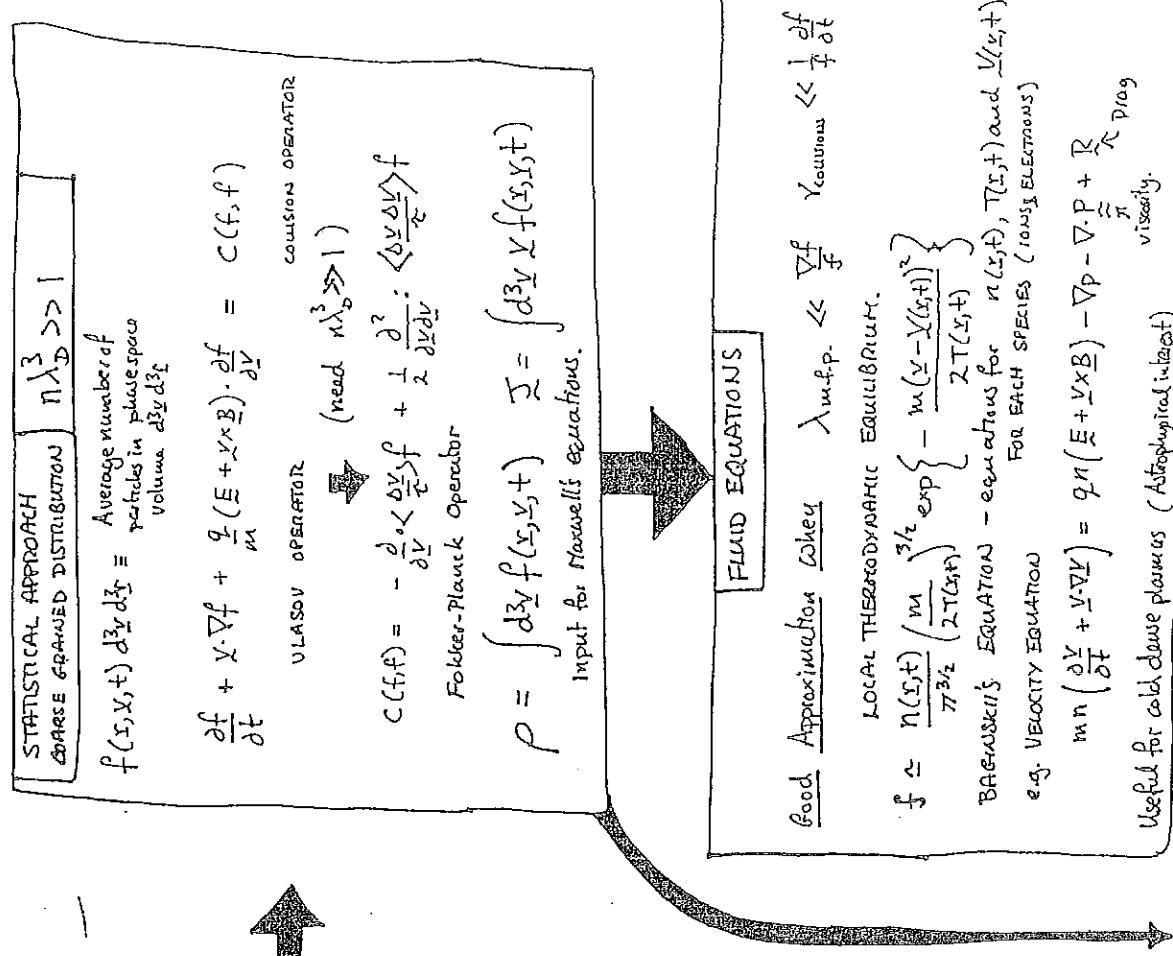
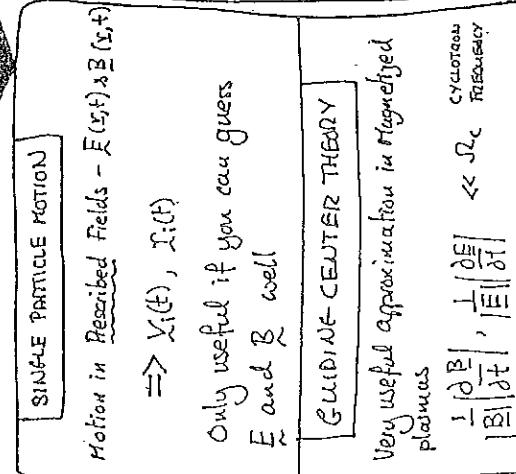
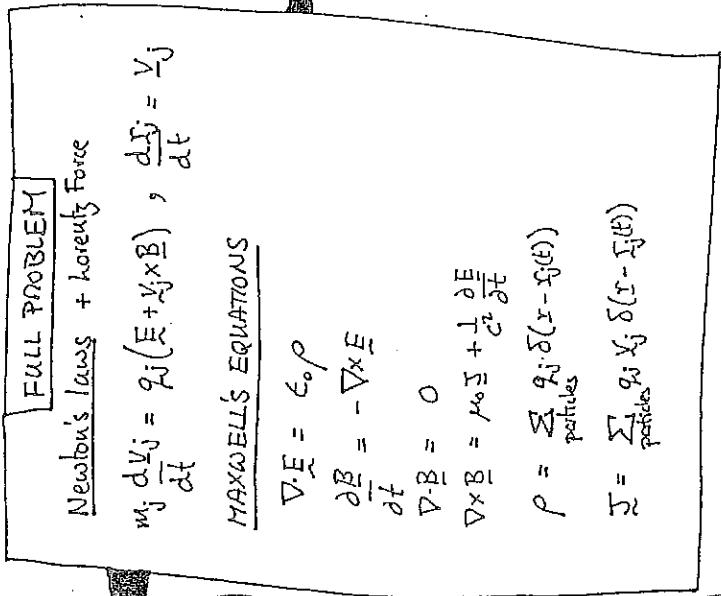


