\{ \frac{\partial \underline{B}}{\partial t} - \nabla \times (\underline{V} \times \underline{B}) \right\} \cdot d\underline{s} = 0 \quad \text{Q.E.D.}$$

(vi) Theorem I is called Flux Freezing, for obvious reasons.

(vii) Theorem II. The magnetic field lines are frozen to the plasma/fluid flow.

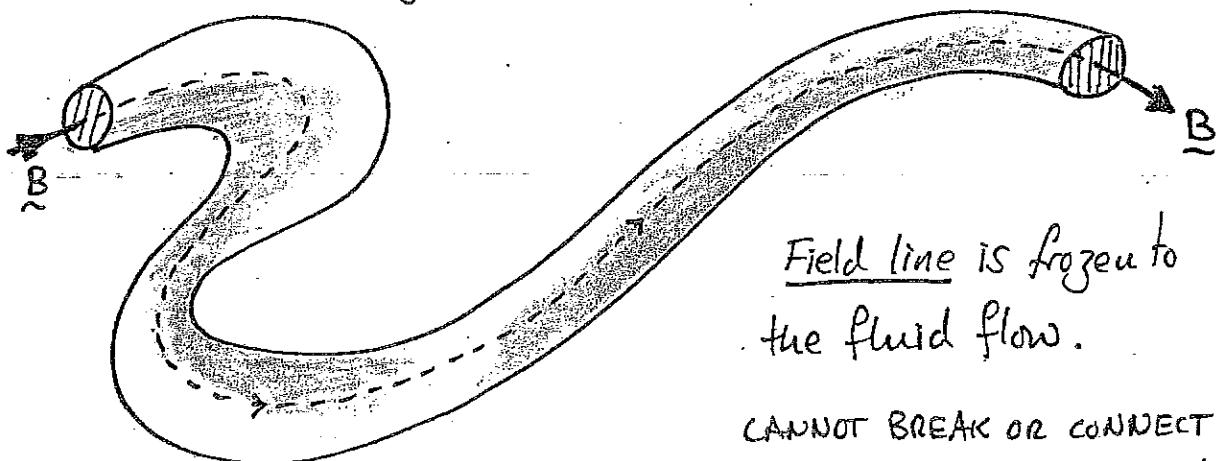
- a) Consider a tube of plasma surrounding a field line at time t .



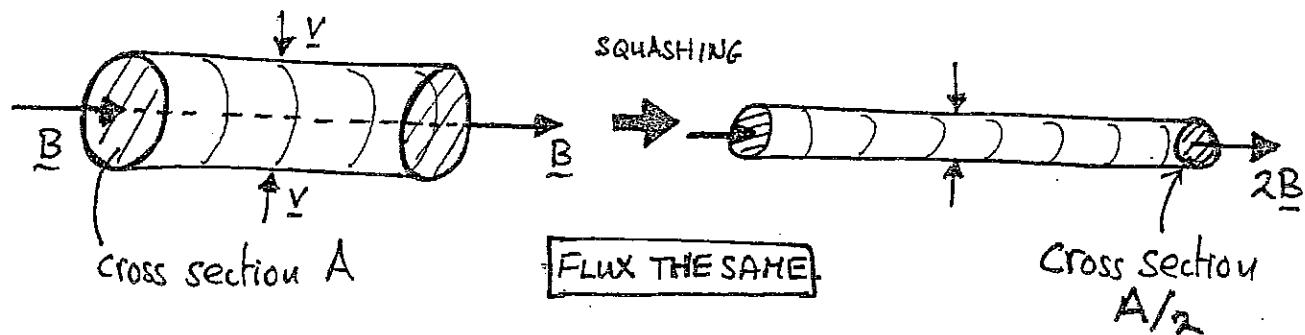
b) Let tube move with velocity v - i.e. with plasma/fluid.

- From THEOREM I.
- At time t' the flux through S_1 must still be zero - this must be true for all parts of the tube sides.
 - the flux through the ends must still be Φ .

\Rightarrow Field line still goes down tube

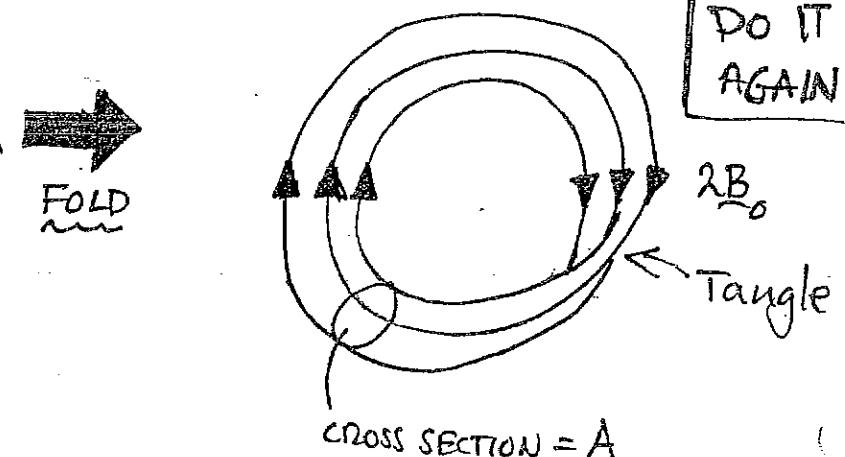
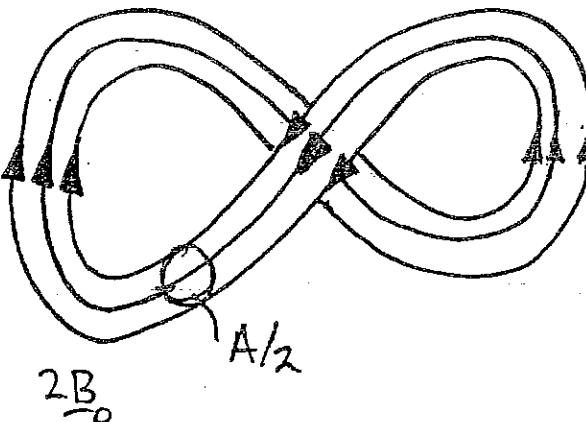
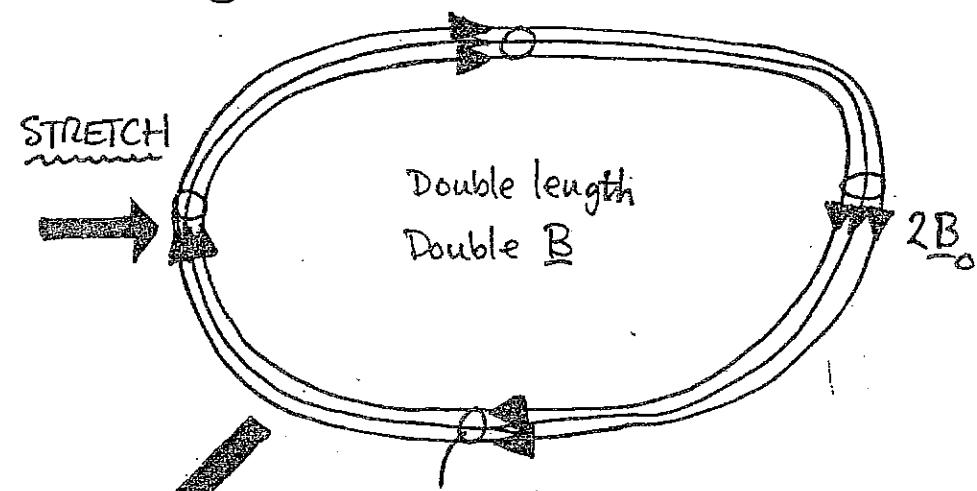
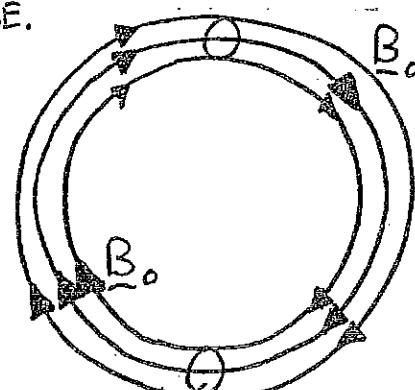


Amplifying Field by squashing.



Zeldovitch's Rope Dynamo:

CIRCULAR FLUX
TUBE.



Nearly the same as the start except \underline{B} has doubled.
and there is a tangle.

Homework #8. Magnetic Fields in Stars.

A simple model of star formation is that a sphere of galactic plasma (density $n=1$ particle per cm^{-3}) collapses under gravity to a sphere of density $n = 10^{26} \text{ cm}^{-3}$. If the initial magnetic field is taken to be $3\mu\text{G} = 3 \times 10^{-10} \text{ Tesla}$ - the typical galactic field - calculate the field in the star if we assume that flux is frozen.

222a. Lecture II. MHD Forces and Equilibrium.

(i) MHD Equations.

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{v}) = 0$$

$$\frac{\partial \underline{B}}{\partial t} = -\nabla \times \underline{E}$$

$$\underline{E} + \underline{v} \times \underline{B} = \eta \underline{J}$$

$$\nabla \times \underline{B} = \mu_0 \underline{J}$$

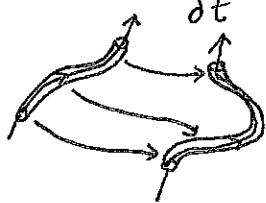
$$mn \left[\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right] = -\nabla p + \underline{J} \times \underline{B} \quad \frac{\partial p}{\partial t} + \underline{v} \cdot \nabla p = -\frac{5}{3} p \nabla \cdot \underline{v} + \text{heating.}$$

(ii) Last time: $\eta \rightarrow 0$ IDEAL MHD.

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) \Leftrightarrow$$

a) Flux through loop moving with plasma is constant.

b) Field lines "frozen to the flow."



(iii) This time: Force Balance \Rightarrow Equilibrium.

Not thermodynamic equilibrium.

$$\frac{\partial}{\partial t} = 0 \quad \underline{v} = 0 \quad \text{Stationary Equilibrium.}$$

$$\nabla p = \underline{J} \times \underline{B}$$

Equilibrium - Force Balance

(iv) Before we examine solutions to this equation let us look at the forces in a physical way.

(v) Pressure force on a volume of plasma.

$$\int_V p dV = \oint_S p d\underline{s} = \text{pressure forces on the surface of plasma.}$$

2

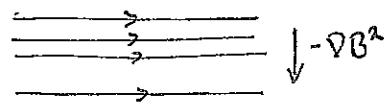
(VI) Magnetic Forces :- $\underline{J} \times \underline{B}$ - MAGNETIC NO FORCE ALONG \underline{B}

$$\text{But } \underline{J} = \frac{\nabla \times \underline{B}}{\mu_0}$$

Ignore components along \underline{B}

$$\underline{J} \times \underline{B} = -\nabla \left(\frac{B^2}{2\mu_0} \right) + \frac{\underline{B} \cdot \nabla \underline{B}}{\mu_0}$$

"Magnetic Pressure"



"Magnetic Curvature Force"



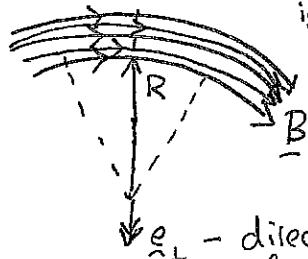
$$\underline{B} = B \underline{b}$$

unit vector

Magnetic Pressure Force tries to make B uniform.

$$\frac{\underline{B} \cdot \nabla \underline{B}}{\mu_0} = \frac{(\underline{B} \cdot \nabla \underline{B}) \underline{b}}{\mu_0} + \frac{B^2}{\mu_0} \underline{b} \cdot \nabla \underline{b}$$

Along \underline{B}
ignore.



R = Radius of Curvature

$$\underline{b} \cdot \nabla \underline{b} = \frac{\underline{e}_\perp}{R}$$

\underline{e}_\perp - direction
of curvature force.

Magnetic curvature force tries to straighten the field lines.

(V&P) In a sense the magnetic field behaves like a collection of buoyant cords - hard to bend or compress.



(viii) Simple Equilibria: a) \underline{z} pinch:

current in the \underline{z} direction: $\underline{J} = J(r) \hat{\underline{z}}$
vaporized wires!

$$\nabla \times \underline{B} = \mu_0 \underline{J} \rightarrow \frac{1}{r} \frac{d(r B_\theta)}{dr} = \mu_0 J(r) \quad B_r = B_z = 0$$

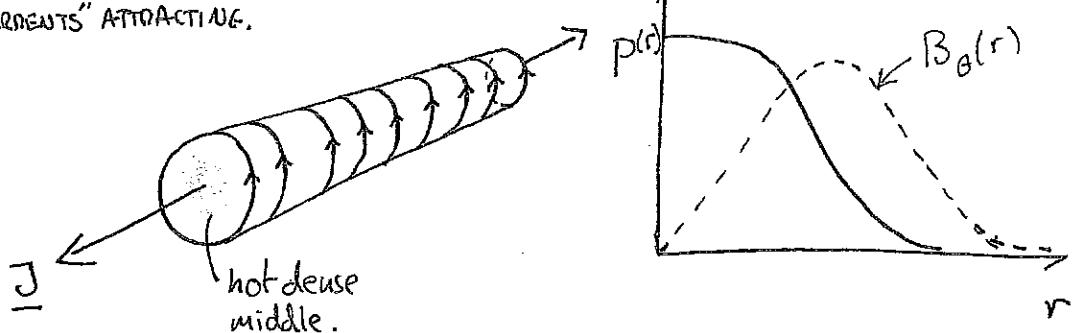
$$B_\theta(r) = \frac{\mu_0}{r} \int_0^r r' J(r') dr' \quad \underline{B} \text{ in the } \theta \text{ direction.}$$

By symmetry $P = P(r)$ - cylindrical.

$$\nabla P = \underline{J} \times \underline{B} \rightarrow \frac{dp}{dr} = - \frac{J(r)}{\mu_0 r} \int_0^r dr' r' J(r') \quad$$

PRESSURE FORCE PUSHES OUTWARDS $\underline{J} \times \underline{B}$ "PINCH" FORCE PUSHES INWARDS.