## PLASMA $\mu_F$ - :: a map. ]



## EQUILIBRIUM

Only nontrivial when highly magnetized.

Relaxation along field is so fast and perpendicular relaxation so slow that we usually establish isotropic pressure thus

$$\nabla p = \mathbf{J} \times \mathbf{B} + \rho g$$

even when MHD is not valid, i.e.  $\lambda_{mfp}$  is large.

## WAVES AND INSTABILITIES

Small deviations/excitation from equilibrium

### WAVES

Stable excitations

#### COLD PLASMA WAVES

Usually fastest appears  
needs  $V_{\text{phase}} \gg V_{\text{th}}$

#### HO TYPE - ALFVEN WAVES HOT PLASMA WAVES

$\omega \neq \Omega_c$

#### KINETIC TYPE

$\omega \gg V_c$

#### LANDAU DAMPING

- Quasi-linear theory  
for nonlinear resonant behavior.

#### wave

## MHD

Needs same conditions for fluid equations

plus: small drag - resistance

$$\frac{\partial \mathbf{B}}{\partial t} \gg \eta \nabla^2 \mathbf{B} \sim \frac{\eta}{L^2} \mathbf{B} \quad \eta = \text{resistance} \propto \frac{m_e}{q^2 n}$$

very small in hot plasma.

## FLUX IN FLUX

### Electron Eqn.

$$\mathbf{E} + \mathbf{v}_e \times \mathbf{B} = \frac{\partial \mathbf{J}_e}{\partial t} - \frac{\nabla p_e}{m_e} - \frac{ne}{\epsilon} \frac{\partial \mathbf{E}}{\partial t} + \nu_{pe} \mathbf{E}$$

small mass

$\nabla \cdot \mathbf{B}_e = 0$

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$\nabla \cdot \math$

## 222a. Plasma Physics. Lecture #1. What is a Plasma?

Lecture notes will be provided - errors in these notes are inevitable please accept a blanket disclaimer. They are intended to convey my thoughts.

- (i) DEFINITION OF A PLASMA -      a) Gas that is significantly ionized  
no good definition!
- b) Ionized gas that displays collective behaviour. WISHLY WASHY
- (ii) Plasma is very common in nature - more common than all other forms except "dark matter".