PINCH COMES FROM "LIKE CURRENTS" ATTRACTING.



At Sandia National lab they put $\sim 30\text{MA}$ of current through wires squeezing them together making lots of x rays.

4

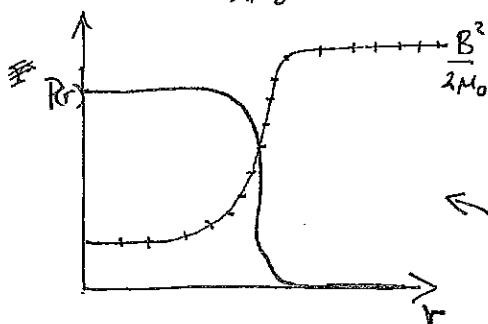
b) θ pinch. \exists , Current in θ direction. \underline{B} in $\cancel{\exists}$ direction.

$$\underline{B} = B(r) \hat{\underline{z}} \quad \underline{B} \cdot \nabla \underline{B} = 0$$

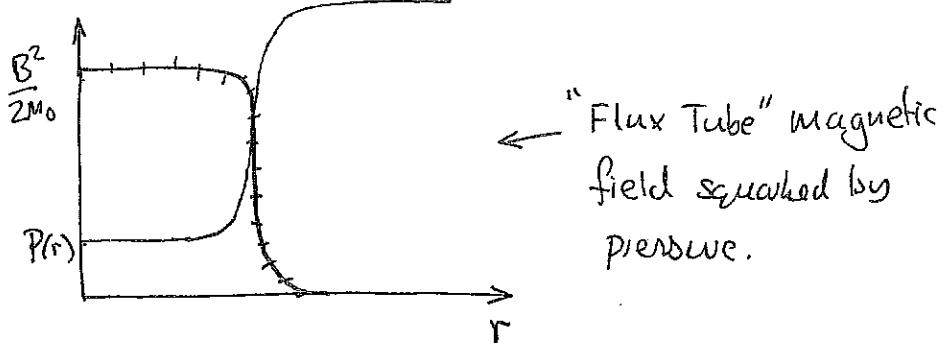
$$\nabla p = \underline{J} \times \underline{B} = -\nabla \left(\frac{B^2}{2\mu_0} \right) + \cancel{\frac{\underline{B} \cdot \nabla \underline{B}}{\mu_0}}$$

$$\nabla \left(p + \frac{B^2}{2\mu_0} \right) = 0 \therefore \text{TOTAL PRESSURE CONSTANT}$$

$$p + \frac{B^2}{2\mu_0} = \text{constant.}$$

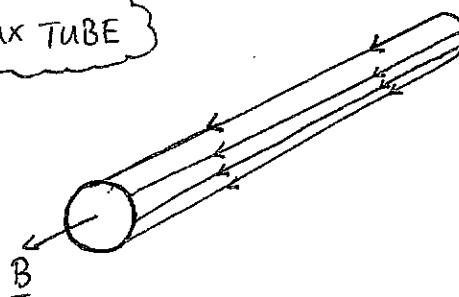


← Pressure squashed by magnetic field.



← "Flux Tube" magnetic field squashed by pressure.

FLUX TUBE



Inside flux tube
the pressure is smaller.

← flux tubes
"pop out" of sun.

(Ex) In sun flux tubes rise because thermal conduction makes temperature constant across tube so to get low pressure in tube we must have low density in tube. ∵ by Archimedes they rise.

(X) SCREW PINCH. Cylinder of length R .

$$\underline{B} = B_\theta(r) \hat{\theta} + B_z(r) \hat{z}$$

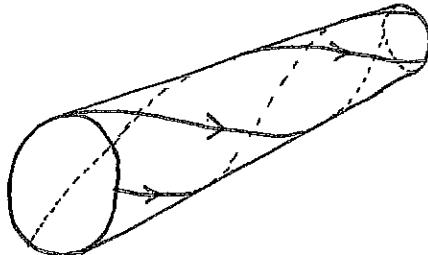
SOMETIMES B_θ IS
CALLED POLOIDAL FIELD
AND B_z TOROIDAL FIELD.

$$\nabla \times \underline{B} = \mu_0 \underline{J} = - \frac{\partial B_z}{\partial r} \hat{\theta} + \frac{1}{r} \frac{\partial (r B_\theta)}{\partial r} \hat{z}$$

$$\nabla p = \underline{J} \times \underline{B}$$

$$\frac{dp}{dr} = - \frac{d}{dr} \left(\frac{B_z^2}{2\mu_0} \right) - \frac{B_\theta d(r B_\theta)}{\mu_0 r dr}$$

Given/specify
 B_θ & B_z and
find p .



Helical field lines
wrapped around cylindrical
pressure surfaces.

How Helical?

Equation for field lines

$$\frac{dr}{de} = \frac{B}{B_\theta}$$

$$dr \cdot \nabla \theta = d\theta$$

$$\underline{B} \cdot \nabla \theta = \frac{B_\theta}{r} \rightarrow$$

$$\frac{d\theta}{de} = \frac{B_\theta}{r B}$$

$$dr \cdot \hat{z} = dz$$

$$\underline{B} \cdot \hat{z} = B_z$$

$$\frac{dz}{de} = \frac{B_z}{B}$$

$$\rightarrow \frac{dz}{d\theta} = \frac{r B_z}{B_\theta} \rightarrow$$

$$z = \frac{r B_z}{B_\theta} \theta + z_0$$

Equation of
a field line.

ANGLE IN "θ" DIRECTION AFTER z HAS GONE ONE

FULL LENGTH ($\Delta z = R$) -

$$\Delta \theta = \frac{R B_\theta}{r B_z} = \frac{1}{q(r)} = i(r)$$

$q(r) \equiv$ SAFETY FACTOR

$i(r) \equiv$ ROTATIONAL TRANSFORM.

Plasma Physics, 222: Homework 9.

Lecturer: Steve Cowley

Question 1. Z-pinch.

A simple model of a z-pinch equilibrium has current density:

$$\mathbf{J} = J_0 \mathbf{z} \quad \text{for } r < a, \quad \mathbf{J} = 0 \quad \text{for } r > a. \quad (1)$$

where J_0 is a constant.

(i) Calculate the magnetic field for this current.

(ii) Calculate the equilibrium pressure.

(iii) Harder Suppose we take a aluminium wire of initial radius 0.1mm and pass a total current of 30MA through it. Suppose also that the wire heats to a uniform temperature of 300eV. Take the current density to have the form given above. Calculate the density as a function of r and the radius a in equilibrium.

122a Lecture #MHD Waves.

READ THE TEXT BOOK ON MHD.

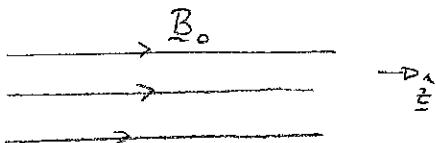
(i) Last time we looked at MHD equilibrium:

$$\nabla p = \underline{J} \times \underline{B}$$

$$\underline{v} = \underline{0}$$

$$\nabla \times \underline{B} = \mu_0 \underline{J}$$

this is the condition for a stationary plasma in force balance.

Note. in equilibrium $\underline{B} \cdot \nabla p = 0 \Rightarrow$ pressure is constant along field lines.(ii) TODAY we look at waves propagating in a stationary (homogeneous) plasma with $\eta = 0$ HOMOGENEOUS EQUILIBRIUM.

$$\underline{B}_0 = \text{constant} = B_0 \hat{\underline{z}}, \underline{v} = \underline{0}$$

$$\rho = \rho_0 = \text{constant}, p = p_0 = \text{constant}$$

(iii) Perturb about this equilibrium

$$\underline{B} = B_0 \hat{\underline{z}} + \delta \underline{B}$$

$$|\delta \underline{B}| \ll B_0$$

$$\rho = \rho_0 + \delta \rho$$

$$\delta \rho \ll \rho_0$$

$$p = p_0 + \delta p$$

$$\delta p \ll p_0$$

$$\underline{v} = \delta \underline{v}$$

no unperturbed velocity.

SMALL PERTURBATION

(iv) Linearize the MHD equations to obtain evolution equations for $\delta \underline{B}$, $\delta \underline{v}$, $\delta \rho$ and δp . Linearize means keep terms linear in the perturbed quantities but drop quadratic and higher order terms.FOR EXAMPLE:

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) \Rightarrow \cancel{\frac{\partial B_0}{\partial t}} \frac{\partial \delta \underline{B}}{\partial t} = \nabla \times (\delta \underline{v} \times \underline{B}_0) + \nabla \times (\delta \underline{v} \times \delta \underline{B})$$

" 0 "

drop because it
is quadratic in
small perturbations

(v) Thus we get the linearized equations:

$$\frac{d\tilde{B}}{dt} = \nabla \times (\underline{v} \times \underline{B}) \Rightarrow \boxed{\frac{d\delta\tilde{B}}{dt} = \nabla \times (\delta\underline{v} \times \underline{B}_0)} \quad (2)$$

$$\rho \left[\frac{d\underline{v}}{dt} + \underline{v} \cdot \nabla \underline{v} \right] = -\nabla p + \underline{j} \times \underline{B} \Rightarrow \boxed{\rho_0 \frac{d\delta\underline{v}}{dt} = -\nabla(\delta p + \frac{\underline{B}_0 \cdot \delta\tilde{B}}{\mu_0}) + \frac{\underline{B}_0 \cdot \nabla \delta\tilde{B}}{\mu_0}} \quad (2)$$

$$\frac{dp}{dt} + \underline{v} \cdot \nabla p = -Y_p \nabla \cdot \underline{v} \Rightarrow \boxed{\frac{d\delta p}{dt} = -Y_{p0} \nabla \cdot (\delta \underline{v})} \quad (3) \quad Y = \frac{5}{3}$$

usually

$$\frac{dp}{dt} + \underline{v} \cdot \nabla p = -p \nabla \cdot \underline{v} \Rightarrow \boxed{\frac{d\delta p}{dt} = -\rho_0 \nabla \cdot \delta \underline{v}} \quad (4)$$

Not needed as δp doesn't appear in (1) - (3).

(vi) Displacement. It is common to write the linearized equations in terms of the displacement ξ of the plasma from its original position.

$$\delta\underline{v} = \frac{d\xi}{dt} = \frac{\partial \xi}{\partial t} + \underline{v} \cancel{\cdot \nabla \xi} \Leftrightarrow \delta\underline{v} = \frac{\partial \xi}{\partial t}$$

STRICT
LINEARIZE

Then the linearized equations become:

$$\delta\tilde{B} = \nabla \times (\xi \times \underline{B}_0)$$

$$\rho_0 \frac{d^2\xi}{dt^2} = -\nabla(\delta p + \frac{\underline{B}_0 \cdot \delta\tilde{B}}{\mu_0}) + \frac{\underline{B}_0 \cdot \nabla \delta\tilde{B}}{\mu_0}$$

$$\delta p = -Y_{p0} \nabla \cdot \xi$$

$$\delta p = -\rho_0 \nabla \cdot \xi$$

the only equation with a time derivative.

(vii) We look for plane wave solutions.

$$\hat{\xi}(r, t) = \hat{\xi} e^{ik \cdot r - i\omega t} + \text{complex conjugate.}$$

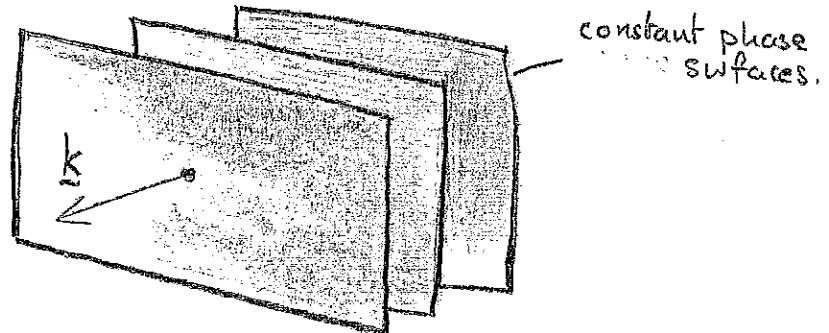
$$\hat{\xi}^* e^{-ik \cdot r + i\omega t}$$

and similarly for δB , δv , p etc.

$\hat{\xi}$ = a constant complex number.

DROP COMPLEX CONJUGATE
PART AND ADD IT IN AT END.

k = WAVE VECTOR
perpendicular
to phase surfaces.



(viii) NOTE:

$$\nabla \xi_x = ik \hat{\xi}_x e^{ik \cdot r - i\omega t} = ik \hat{\xi}_x$$

$$\frac{\partial \xi_x}{\partial t} = -i\omega \xi_x$$

SO EVERY WHERE WE SEE ∇ IN THE EQUATION WE CAN REPLACE IT BY ik
AND $\frac{\partial}{\partial t}$ \rightarrow $-i\omega$

(ix) so FOR EXAMPLE,

$$\delta \underline{B} = \nabla \times (\underline{\xi} \times \underline{B}_0) \Rightarrow \delta \underline{B} e^{ik \cdot r - i\omega t} = ik \times (\hat{\underline{\xi}} \times \underline{B}_0) e^{ik \cdot r - i\omega t}$$

$$\Rightarrow \boxed{\delta \underline{B} = ik \times (\hat{\underline{\xi}} \times \underline{B}_0) = i(k \cdot \underline{B}_0) \hat{\underline{\xi}} - i\underline{B}_0 (k \cdot \hat{\underline{\xi}})}$$

$$\delta \hat{p} = -i p_0 ik \cdot \hat{\underline{\xi}}$$

$$\delta \hat{p} = -p_0 ik \cdot \hat{\underline{\xi}}$$

and note

$$\frac{\partial^2 \hat{\underline{\xi}}}{\partial t^2} = -\omega^2 \hat{\underline{\xi}} e^{ik \cdot r - i\omega t}$$

(X) Now the momentum equation becomes

$$\omega^2 \rho_0 \hat{\xi} = -k \left[(\gamma p_0 + \frac{B_0^2}{\mu_0}) (\underline{k} \cdot \hat{\xi}) - \left(\frac{\underline{k} \cdot \underline{B}_0}{\mu_0} \right) (\hat{\xi} \cdot \underline{B}_0) \right] \\ + \left(\frac{\underline{k} \cdot \underline{B}_0}{\mu_0} \right)^2 \hat{\xi} - \left(\frac{\underline{k} \cdot \underline{B}_0}{\mu_0} \right) (\underline{k} \cdot \hat{\xi}) \underline{B}_0. \quad (6)$$

a vector equation for $\hat{\xi}$.

BEFORE WE DO THE GENERAL CASE LETS LOOK AT WAVES PROPAGATING PARALLEL AND PERPENDICULAR TO \underline{B}_0 .

(X') PARALLEL PROPAGATION. $\underline{k} = k_{||} \hat{z}$ $\hat{\xi} = \hat{\xi}_{||} \hat{z} + \hat{\xi}_{\perp}$
From (6)

$$(7) \quad \omega^2 \rho_0 \hat{\xi}_{||} = k_{||}^2 \gamma p_0 \hat{\xi}_{||} \quad \leftarrow \text{SOUND WAVES}$$

$$(8) \quad \omega^2 \rho_0 \hat{\xi}_{\perp} = k_{||}^2 B_0^2 \hat{\xi}_{\perp} \quad \leftarrow \text{MAGNETIC WAVES.}$$

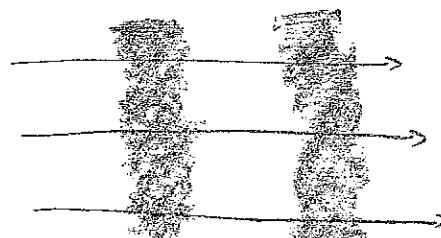
From (7)

$$c_s = \sqrt{\frac{\gamma p_0}{\rho_0}} \quad \text{SOUND SPEED}$$

DISPERSION RELATION.

$$\omega = \pm k_{||} c_s \quad \text{parallel propagating sound waves.}$$

• LONGITUDINAL WAVE



$$\delta \underline{B} = 0 \quad \text{Field lines unperturbed.}$$

$$\hat{\xi}_{\perp} = 0$$

From (8)

$$V_A = \sqrt{\frac{B_0^2}{\mu_0 \rho_0}} \quad \text{ALFVÉN SPEED.}$$

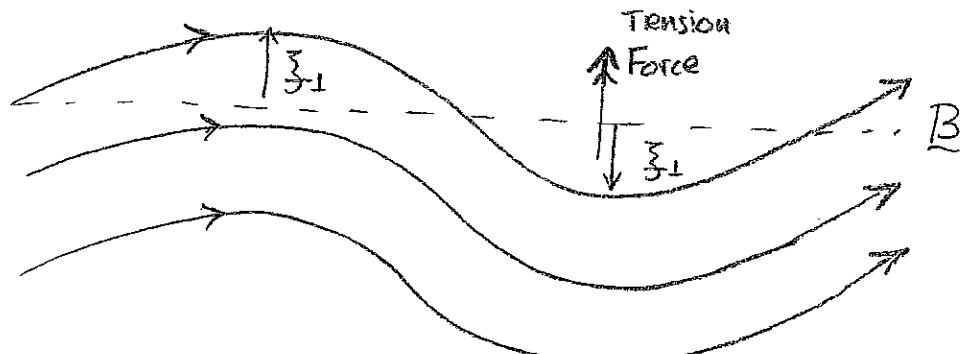
"Alfvén Waves"

DISPERSION RELATION.

$$\omega = \pm k_{||} V_A$$

$$\delta \underline{B} = i k_{||} B_0 \hat{\xi}_{\perp}$$

• TRANSVERSE WAVES



2 polarizations.

"Tension" of magnetic field lines makes them act like strings.
Waves "Alfvén" waves are transverse waves propagating due to magnetic tension along the field.

(Xii) Perpendicular Propagation: $\underline{k} \cdot \underline{B}_0 = 0$ $\underline{k} = \underline{k}_\perp$

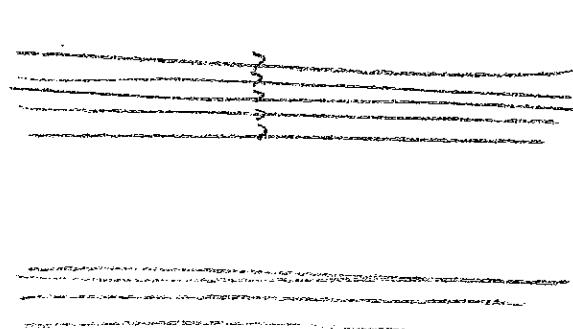
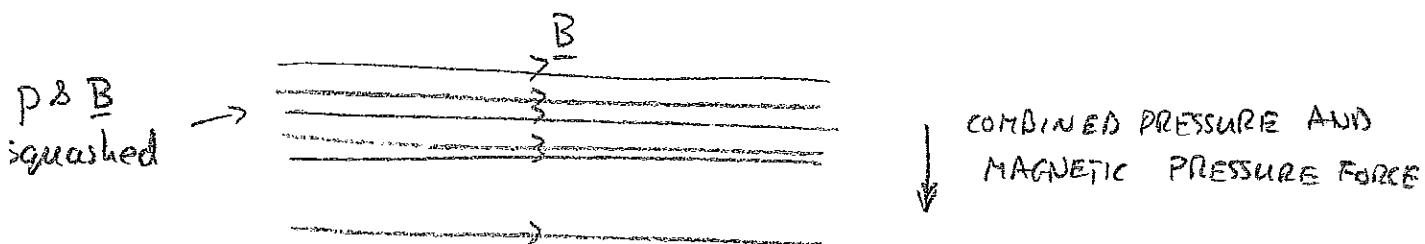
Parallel part of ⑥ $\Rightarrow \omega^2 \rho_0 \xi_{\parallel} = 0 \Rightarrow \omega^2 = 0$ "Zero frequency wave"

Perpendicular part of ⑥ $\Rightarrow \omega^2 \rho_0 \xi_\perp = k_\perp \left(\gamma p_0 + \frac{B_0^2}{\mu_0} \right) \underline{k}_\perp \cdot \underline{\xi}_\perp$

two waves from this

$$\omega^2 = 0 \quad \underline{k}_\perp \cdot \underline{\xi}_\perp = 0 \Rightarrow \underline{\xi}_\perp \propto (\underline{k}_\perp \times \underline{B}_0)$$

$$\omega^2 = k_\perp^2 (c_s^2 + V_A^2) \quad \text{MAGNETOSONIC WAVE.}$$



Acts like a sound wave with pressure repeated by total pressure.

(xiii) Now the tedious general case: ⑥ is really a matrix equation

$$(\rho_0 \omega^2 \underline{\underline{I}} + \underline{\underline{M}}) \cdot \underline{\xi} = 0$$

DISPERSION RELATION $|\rho_0 \omega^2 \underline{\underline{I}} + \underline{\underline{M}}| = 0$ determines $\omega = \omega(k)$

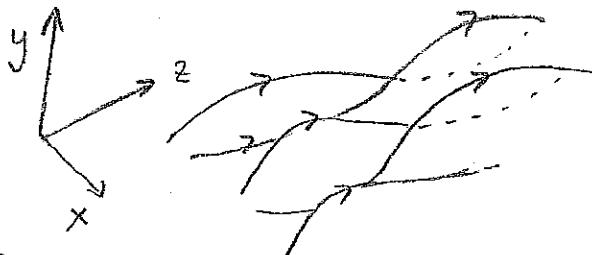
It is easier to separate if we write. $\underline{k} = k_{\parallel} \hat{\underline{z}} + k_{\perp} \hat{\underline{x}}$

y-component of ⑥.

$$\omega^2 \rho_0 \xi_y = \frac{k_{\parallel}^2 B_0^2}{\mu_0} \xi_y \Rightarrow \boxed{\omega = \pm k_{\parallel} V_A}$$

STILL SHEAR ALFVÉN WAVES PROPAGATING ALONG \underline{B} INDEPENDANT OF k_{\perp}

POLARIZED PERPENDICULAR TO \underline{k} AND \underline{B}



$$\delta \underline{B} = i k_{\parallel} B_0 \xi_y \hat{\underline{y}}$$

$$\underline{B} \cdot \delta \underline{B} = \delta B_{\parallel} = 0$$

$$\delta p = \delta \rho = 0$$

X COMPONENT OF ⑥

$$\omega^2 \rho_0 \xi_x = k_{\perp} \left[\left(\gamma \rho_0 + \frac{B_0^2}{\mu_0} \right) (k_{\perp} \xi_x + k_{\parallel} \xi_z) - \frac{k_{\parallel}^2 B_0^2}{\mu_0} \xi_z \right] + \frac{k_{\parallel}^2 B_0^2}{\mu_0} \xi_x - \ddot{\xi}_x$$

Z COMPONENT OF ⑥

$$\omega^2 \rho_0 \xi_z = k_{\perp} \left[\gamma \rho_0 (k_{\perp} \xi_x + k_{\parallel} \xi_z) \right]$$

only pressure forces along field lines.

Eliminating ξ_x & ξ_z we get the dispersion relation

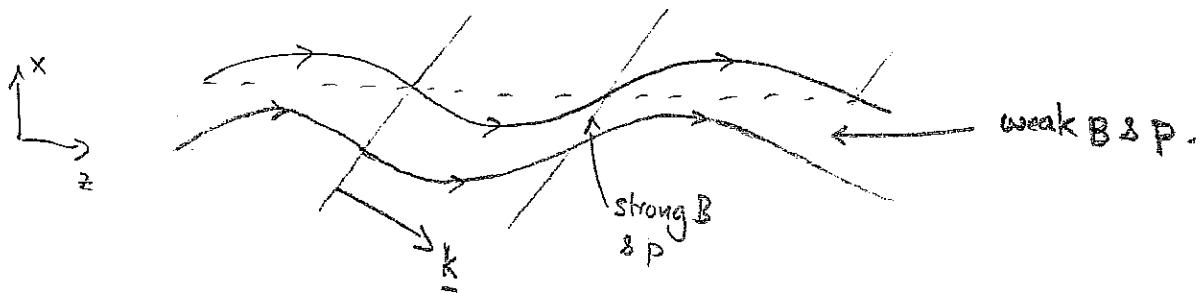
$$\omega^4 - k^2 (c_s^2 + V_A^2) \omega^2 + k_{\parallel}^2 c_s^2 (k_{\parallel}^2 V_A^2 + k_{\perp}^2 c_s^2) = 0$$

Solving the Quadratic we get.

FAST
WAVE

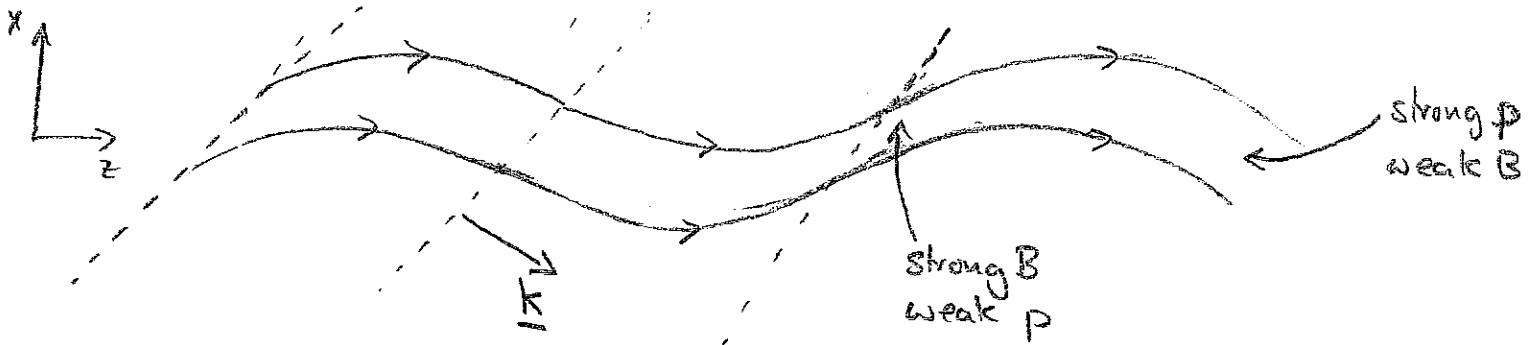
$$\omega^2 = \frac{k^2(c_s^2 + V_A^2)}{2} + \frac{1}{2}\sqrt{k^4(c_s^2 + V_A^2)^2 - 4k_{\parallel}^2c_s^2(k_{\parallel}^2V_A^2 + k_{\perp}^2c_s^2)}$$

- COMBINATION OF PRESSURE, MAGNETIC PRESSURE AND TENSION FORCES.
- WHERE B GOES UP SO DOES P .