**STELLAR INTERIORS**

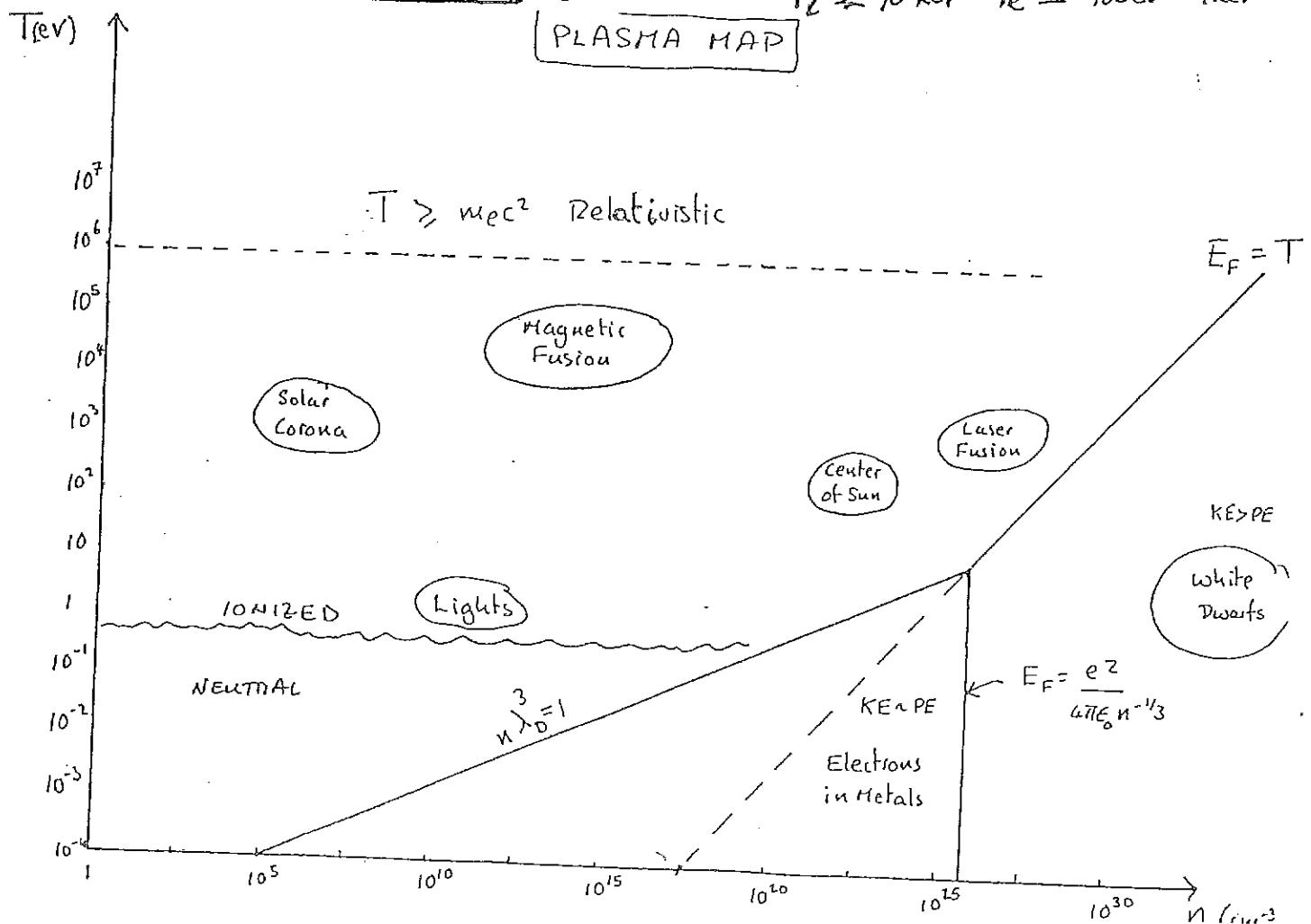
$$n_e = \# \text{ of electrons per cm}^3 \sim 10^{23-24}$$

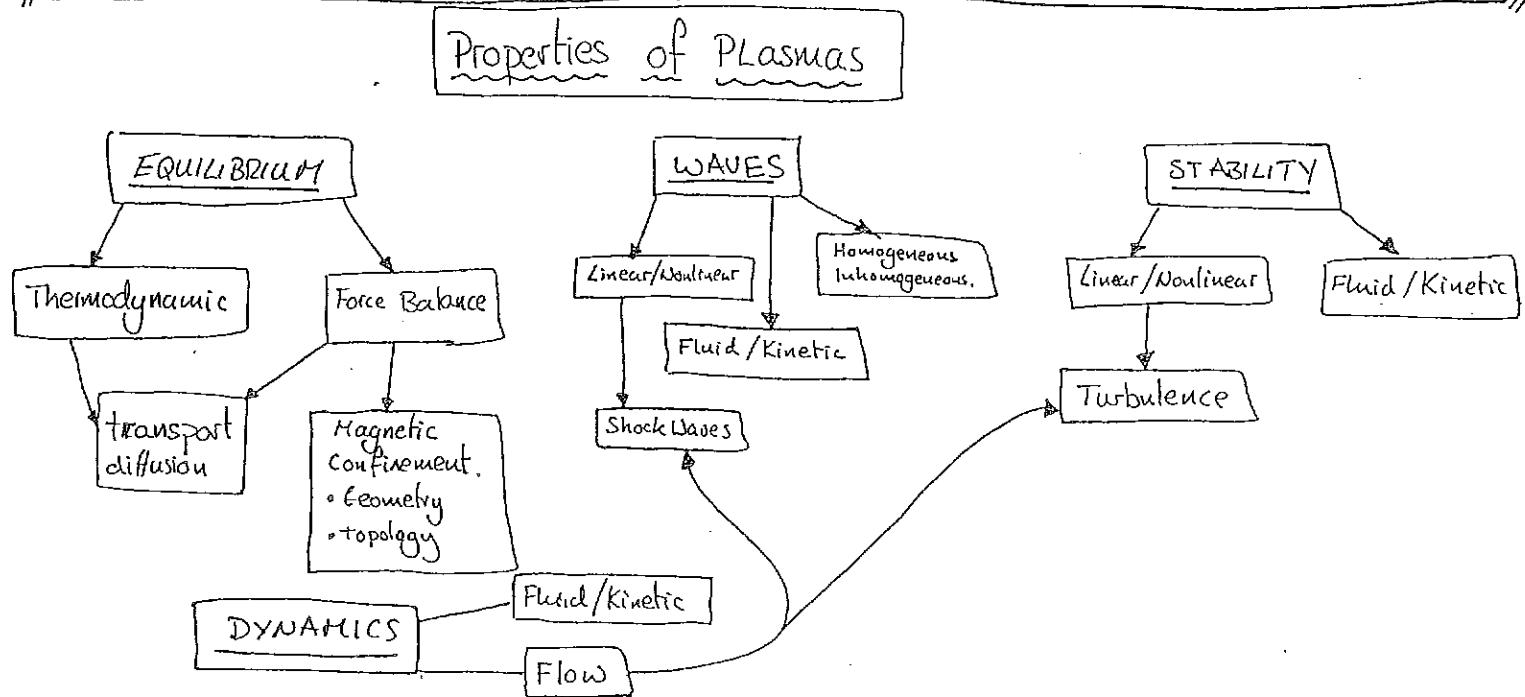
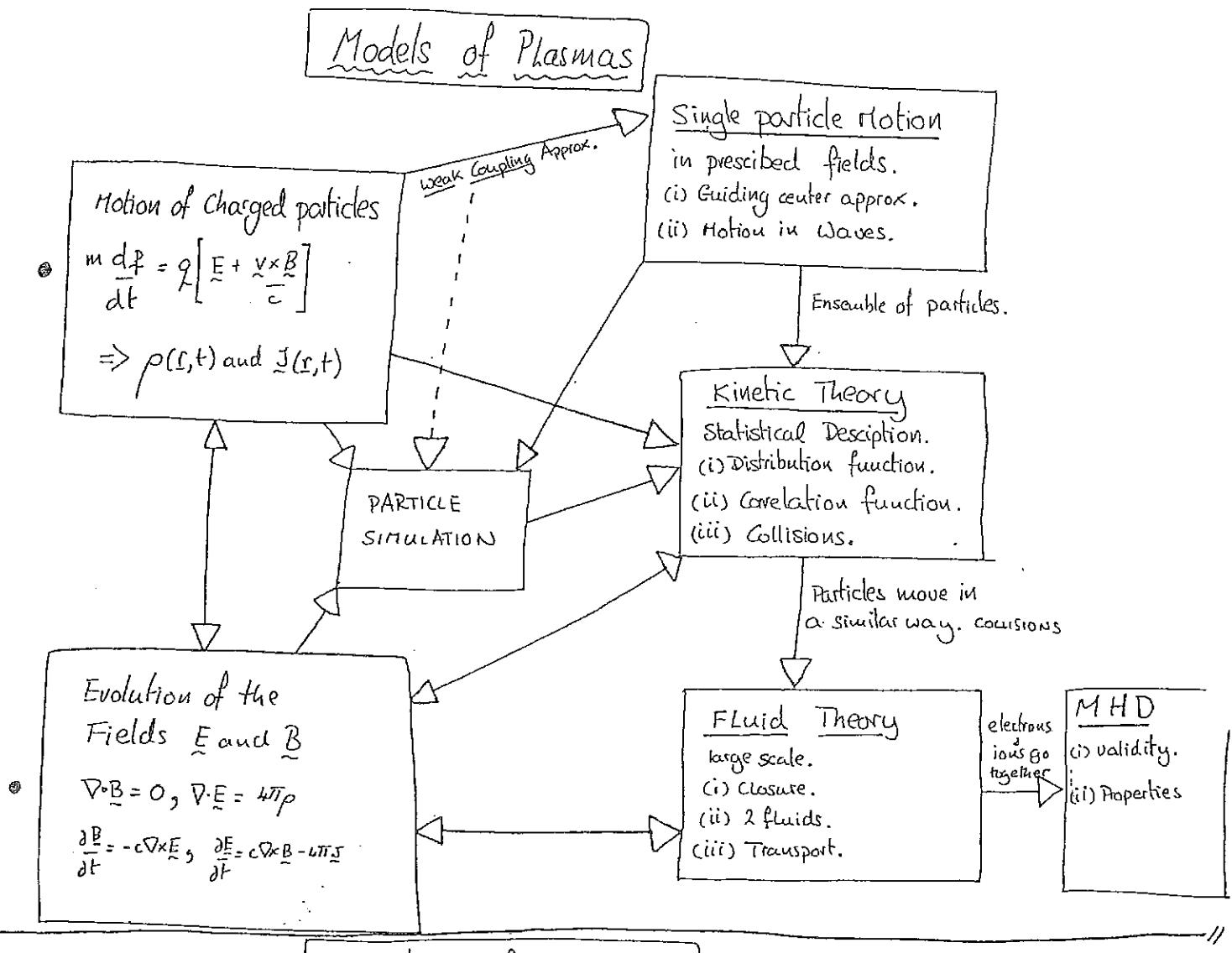
$$\text{electron temperature} \approx T_e \approx 100 \text{ eV} \quad (1 \text{ eV} \approx 1.16 \times 10^4 \text{ Kelvin})$$