SLOW
AWE

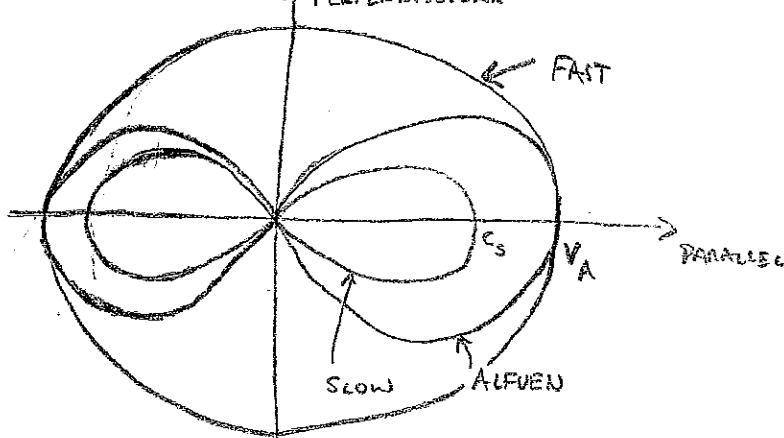
$$\omega^2 = \frac{k^2(c_s^2 + V_A^2)}{2} - \frac{1}{2}\sqrt{k^4(c_s^2 + V_A^2)^2 - 4k_{\parallel}^2c_s^2(k_{\parallel}^2V_A^2 + k_{\perp}^2c_s^2)}$$



POLAR PLOT

$$k = k(\cos\theta \hat{z} + \sin\theta \hat{x})$$

↑ PERPENDICULAR



radius = $\frac{\omega}{k}$ = phase velocity.
angle = θ

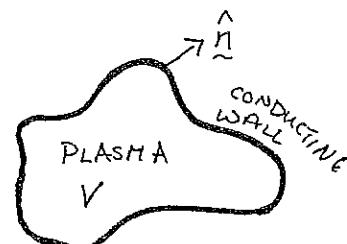
for $V_A \geq c_s$ $\beta = \frac{kp}{B^2} \propto 1$

Physics 222a. Lecture #13: Lagrangian MHD and Energy Principle

(i) MHD has a conserved energy:-

$$E = \int_V \left\{ \frac{\rho v^2}{2} + \frac{B^2}{8\pi} + \frac{P}{\gamma - 1} \right\} d^3\Sigma$$

KINETIC ENERGY MAGNETIC ENERGY PRESSURE ENERGY

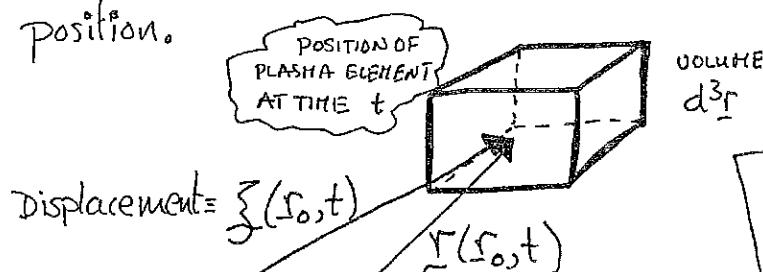


Take simple boundary conditions - no flow out of walls $v \cdot \hat{n}$

HOMEWORK Q.1.

Using the MHD Equations show that $\frac{dF}{dt} = 0$ with the simple boundary conditions given above.

(ii) LAGRANGIAN MHD: a convenient way to look at stability is using LAGRANGIAN variables. We write everything in terms of the displacement vector $\xi(r_0, t)$ of each piece of plasma from its original position.



Displacement = $\xi(r_0, t)$

$$\begin{aligned} \underline{r} &= \underline{r}_0 + \xi(r_0, t) \\ \underline{r} &\equiv \underline{r}(r_0, t) \\ \underline{r}_0 &\equiv \underline{r}_0(r, t) \end{aligned}$$

: PLASMA VELOCITY = $v(r_0, t) = \left(\frac{\partial \underline{r}}{\partial t} \right)_{r_0} = \left(\frac{\partial \xi}{\partial t} \right)_{r_0}$

: ∇_r = gradient with respect to r .

: $\nabla_r \underline{r} \equiv \text{"strain matrix"} = \underline{\underline{I}} + \nabla_r \xi$

note $\nabla_r \underline{r} \cdot \nabla_r \underline{r}_0 = \underline{\underline{I}}$ by chain rule.

$$\begin{aligned} \underline{\underline{I}} &= \text{unit, "identity" matrix} \\ &\equiv \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(iii) All quantities can be thought of as either functions of \underline{x} or \underline{x}_0 and t . For example density in the displaced box can be written as

$$\rho = \rho_e(\underline{x}, t) \quad \text{or} \quad \rho = \rho_L(\underline{x}_0, t)$$

so the functions are related by

$$\rho_e(\underline{x}_0 + \underline{\xi}(\underline{x}_0, t), t) = \rho_L(\underline{x}_0, t)$$

For small displacements:- ie. $\underline{\xi} \ll L \equiv |\nabla \underline{\xi}|^{-1}$

$$\rho_L(\underline{x}_0, t) \approx \rho_e(\underline{x}_0, t) + \underline{\xi} \cdot \nabla \rho_e \dots \dots \dots$$

(iv) We can integrate the equations of MHD using the Lagrangian displacement.

Density - Conservation of Mass define: $\rho_0(\underline{x}_0) = \rho_L(\underline{x}_0, 0)$

$$\therefore \text{Mass in piece of plasma} = dm = \rho_0(\underline{x}_0) d^3 \underline{x}_0 = \rho_L(\underline{x}_0, t) d^3 \underline{x}$$

BUT from the rules of partial differentiation

$$d^3 \underline{x} = J d^3 \underline{x}_0$$

where $J = \text{JACOBIAN} = |\nabla_{\underline{x}_0} \underline{x}| \equiv \text{DETERMINANT OF STRAIN MATRIX}$

(v) THUS

①

$$\rho_L(\underline{x}_0, t) = \frac{\rho_0(\underline{x}_0)}{J(\underline{x}_0, t)}$$

gives density in terms of $\underline{\xi}(\underline{x}_0, t)$

i) PRESSURE - conservation of Entropy recall $\frac{d}{dt} \left(\frac{P}{\rho^\gamma} \right) = \left(\frac{d}{dt} \left(\frac{P}{\rho^\gamma} \right) \right)_{r_0} = 0$

$\frac{P}{\rho^\gamma}$ = constant for element of plasma.

$$\Rightarrow P_L(r_0, t) = \frac{P_0(r_0)}{\gamma^\gamma} \quad \text{--- (2)}$$

(vi) Acceleration $\left(\frac{\partial}{\partial t} \right)_r + \underline{v} \cdot \nabla = \left(\frac{\partial}{\partial t} \right)_{r_0}$

$$\Rightarrow \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = \frac{\partial^2 \underline{x}}{\partial t^2}$$

(vii) MAGNETIC FIELD - FLUX CONSERVATION

Homework Q.2. From $\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B})$ show that

Reference Horace Lamb, Hydrodynamics, 1879

$$\underline{B}_L(r_0, t) = \frac{(\underline{B}_0(t_0) \cdot \nabla_r t_0)}{\gamma} \quad \text{--- (3)}$$

this is called "Lundquist's identity" by some.

or If you like you can prove this geometrically.

(viii) We have "integrated" all the equations of MHD except the force^{or momentum} equation: The force equation yields an equation for \underline{x} the displacement.

4

(ix) Substituting ①, ② & ③ into the force equation we obtain

$$\boxed{\frac{\rho_0}{J} \frac{\partial^2 \xi}{\partial t^2} = -\nabla_{\xi_0} \cdot \nabla_{\xi_0} \left\{ \rho_0 J^{-\gamma} + \frac{(B_0 \cdot \nabla_{\xi_0} \xi)^2}{J^2 8\pi} \right\} + \frac{B_0 \cdot \nabla_{\xi_0}}{J} \left(\frac{B_0 \cdot \nabla_{\xi_0} \xi}{J} \right)}$$

This is the equation of motion for a sort of elastic medium where the right hand side is a force dependant on the displacement $\underline{F}(\xi_0, \xi, t)$.

(x) The ENERGY is also easily written as:

$$E = \int_V \left\{ \frac{\rho_0}{2} \left(\frac{\partial \xi}{\partial t} \right)^2 + \frac{(B_0 \cdot \nabla_{\xi_0} \xi)^2}{J 8\pi} + \frac{\rho_0 J^{(1-\gamma)}}{\gamma-1} \right\} d^3 \xi_0$$

KINETIC ENERGY POTENTIAL ENERGY

AN ASIDE FOR THE THEORETICALLY Minded.

The equation of motion can be derived from an action principle

$$S(\xi) = \int_{t_0}^t dt \int_V d^3 \xi_0 L \quad \text{with} \quad \frac{\delta S}{\delta \xi} = 0$$

variation
with end
points fixed.

where

$$L = \frac{\rho_0}{2} \left(\frac{\partial \xi}{\partial t} \right)^2 - \frac{(B_0 \cdot \nabla_{\xi_0} \xi)^2}{J 8\pi} - \frac{\rho_0 J^{1-\gamma}}{\gamma-1}$$

"Lagrangian density"

(xi) For small displacements (ξ) we can expand the energy to quadratic order to get.

$$E_0 = \int d^3 r_0 \left\{ \frac{p_0}{\gamma - 1} + \frac{B_0^2}{8\pi} \right\}$$

$$E_1 = \int d^3 r_0 \xi \cdot \left\{ \nabla_0 p_0 - \frac{\underline{J}_0 \times \underline{B}_0}{c} \right\}$$

$$\underline{J}_0 = \frac{\nabla_0 \times \underline{B}_0}{c}$$

of course
NOT THE JACOBIAN!

$$E_2 = \int d^3 r_0 \left\{ \frac{p_0}{2} \left(\frac{\partial \xi}{\partial t} \right)^2 \right\} + \delta W(\xi, \dot{\xi})$$

"called delta W" For us to be given

If we assume the system starts in an equilibrium

then $\nabla_0 p_0 - \underline{J}_0 \times \underline{B}_0 = 0$ and therefore neglecting higher orders

$$E_2 = \text{constant}$$

since $E_0 = \text{constant}$ and $E = \text{constant}$.

(xii) FORMS FOR δW equivalent with integrations by parts

$$\delta W = \frac{1}{2} \int d^3 r_0 \left[|Q + \hat{n} \cdot \xi (\underline{J}_0 \times \hat{n})|^2 + \gamma p_0 |\nabla_0 \cdot \xi|^2 - 2 (\underline{J}_0 \times \hat{n}) \cdot \underline{B}_0 \nabla_0 \cdot \hat{n} |\hat{n} \cdot \xi|^2 \right]$$

$$\text{where } Q = \nabla_0 \times (\xi \times \underline{B}_0) \quad \hat{n} = \frac{\nabla_0 p_0}{|\nabla_0 p_0|} \quad K = \underline{b}_0 \cdot \nabla_0 \underline{b}_0$$

$$\text{or } \delta W = \frac{1}{2} \int d^3 r_0 \left[|Q_1|^2 + B_0^2 |\nabla_0 \cdot \xi_1 + 2 \xi_1 \cdot K|^2 + \gamma p_0 |\nabla_0 \cdot \xi_1|^2 - 2 \xi_1 \cdot \nabla_0 K \cdot \xi_1 \right.$$

"KINK DRIVE"

$$\left. - \underline{J}_0 \cdot \underline{B}_0 \cdot \xi_1 \times \underline{B}_0 \cdot Q_1 \right]$$

(xiii) The force equation may be linearized. (some algebra.

$$\boxed{\rho_0 \frac{\partial^2 \xi}{\partial t^2} = E(\xi)} - ④$$

$$E(\xi) = (\nabla_0 \times B_0) \times Q + (\nabla_0 \times Q) \times B_0 + \nabla_0 \left(\xi \cdot \nabla_0 p_0 + \gamma p_0 \nabla_0 \cdot \xi \right)$$

(xiv) we differentiate

$$\frac{dE_2}{dt} = \int \rho_0 \frac{\partial \xi}{\partial t} \cdot \frac{\partial^2 \xi}{\partial t^2} d^3 r_0 + \delta W \left(\frac{\partial \xi}{\partial t}, \xi \right) + \delta W \left(\xi, \frac{\partial \xi}{\partial t} \right)$$

using ④ we obtain

$$\int_V \rho_0 \frac{\partial \xi}{\partial t} \cdot F(\xi) d^3 r_0 = - \left\{ \delta W \left(\xi, \frac{\partial \xi}{\partial t} \right) + \delta W \left(\frac{\partial \xi}{\partial t}, \xi \right) \right\}$$

THIS MUST BE TRUE AT $t=0$ WHEN I CAN CHOOSE ξ AND $\frac{\partial \xi}{\partial t}$
ARBITRARILY HENCE FOR ALL η AND $\xi(r_0)$

$$\int \rho_0 \eta \cdot F(\xi) d^3 r_0 = - \left\{ \delta W \left(\xi, \eta \right) + \delta W \left(\eta, \xi \right) \right\}$$

clearly then $F(\xi)$ is what we call "SELF ADJOINT"

$$\int \eta \cdot F(\xi) d^3 r_0 = \int \xi \cdot F(\eta) d^3 r_0$$

$$\frac{1}{2} \int d^3 r_0 \xi \cdot F(\xi) = - SW(\xi, \xi)$$

222a. Lecture #14: MHD Stability Part II.