**EARTH'S MAGNETOSPHERE**

$$n_e = 1 \text{ cm}^{-3} \quad T_i \gtrsim 10 \text{ keV} \quad T_e \approx 100 \text{ eV} - 1 \text{ keV}$$

**PLASMA MAP**





(vi) Imagine placing a <sup>hydrogen</sup> ion in a plasma made up of smoothly distributed charge (neutral). In thermodynamic equilibrium:-

$$n_e = \text{# density of electrons} = n_0 e^{-\frac{E}{T}} = n_0 e^{\frac{e\phi}{T}}$$

$\phi$  = electrostatic potential.

$$n_i = n_0 e^{-\frac{e\phi}{T}}$$

Putting these formulae into POISSON'S EQUATION we get

$$\nabla^2 \phi = \frac{1}{r} \frac{d^2(r\phi)}{dr^2} = -4\pi n_0 e \left\{ e^{-\frac{e\phi}{T}} - e^{\frac{e\phi}{T}} \right\} - 4\pi e \delta(\Sigma)$$

spherical symmetry

↑  
charge density of  
hydrogen ion.

If  $\frac{e\phi}{T} \ll 1$  we can expand the exponentials.

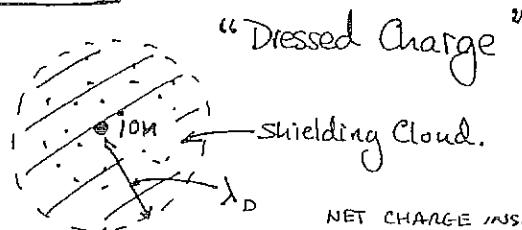
$$\frac{1}{r} \frac{d^2(r\phi)}{dr^2} = \frac{4\pi n_0 e^2}{T} \phi - 4\pi e \delta(\Sigma)$$

(vii)

$$\rightarrow \boxed{\phi(r) = \frac{e}{r} e^{-\frac{\sqrt{2}r}{\lambda_D}}}$$

shielded Coulomb field.

$$\boxed{\lambda_D = \sqrt{\frac{T}{4\pi n_0 e^2}}}$$



NET CHARGE INSIDE  $\lambda_D = 0$ .

(viii) Taking the charges as smoothed out we need many particles in a Debye sphere:-

$$\frac{4\pi}{3} \lambda_D^3 n = g \equiv \text{PLASMA PARAMETER}$$

$$\propto \frac{T^{3/2}}{n v_z}$$

# of electrons in a Debye sphere.

- (ix) WHEN:  $g \gg 1$  Ideal plasma - "Weak Coupling" - THIS COURSE FROM NOW ON.  
 $g \lesssim 1$  "Strong Coupling" - Stellar interiors, etc. Very dense plasmas  
 solids have  $g \sim 1$ .

(x) We should first check the assumption  $\frac{e\phi}{T} \ll 1$

$$\frac{e\phi}{T} \sim \frac{e^2}{T\lambda_D} \sim \frac{4\pi n e^2}{4\pi T\lambda_D n} \sim \frac{1}{4\pi\lambda_D^3 n} \sim \frac{1}{3g} \ll 1$$

(xi) It is also instructive to see how  $g$  measures the ratio of kinetic to potential energy

$$\langle KE \rangle = \text{Av. kinetic energy per. ion} = \frac{3}{2}T$$

$$\langle PE \rangle = \text{Av. potential energy per. ion} = \int \rho \phi d^3r$$

$$\rho \sim \frac{\phi}{\lambda_D^2}$$

$$\phi \approx \frac{ee^{-\frac{r}{\lambda_D}}}{r}$$

$$\frac{\langle KE \rangle}{\langle PE \rangle} \approx g \gg 1$$

(xii) Since the potential energy is small the particles move largely unimpeded  
 - collisions are rare.

(xiii) Closest particles at a distance  $n^{-1/3}$  roughly - most of the PE comes from particles at distances bigger than  $n^{-1/3}$  and less than  $\lambda_D$ . Smoothed field of many particles dominates not DISCREET PARTICLES. STATISTICALLY CORRECT

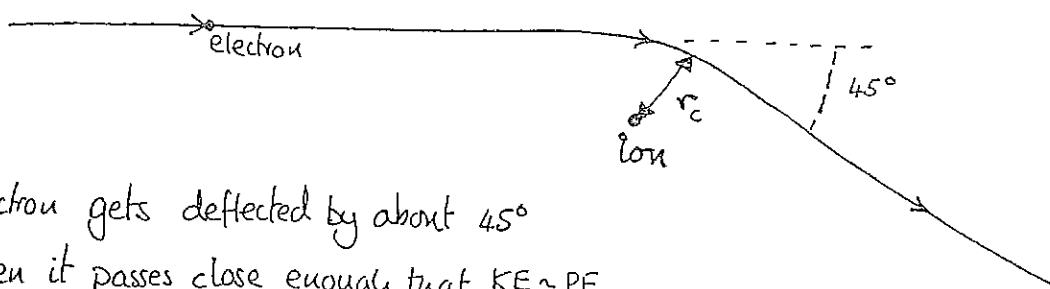
NOTE THOUGH:  $\frac{e^2}{n^{1/3}} \sim \text{PE of interaction with closest particle} \gg \frac{e^2}{\lambda_D}$

STATISTICALLY THIS PE IS CANCELLED BY OPPOSITE CHARGES TO A GOOD APPROX.