(i) Last time, we looked at the formal theory of MHD stability:

$$E_2 = \text{CONSERVED TOTAL ENERGY} = \int d^3r_0 \left\{ \rho_0 \left(\frac{\partial \xi}{\partial t} \right)^2 \right\} + \delta W(\xi, \dot{\xi})$$

KINETIC ENERGY

POTENTIAL ENERGY.

ξ = plasma displacement

For small displacements

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = F(\xi) \quad \leftarrow \text{FORCE OPERATOR}$$

The exact form of δW , F are not important here just two facts.

$$(a) \delta W = -\frac{1}{2} \int \xi \cdot F(\xi) d^3r_0 \quad \leftarrow \text{work done by plasma}$$

$$(b) F \text{ is SELF ADJOINT, i.e. } (\text{real version of Hermitian})$$

$$\int \eta \cdot F(\xi) d^3r_0 = \int \xi \cdot F(\eta) d^3r_0$$

(ii) NORMAL MODES: Since the equation of motion is time independent we would like to write:-

$$\xi = \hat{\xi}(r) e^{\gamma t}$$

SEPERABLE SOLUTION.
 γ = GROWTH RATE.

$$\Rightarrow \rho_0 \gamma^2 \hat{\xi} = F(\hat{\xi})$$

Equation for the eigenvalue γ^2
 the growth rate squared

(iii) We would like to write a general displacement in terms of an expansion in NORMAL MODES.

$$\xi(r,t) = \sum_{n=1}^{\infty} a_n \xi_n(r) e^{\gamma_n t}$$

CAN DO IF THEY ARE COMPLETE AND DISCRETE
— NOT ALWAYS DISCRETE

PROPERTIES

♣ (c) NORMAL MODES ARE ORTHOGONAL:

PROOF: put $\eta = \xi_m e^{\gamma_m t}$ $\bar{\xi} = \xi_n e^{\gamma_n t}$ in SELF-ADJOINT EQN.

$$(\gamma_m^2 - \gamma_n^2) \int \rho_0 \xi_m \bar{\xi}_n d^3 r_0 = 0 \Rightarrow \int \rho_0 \xi_m \bar{\xi}_n d^3 r_0 = 0 \text{ if } \gamma_m \neq \gamma_n.$$

(d) EIGENVALUES γ_n^2 ARE REAL

$$\eta = \xi_n^* e^{\gamma_n^* t} \quad \bar{\xi} = \xi_n e^{\gamma_n t}$$

$$\Rightarrow (\gamma_n^{**2} - \gamma_n^2) \int \rho_0 |\xi_n|^2 d^3 r_0 = 0$$

$\Rightarrow \gamma_n^2$ is Real.

If $\gamma_n^2 > 0$ we have γ_n Real Growing modes $\gamma_n > 0$ UNSTABLE

If $\gamma_n^2 < 0$ $\gamma_n = \pm i\omega_n$ oscillating stable solution.

(iv) ENERGY PRINCIPLE:

$$\delta W = -\frac{1}{2} \int \xi \cdot F(\xi) d^3 r_0 = - \sum_{n=1}^{\infty} a_n^2 \gamma_n^2 \left(\rho_0 |\xi_n|^2 d^3 r_0 \right)$$

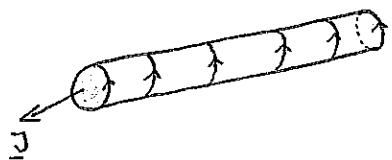
⇒ If we can find a ξ that makes $\delta W < 0$ then there must be at least one NORMAL MODE WITH $\gamma_n^2 > 0$ i.e. UNSTABLE PLASMA

If we can't find any ξ that makes $\delta W < 0$ then there cannot be a growing NORMAL MODE i.e. STABLE PLASMA.

WE SAY

$\delta W > 0$ FOR ALL ξ IS A NECESSARY AND SUFFICIENT CONDITION FOR INSTABILITY.

(V) Z-PINCH STABILITY.



$$\underline{J} = J_0(z) \hat{z} \quad \underline{B} = B_0(r) \hat{\theta}$$

$$\nabla p = \underline{J} \times \underline{B} \Rightarrow \boxed{\frac{d}{dr} \left(p_0 + \frac{B_0^2}{2} \right) = - \frac{B_0^2}{r}}$$

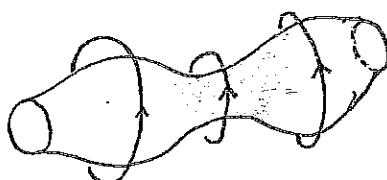
(vi) We want to look at behavior when plasma is perturbed by a perturbation (displacement)

$$\xi(r, \theta, z, t) = \xi(r) e^{ikz + i m \theta + \gamma t}$$

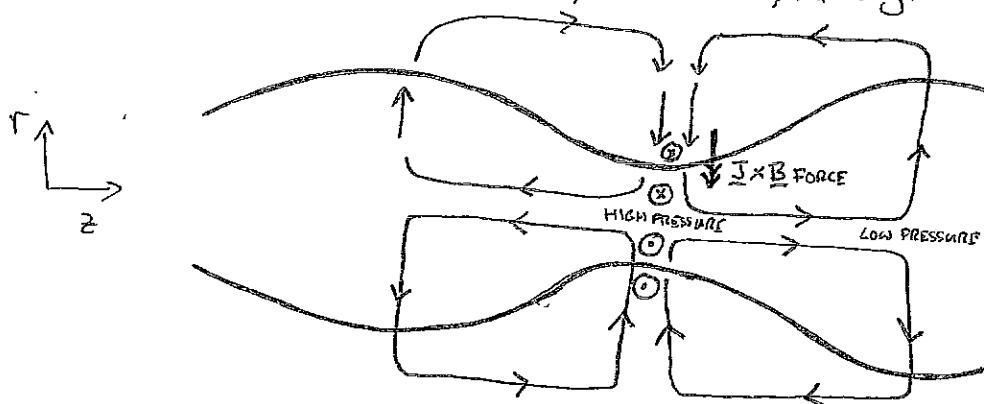
γ = GROWTH RATE.
mode numbers
k and m.

(vii) SAUSAGE INSTABILITY. m=0

only consider this case for simplicity.



displacement/velocity.



- CURRENT IS STRONGER IN THE NECK $\therefore \underline{J} \times \underline{B}$ FORCE IS STRONGER. $(p_0 \frac{B^2}{2})$
- PRESSURE IN THE NECK IS HIGHER PUSHES PLASMA IN Z DIRECTION OUT OF THE NECK.
- CAN BE UNSTABLE IF CURRENT PINCH/UK FORCE EXCEEDS PRESSURE FORCES.

Now some Algebra.

$$(viii) \quad \underline{\xi} = \underline{\xi}(r) e^{ikz + \delta t}$$

From 8- $\delta \underline{B} = \nabla \times (\underline{\xi} \times \underline{B}_0) = \underline{B}_0 \cdot \nabla \underline{\xi} - \underline{\xi} \cdot \nabla \underline{B}_0 - \underline{B}_0 (\nabla \cdot \underline{\xi})$ vector identity.
after a little algebra

$\delta B_r = 0, \delta B_z = 0$ as expected field remains circles.

$$\delta B_\theta = \frac{B_0 \xi_r}{r} - \xi_r \frac{dB_0}{dr} - B_0 \left[\frac{1}{r} \frac{d(r\xi_r)}{dr} + \frac{d\xi_z}{dz} \right] \quad (1)$$

$$\begin{aligned} \text{From 8- } \delta p &= -\underline{\xi} \cdot \nabla p_0 - \gamma p_0 \nabla \cdot \underline{\xi} \\ &= -\xi_r \frac{dp_0}{dr} - \gamma p_0 \left[\frac{1}{r} \frac{d(r\xi_r)}{dr} + \frac{d\xi_z}{dz} + \frac{1}{r} \frac{d\xi_\theta}{d\theta} \right] \end{aligned} \quad (2)$$

(ix) FORCE EQUATION

$$\rho_0 \frac{\partial^2 \underline{\xi}}{\partial t^2} = \underline{F}(\underline{\xi})$$

$\hat{\theta}$ component:

$$\rho_0 \frac{\partial^2 \xi_\theta}{\partial t^2} = 0 \quad \text{No forces (as expected) accelerating plasma in } \theta \text{ direction.} \quad \Rightarrow \boxed{\xi_\theta = 0}$$

\hat{z} component

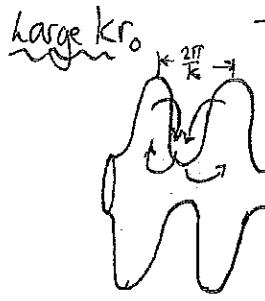
$$(3) \quad \rho_0 \frac{\partial^2 \xi_z}{\partial t^2} = - \frac{\partial}{\partial z} (\delta p + B_0 \delta B_\theta) = - ik (\delta p + B_0 \delta B_\theta)$$

Acceleration in z direction due to p.s B^2 pressure.

\hat{r} component

$$(4) \quad \rho_0 \frac{\partial^2 \xi_r}{\partial t^2} = - \frac{\partial}{\partial r} (\delta p + B_0 \delta B_\theta) - 2 B_0 \frac{\delta B_\theta}{r} \quad \swarrow \text{curvature "pinch" force, circles try to shrink.}$$

(x) The \hat{z} -component and the \hat{r} component field equations for ξ_r and ξ_z - with (1) and (2)
but they are complicated and we simplify further by taking $kr_0 \ggg 1$



Large k_r . This is the most unstable case for $m=0$ (no proof here).

From ③

Force in z direction $-ik(\delta p + B_0 \delta B_0)$ is very large - unless $\delta p + B_0 \delta B_0 \rightarrow 0$

Almost pressure balance - a little is needed to accelerate in z direction - note it doesn't have to go far in z direction.

$$(xi) \quad [\delta p + B_0 \delta B_0 = 0]$$

$$= -\xi_r \left[\frac{dp_0}{dr} + \frac{d^2 B_0^2}{dr^2} - \frac{B_0^2}{r} \right] - (\gamma p_0 + B_0^2) \left[\frac{1}{r} \frac{d(\xi_r)}{dr} + \frac{d\xi_z}{dz} \right]$$

USING EQUILIBRIUM

$$= -\frac{2B_0^2}{r}$$

We can now use this to eliminate ξ_z from ④ To get.

$$\rho_0 \frac{d^2 \xi_r}{dr^2} = \gamma^2 \rho_0 \xi_r = - \frac{d}{dr} (\delta p + B_0 \delta B_0) - 2 \frac{B_0 \delta B_0}{r}$$

after a little algebra

↑ will destabilize if
line displaced outwards decreases
in field strength i.e. $\delta B_0 < 0$.

$$\gamma^2 \rho_0 \xi_r = -\xi_r \left[2r \frac{dp_0}{dr} + \frac{4\gamma p_0 B_0^2}{\gamma p_0 + B_0^2} \right]$$

⇒ THEREFORE:



LOCAL GROWTH $\gamma^2 > 0$ IF $[] < 0$ or

$$-2r \frac{dp_0}{dr} > \frac{4\gamma p_0 B_0^2}{\gamma p_0 + B_0^2}$$

CONDITION FOR A LOCAL
SAUSAGE INSTABILITY.

↑ proportional to the local current.

Plasma Physics, 222: Homework for Lecture 14..

Lecturer: Steve Cowley

Question 1. θ -pinch stability.

A standard form of δW is:

$$\delta W = \frac{1}{2} \int d^3r \left[\frac{|\mathbf{Q}_\perp|^2}{\mu_0} + \frac{B_0^2}{\mu_0} (\nabla \cdot \xi_\perp + 2\xi_\perp \cdot \kappa)^2 + \gamma p_0 (\nabla \cdot \xi)^2 - 2(\xi \cdot \nabla p_0)(\kappa \cdot \xi_\perp) - \mathbf{J} \cdot \mathbf{b} (\xi_\perp \times \mathbf{b}) \cdot \mathbf{Q} \right] \quad (1)$$

where $\mathbf{Q} = \nabla \times (\xi \times \mathbf{B}_0)$, $\mathbf{b} = \mathbf{B}_0/B_0$ and $\kappa = \mathbf{b} \cdot \nabla \mathbf{b}$ and \perp refers to components perpendicular to \mathbf{B}_0 .

Using this form of δW show that θ pinches (where $\mathbf{B}_0 = B_0(r)\mathbf{z}$ and $p_0 = p_0(r)$) are *always* stable.

222a. Lecture #14: Accretion Disks and MHD Instability.

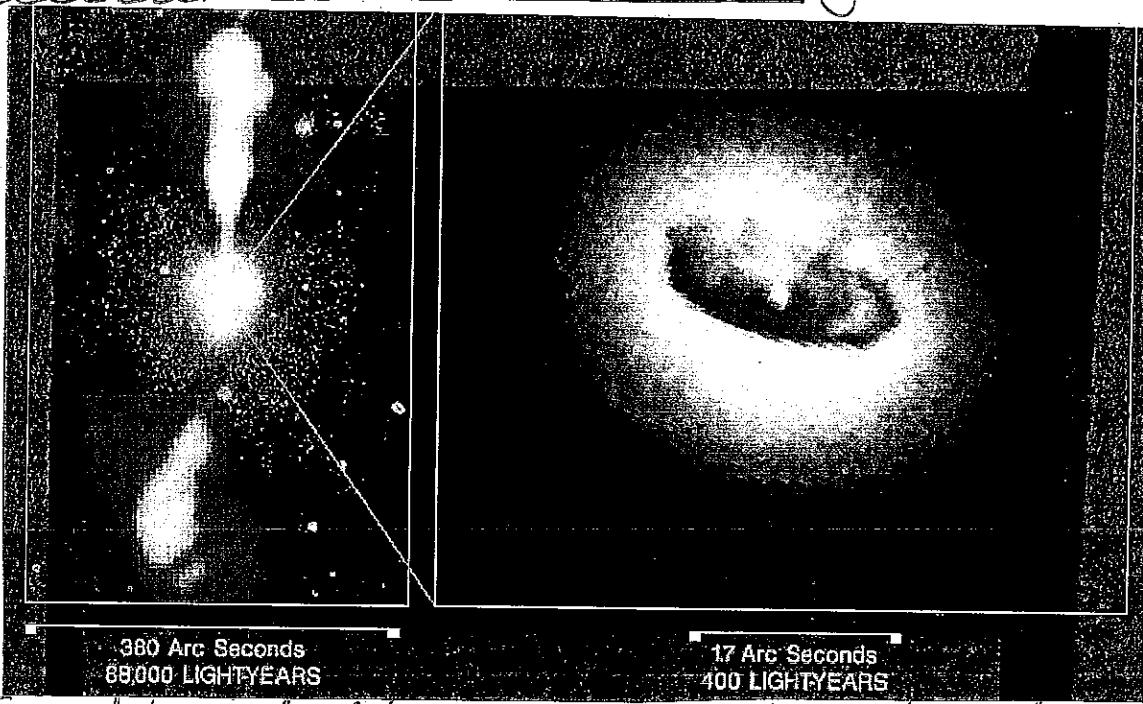


Image of the core of a Galaxy NGC 4261 - jet seems to come from disk around a central object which is probably a black hole. Glow is from hot plasma - we presume the plasma is falling into a black hole gaining heat as it loses potential energy. The disk is called an accretion disc.

(i) In the disc it is a hot plasma (many keV) we use MHD to describe disk.

$$m_p n \left(\frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right) = -\nabla p + \underline{J} \times \underline{B} + m_p n g \quad - (1)$$

g = local gravitational acceleration - due to black hole

m_p = proton mass
disc is mainly hydrogen.

$$\underline{g} = \frac{GM}{4\pi R^2} \hat{\underline{R}}$$

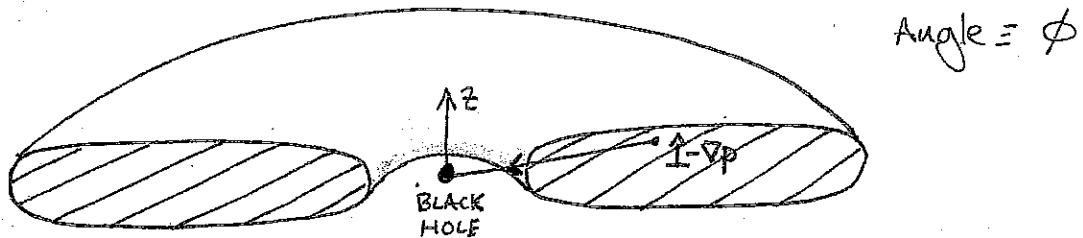
$\hat{\underline{R}}$ = radial unit vector.

M = Mass of Black Hole.

$$(ii) It seems from measurements that \frac{|\nabla p|}{|\underline{v} \cdot \nabla \underline{v}|} \sim \frac{c_s^2}{V_\phi^2} \ll 1$$

sound speed
SUPersonic CIRCULAR
FLOW

3



(iii) If we start with the assumption that the magnetic field is small and therefore $\underline{J} \times \underline{B}$ is small we obtain in steady rotation.

$$m_p \underline{v} \cdot \nabla V \approx - m_p \frac{GM}{R^2} \hat{R}$$

with $\underline{v} = \Omega(R) R \hat{e}_\phi$ $\nabla V = - \Omega^2 R \hat{R}$

CIRCULAR MOTION.

$$\boxed{-\Omega^2 = \frac{GM}{R^3}}$$

KEPLERIAN DISK. $\Omega \propto R^{-3/2}$

- INNER PART OF THE DISC ROTATES FASTER.

It is believed that the discs are dominantly Keplerian.

(iv) The thickness of the disc is determined by pressure.

$$\frac{\partial P}{\partial z} = - \frac{GMnm}{R^2} (\hat{z} \cdot \hat{R})$$

z component of ∇P .

$$= - nm \Omega^2 z$$

$P = nT \Leftrightarrow$ Assume $T = \text{constant}$ over disk thickness.

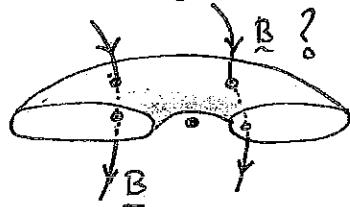
$$n = \text{DENSITY} = n_0(R) \exp \left\{ - \frac{\Omega^2 z^2}{2 C_s^2} \right\}$$

$$C_s^2 = \frac{T}{m_p}$$

$$H = \text{SCALE HEIGHT} \sim \sqrt{\frac{C_s^2}{2R}} \sim \left(\frac{C_s}{V_\phi} \right) R \ll R \quad \text{THIN DISK.}$$

(v) The energetics of the disk are interesting:-

- a) Disk is radiating energy rapidly.
- b) The only source of such energy is the gravitational potential energy. PLASMA MUST BE FALLING INTO BLACK HOLE.
- c) But to fall into the Black Hole the gas must lose angular momentum - to do so requires more than the Kepler flow - it requires some kind of viscosity to transfer angular momentum to gas further out.
- d) Ordinary viscosity is much too small to explain the in flow: Astrophysicists have typically invoked some TURBULENT VISCOSITY.
- e) It is believed that the Keplerian disk is stable without \mathbf{B} fields.
- f) Accretion disks seem to have \mathbf{B} fields - they emit synchrotron radiation.



g) The fashionable view is that MHD instability of the disk - called the Magneto Rotational Instability or MRI - causes the turbulence.

MRI - discovered by Velikov in 1958 and rediscovered and brought to prominence by Balbus and Hawley.

h) Simple version of instability today.

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(vi) MRI. in a simple disk.

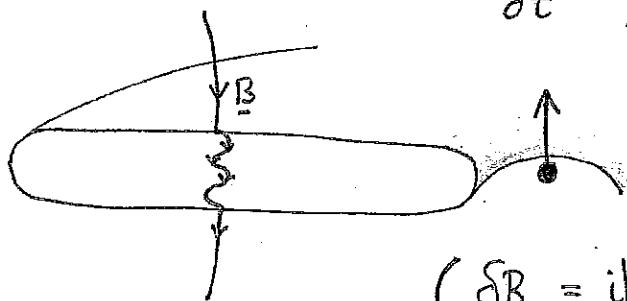
$$\underline{B}_0 = B_0 \hat{\underline{z}}$$

$$\underline{V}_0 = \Omega(R) R \hat{\underline{e}}_\phi$$

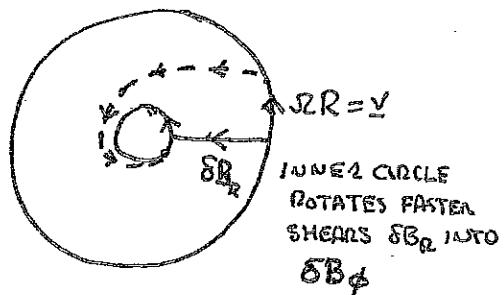
$$\Omega^2 = \frac{GM}{R^3}$$

$$\delta \underline{v} = \frac{\partial \xi}{\partial t} \quad \left. \begin{array}{l} \xi_2 = 0 \text{ for simplicity.} \\ \xi = \xi e^{ikz} \end{array} \right\}$$

IGNORE ANY R DEPENDENCE.



(vii) $\frac{\partial \underline{B}}{\partial t} = \nabla(\underline{V} \times \underline{B}) \Rightarrow \left\{ \begin{array}{l} \delta B_R = ik B_0 \xi_R, \quad \delta B_z = 0 \quad (2, 3) \\ \frac{\partial \delta B_\phi}{\partial t} = \delta B_R \frac{R d\Omega}{dR} + ik B \frac{\partial \xi_\phi}{\partial t} \quad (4) \end{array} \right.$

BENDING OF FIELD BY ξ_R SHEARING OF δB_R BY SHEARED ROTATION(viii) $\delta n = 0$ if $\nabla n = 0$ is assumed so no perturbation of gravity.(ix) FORCE EQUATIONS.

$$\frac{\partial V_R}{\partial t} + (\underline{V} \cdot \nabla \underline{V})_R = \frac{(\underline{B}_0 \cdot \nabla \underline{\delta B})_R}{mn \mu_0}$$

$$\frac{\partial V_\phi}{\partial t} + (\underline{V} \cdot \nabla \underline{V})_\phi = \frac{(\underline{B}_0 \cdot \nabla \underline{\delta B})_\phi}{mn \mu_0}$$

(X) After a little Algebra.

$$\frac{\partial^2 \hat{\xi}_R}{\partial t^2} - 2\Omega \frac{\partial \hat{\xi}_\phi}{\partial t} = \frac{i k B_0}{\mu_0 \text{nm}} \delta B_R$$

$$\frac{\partial^2 \hat{\xi}_\phi}{\partial t^2} + \frac{K^2}{2\Omega} \frac{\partial \hat{\xi}_R}{\partial t} = \frac{i k B_0}{\mu_0 \text{nm}} \delta B_\phi$$

$$K^2 = \frac{1}{R^3} \frac{d}{dR} (S \Omega^2 R^3)$$

K = "epicycle frequency"

$K^2 > 0$ stable without B

(XI) Substituting for δB_R & δB_ϕ we get,

$$\ddot{\hat{\xi}}_R - 2\Omega \dot{\hat{\xi}}_\phi = -\omega_A^2 \hat{\xi}_R$$

$$\omega_A^2 = \frac{k^2 B^2}{\mu_0 \text{nm}}$$

$$\ddot{\hat{\xi}}_\phi + 2\Omega \dot{\hat{\xi}}_R = -\left[\omega_A^2 + R \frac{d\Omega^2}{dR}\right] \hat{\xi}_\phi$$

$$\text{TAKE } \hat{\xi}_\phi = \bar{\hat{\xi}}_\phi e^{-i\omega t} \quad \hat{\xi}_R = \bar{\hat{\xi}}_R e^{-i\omega t}$$

$$(XII) \begin{bmatrix} \omega_A^2 - \omega^2, i2\Omega\omega \\ -i2\Omega\omega, \omega_A^2 + R \frac{d\Omega^2}{dR} - \omega^2 \end{bmatrix} \begin{bmatrix} \bar{\hat{\xi}}_R \\ \bar{\hat{\xi}}_\phi \end{bmatrix} = 0 \quad (5)$$

Determinant of matrix must vanish \rightarrow Dispersion Relation

$$(6) \quad (\omega_A^2 - \omega^2)(\omega_A^2 + R \frac{d\Omega^2}{dR} - \omega^2) = 4\Omega^2 \omega^2$$

Quartic
4 roots in
 \pm pairs.

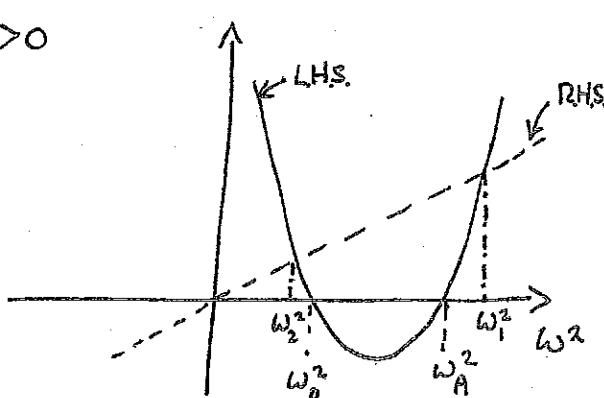
$$\text{Let } \omega_0^2 = \omega_A^2 + R \frac{d\Omega^2}{dR}$$

Find the condition for complex roots \rightarrow complex roots unstable.

6

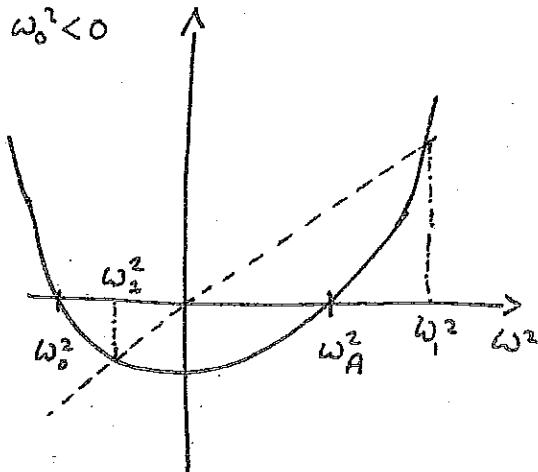
(Xiii) Plot LHS & RHS of Equation ⑥ (2 Cases) Intersections are roots (ω_1^2, ω_2^2)

$$\omega_0^2 > 0$$



In this case:

$$\omega_1^2, \omega_2^2 > 0 \quad \text{ALL ROOTS } \omega_1, \omega_2 \text{ ARE REAL. STABLE}$$



In this case:

$$\omega_1^2 > 0 \quad \omega_2^2 < 0$$

ω_1, ω_2 roots stable.