(xiv) Particles move independently - they don't to a good approximation drag other particles with them.

(X V) COLLISIONS

Only rough idea here the formal theory in the winter.

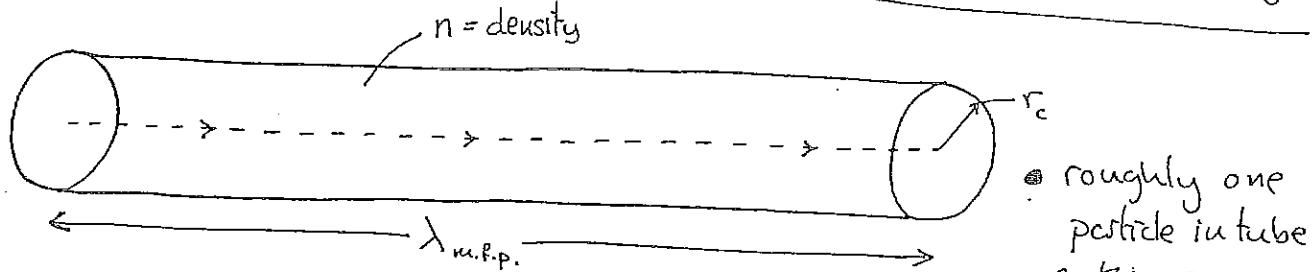


Electron gets deflected by about 45°  
when it passes close enough that  $KE \sim PE$

$$\rightarrow \frac{3}{2} T \approx \frac{e^2}{r_c}$$

$$\boxed{\text{CROSS SECTION}} = \pi r_c^2 \sim \frac{4\pi}{9} \frac{e^4}{T^2}$$

$$\boxed{\text{MEAN FREE PATH}} \quad \lambda_{m.f.p.} \equiv \text{distance between collisions.} \quad = \quad \sim \frac{1}{\lambda_D^4 n^2} \sim \frac{\lambda_D^2}{g^2}$$



• roughly one particle in tube  
so there is one 45° collision.

$$\pi n r_c^2 \lambda_{m.f.p.} = 1$$

$$\lambda_{m.f.p.} = \frac{1}{n \pi r_c^2} \sim 38 \pi n \lambda_D^4 \sim 9 g \lambda_D$$

$$\boxed{\frac{\lambda_{m.f.p.}}{\lambda_D} \sim 9g \gg 1.}$$

Goes along way before colliding

- (XVI) Fields due to smoothed currents and charges, are large enough to bend particle trajectory. We should therefore study the trajectory of SINGLE PARTICLES. in prescribed fields.

from lots of particles



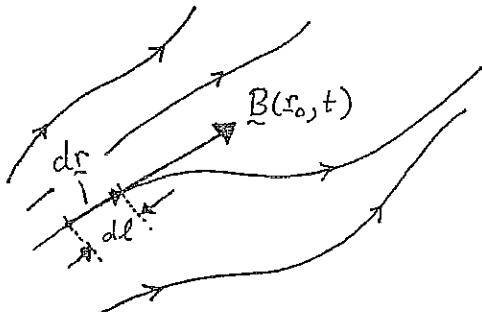
## 222a. Lecture #2. Magnetic Fields.

(i) Very many plasmas of interest are magnetized. We should understand the structure of fields in some detail.

(ii) Field lines.

$$\underline{B} = \underline{B}(r, t)$$

• Freeze  $\underline{B}$  at some instant in time



Field Line Defined by:-

$$\frac{d\underline{r}}{dl} = \frac{\underline{B}(r, t)}{B}$$

Lines tangent to  $\underline{B}$  everywhere - one problem  
Spot where  $B = 0$  - magnetic nulls.