$\omega_2 = i|\omega_2|$ UNSTABLE ROOT

$\omega_2 = -i|\omega_2|$ STABLE ROOT

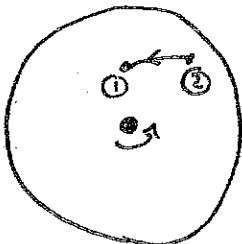
(Xiv) For $\omega_0^2 < 0$ $\omega_A^2 < -12 \frac{d\Omega^2}{dR} = 3\Omega^2$ CRITERION FOR INSTABILITY.
WEAK B FIELD.

For $\omega_A^2 \ll \Omega^2$ it is easy to show that

$$\gamma \equiv \text{growth rate} = \frac{\omega_A}{\sqrt{3}}$$

(Xv) The MRI has a simple physical picture: Displace field line Radially

Tension of B between ① & ② :-



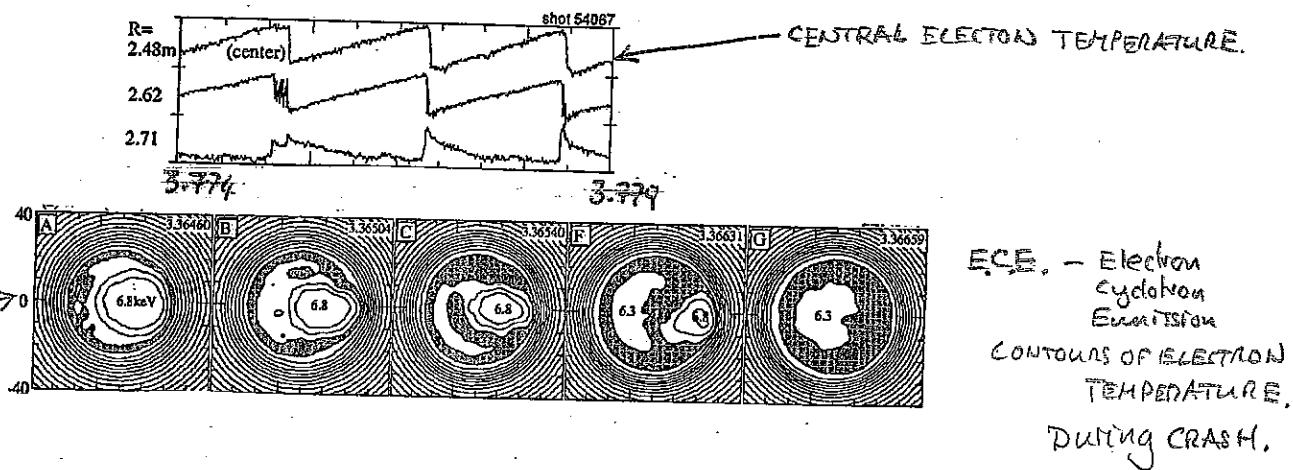
— SPEEDS UP ② SO ② HAS TOO MUCH CENTRIFUGAL FORCE AND MOVES OUTWARDS.

— SLOWS DOWN THE FASTER MOVING ① CAUSING IT TO FALL INWARDS.

BOTH EFFECTS LENGTHEN LINE AND CAUSE AMPLIFY δ_R AND THUS THE INSTABILITY.

22a. Lecture #18. Magnetic Reconnection.

- (i) The sawtooth oscillations in a tokamak are periodic drops in the central temperature.

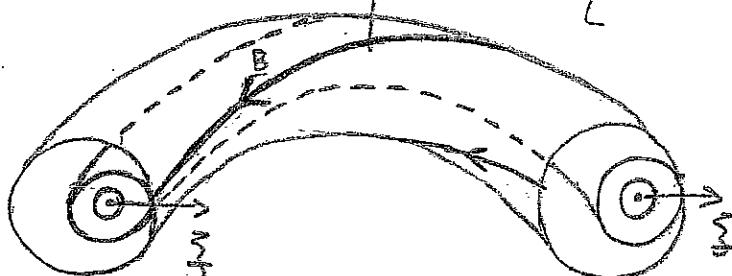


- (ii) The "crash" is the steep drop in T_e - sometimes in $< 50\mu s$.

Associated with an $m=1 \ n=1 \ \delta_B$ perturbation

$$\text{i.e. } \delta_B = \hat{\delta_B}(r,t) e^{i(\theta+\phi)}$$

$$\xi_r \uparrow \xi = \left\{ \xi_r(r,t) \cos(\theta + \phi) \hat{r} + \xi_\theta(r,t) \sin(\theta + \phi) \hat{\theta} \right\}$$

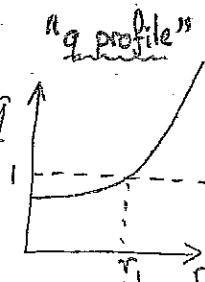


- Shift and tilt of toroidal surfaces. can vary with radius.

- (iii) $q(r)=1$ surface plays an important role.

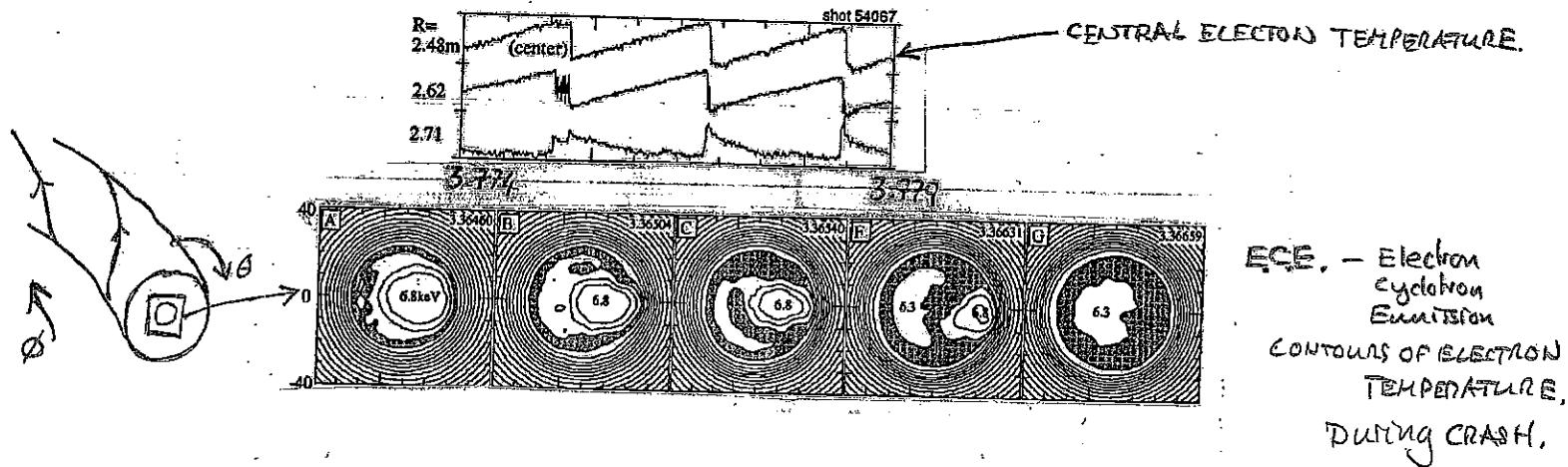
$$q = \frac{T B_T}{R B_p} \quad (\text{in cylinder}) \quad q = q(r)$$

= number of times a field line goes around in the toroidal direction each time it goes around once in the poloidal (θ) direction.



22a. Lecture #16. Magnetic Reconnection.

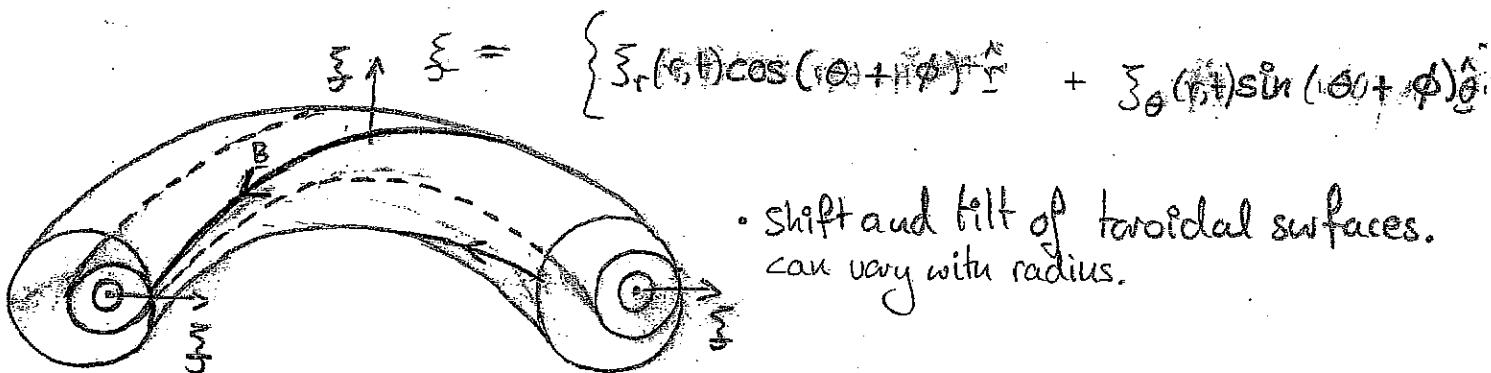
- (i) The sawtooth oscillations in a tokamak are periodic drops in the central temperature.



- (ii) The "crash" is the steep drop in T_e - sometimes in $< 50\mu s$.

Associated with an $m=1 \ n=1 \ \delta B$ perturbation

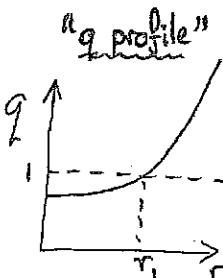
$$\text{i.e. } \delta B = \delta B(r,t) e^{i(\theta + \phi)}$$



- (iii) $q(r)=1$ surface plays an important role.

$$q = \frac{TB_t}{RB_p} \quad (\text{in cylinder}) \quad q = q(r)$$

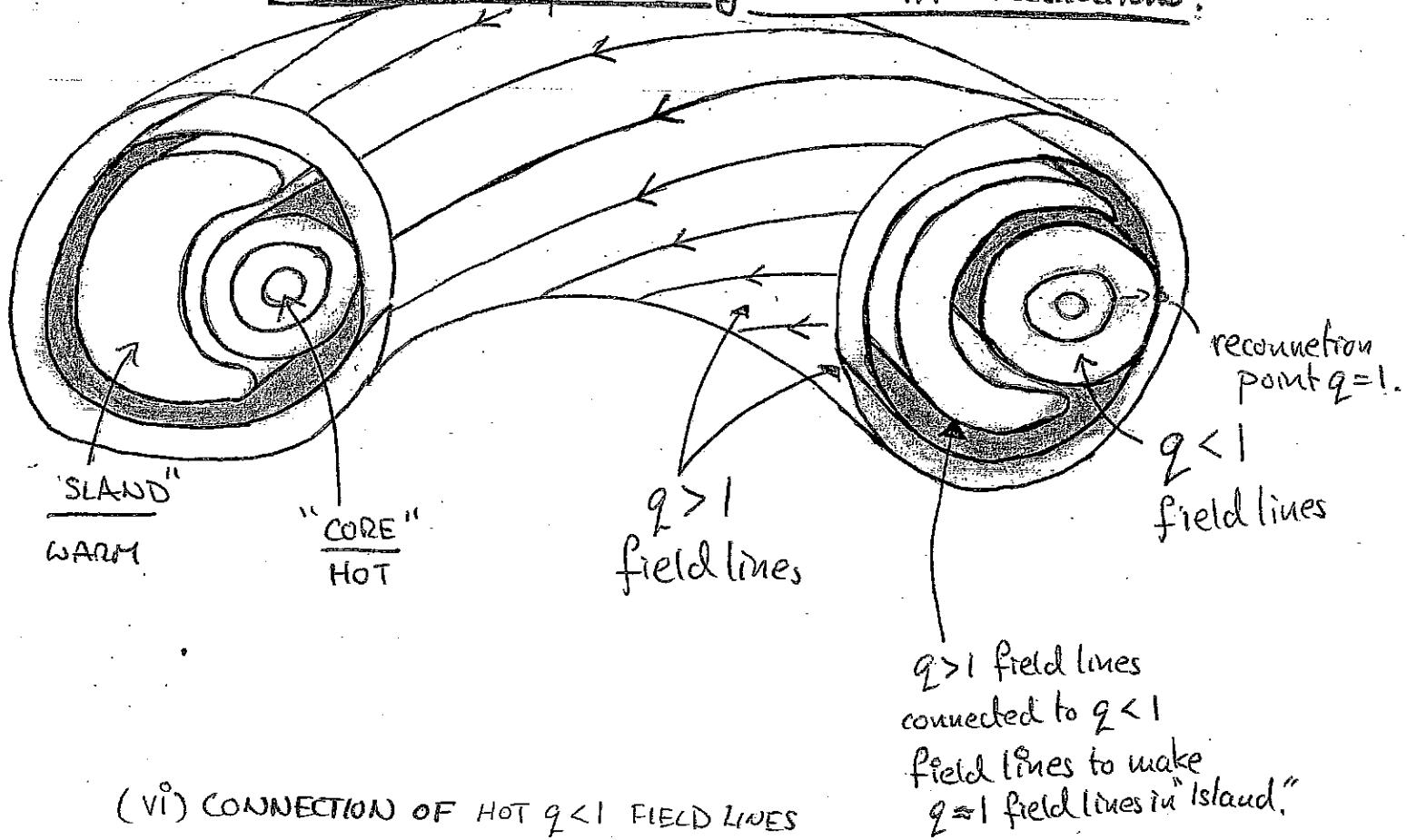
= number of times a field line goes around in the toroidal direction each time it goes around once in the poloidal (θ) direction.



(iv) Perturbation resonates with $q=1$ field lines.

$q(r_i) = 1$ defines the "rational surface".

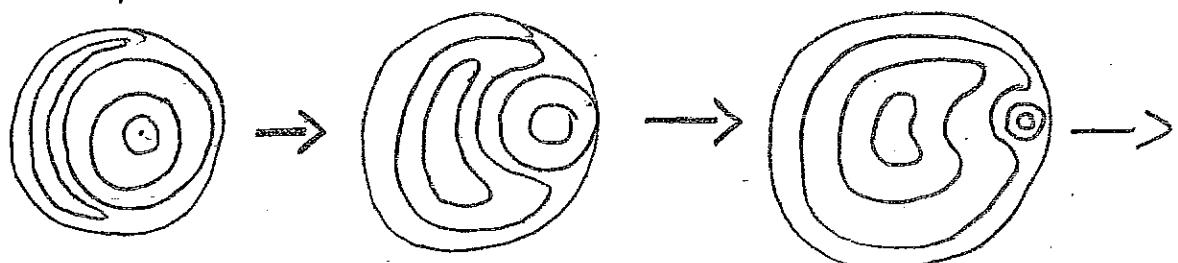
(v) Kadomtsev's picture of sawtooth oscillations.



(vi) CONNECTION OF HOT $q < 1$ FIELD LINES

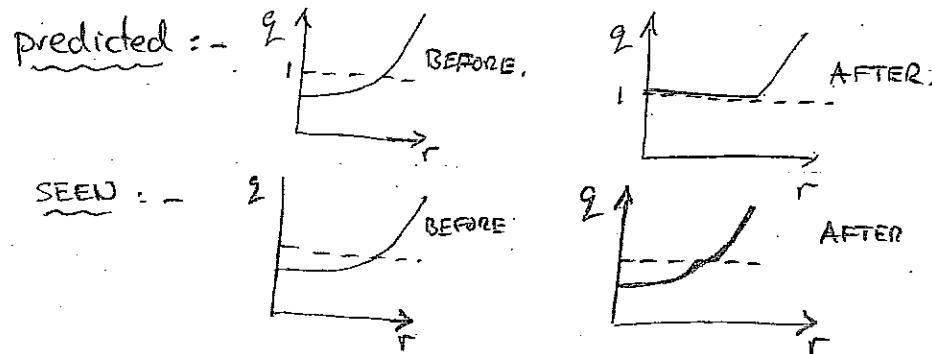
TO COLDER $q > 1$ FIELD LINES MAKES WARM
 $q \approx 1$ FIELD LINES.

(vii) In Kadomtsev's picture the core keeps moving over until all the core field lines have reconnected and the hot plasma has disappeared and island becomes the core



(vii) The problems with Kadomtsev picture became apparent in the mid 1990s. Two observations that don't agree with his picture are:

- (i) CRASH TIME IS TOO SHORT. (reconnection is slow - perhaps not?)
- (ii) Measured q profile before and after crash is similar - Kadomtsev



(viii) It is now believed that there is some reconnection but it doesn't go all the way. Some kind of secondary instability causes rapid loss of heat from core region.

(ix) This has ~~not~~ been a lead in to study reconnection.
How do field lines break and reconnect?

We know that in ideal MHD field lines frozen to the plasma and therefore can't reconnect.

$$\text{i.e. } E + v \times B = 0 \Rightarrow \frac{\partial B}{\partial t} = \nabla \times (v \times B) \rightarrow \begin{array}{l} \text{FLUX FREEZING,} \\ \text{FROZEN IN LINES} \\ \text{NO RECONNECTION.} \end{array}$$

WE NEED SOME EFFECT TO BREAK THE CONSTRAINT - THE SIMPLEST IS RESISTIVITY

$$E + v \times B = \eta J$$

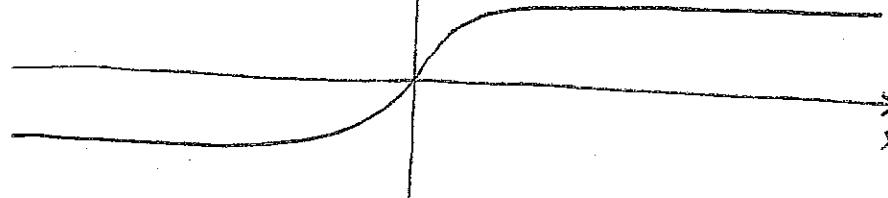
resistive MHD.

Tearing Mode.

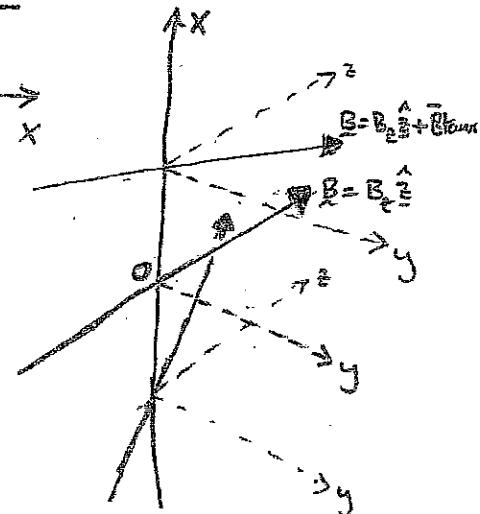
(i) Stability of a current carrying sheet to resistive "tearing modes"

Equilibrium Field: $\underline{B} = B_z \hat{z} + B_y(x) \hat{y}$

For Example: $B_y(x) = \bar{B} \tanh\left(\frac{x}{L}\right)$



SHEARED FIELD: $B_y(0) = 0$



(ii) Perturbation. We choose.

$$\delta \underline{B} = \nabla \delta \psi \times \hat{z} : \quad \delta \psi = \text{FLUX FUNCTION}$$

$$\delta \underline{V} = \nabla \phi \times \hat{z} : \quad \nabla \cdot \underline{V} = 0$$

(iii) CHOOSE 2D PERTURBATION:

$$\delta \psi = \delta \hat{\psi}(x) \cos ky e^{i\theta t} \quad \phi = \frac{\hat{\psi}}{k} e^{i\theta t}$$

\propto GROWTH RATE

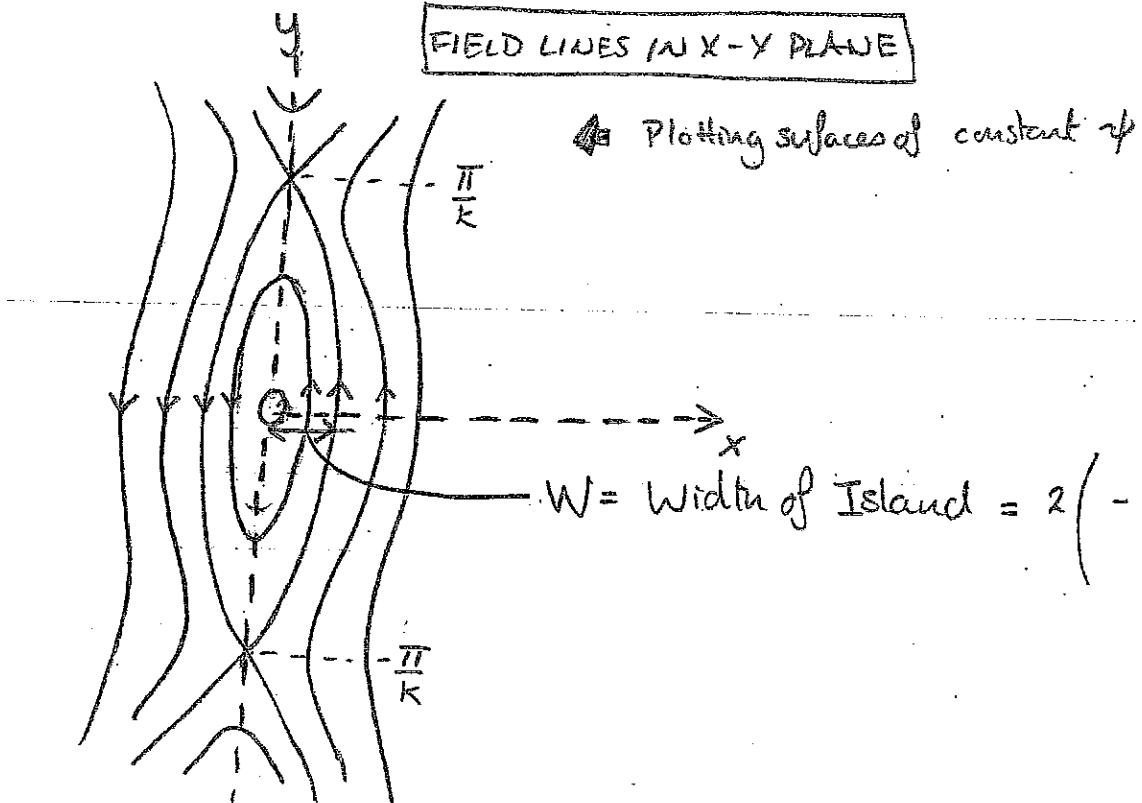
Near $x=0$ $B_y \sim B'_y x \quad \delta \hat{\psi} \sim \delta \psi(0)$

$$\underline{B} \approx \nabla \psi \times \hat{z} + B_z \hat{z}$$

$$\psi \approx B'_y \frac{x^2}{2} + \delta \psi(0) \cos ky.$$

Since $\underline{B} \cdot \nabla \psi = 0$ \underline{B} lies on ψ surfaces.

FIELD LINES IN X-Y PLANE



Plotting surfaces of constant ψ

$$W = \text{Width of Island} = 2 \left(-\frac{\delta\psi(0)}{B'_y} \right)^{1/2}$$

(iv) CALCULATING THE GROWTH RATE:

We substitute δB and δv into the ^{resistive} MHD equations.

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta \nabla^2 B \Rightarrow \delta B - B_y \hat{\phi} = \frac{\eta}{\gamma} \left[\frac{d^2 \delta v}{dx^2} - k^2 \delta \psi \right]$$

$$\rho \frac{dv}{dt} = -\nabla p + J \times B \Rightarrow -\frac{\gamma^2}{k} \left[\frac{d^2 \hat{\phi}}{dx^2} - k^2 \hat{\phi} \right] = B_y(x) \left[\frac{d^2 \delta \psi}{dx^2} - k^2 \delta \psi \right] - \frac{d^2 B_y}{dx^2} \delta \psi$$

Boundary conditions $\delta \psi, \hat{\phi} \rightarrow 0$ as $|x| \rightarrow \infty$.

The mathematical details can be horible so I will try to summarize without too much detail.

(v) ANSATZ 2 / GUESS a) GROWTH TIME ($\frac{1}{\delta}$) LONG COMPARED TO ALFVEN TIME

b) GROWTH TIME ($\frac{1}{\delta}$) SHORT COMPARED TO RESISTIVE TIME τ_r

$$\left(\tau_A \sim \sqrt{\frac{f_0}{B'_y}} \right) \ll \frac{1}{\delta} \ll \left(\frac{1}{R\eta} \approx \tau_r \sim \frac{L^2}{\eta} \right)$$

(vi) With this assumption we have two regions

$x \sim L$: OUTER REGION RESISTIVITY AND INERTIA UNIMPORTANT
Simplified Equations.

$$\delta\psi = B_y \phi$$

$$B_y \left(\frac{d^3\psi}{dx^3} - k^2 \delta\psi \right) - \frac{d^2 B_y \delta\psi}{dx^2} = 0 \quad \left. \begin{array}{l} \\ \text{no } \gamma \\ \text{in there.} \end{array} \right\}$$

OUTER SOLUTION $\delta\psi_{\text{out}}(x)$

$x \sim \delta \ll L$: INNER REGION $B_y \sim B'_y x$

$$\delta = \left(\frac{\delta \rho \eta}{B_y'^2 k^2} \right)^{1/4}$$

$$\delta\psi - B'_y x \hat{\phi} = \frac{1}{8} \frac{d^2 \delta\psi}{dx^2}$$

$$\frac{\delta^2 \rho d^2 \hat{\phi}}{k^2 dx^2} = - B'_y x \frac{d^2 \delta\psi}{dx^2}$$

(vii) MATCH THE SOLUTIONS IN A REGION

$\boxed{\delta \ll x \ll L}$

DEFINE

$$\Delta' = \frac{\frac{d\delta\psi_{\text{out}}}{dx} \Big|_0 - \frac{d\delta\psi_{\text{out}}}{dx} \Big|_{-\delta}}{\delta\psi_{\text{out}}(0)}$$

"DELTA PRIME"

After a lot of work we get:

$\boxed{\gamma > 0 \text{ if } \Delta' > 0}$

STABILITY CRITERION.

$$\boxed{\gamma = c \left(\frac{\Delta'}{k} \right)^{4/5} r_R^{-3/5} r_A^{-2/5}}$$

$$c = \left(\frac{\Gamma(11/4)}{\pi \Gamma(3/4)} \right)^{4/5}$$

CONSTANT

NOT IMPORTANT TO REMEMBER THE CALCULATION, MUST REMEMBER.

- a) Resistance important only in narrow layer b) Growth only if $\Delta' > 0$.
- c) Growth faster than resistive slower than Alfvénic.