- since particles (electrons & ions) tend to follow field lines their structure is important.

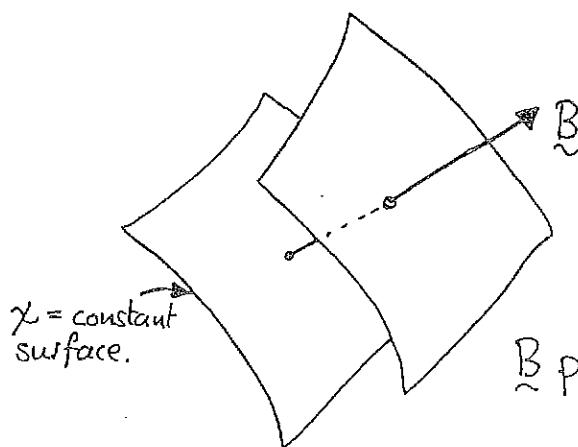
References: Morozov & Soloviev, "Reviews Plasma Physics" Vol. 1. Russian Series on Plasmas.

(iii) In many situations we can represent the field in terms of scalar "potentials" of one sort or another. Obviously the most well known but least useful is the vacuum scalar potential (Assume  $\frac{1}{c} \frac{\partial \underline{E}}{\partial t} = 0$ )

VACUUM FIELD

$$\text{If } \frac{4\pi}{c} \underline{j} = \nabla \times \underline{B} = 0 \text{ then}$$

$$\underline{B} = -\nabla \chi$$

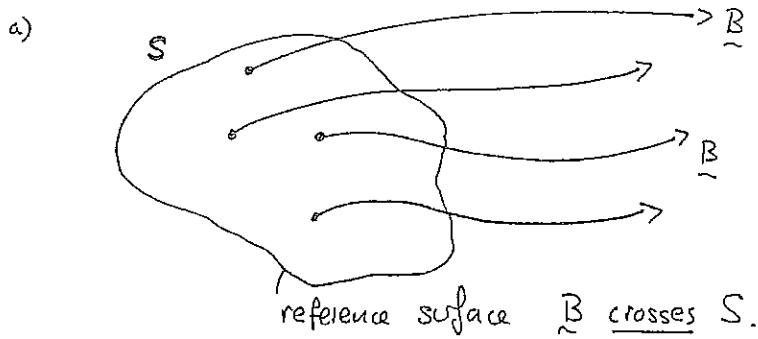


$\underline{B}$  perpendicular to  $\chi$  surfaces.

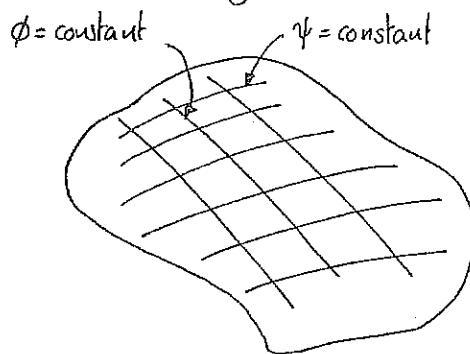
$$\nabla \cdot \underline{B} = 0 \Rightarrow$$

$$\nabla^2 \chi = 0$$

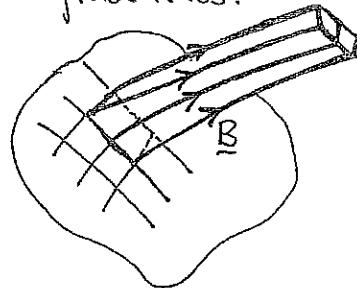
Laplace's Equation.

(iv) CLEBSCH POTENTIALS. General Field.

b) Construct any coordinate system on S i.e. label each point on S with  $\psi$  and  $\phi$



c) Continue  $\psi$  and  $\phi$  off S by making them constant along field lines.



$$\stackrel{\approx}{=} \underline{B} \cdot \nabla \psi = 0$$

and

$$\underline{B} \cdot \nabla \phi = 0$$

so  $\underline{B}$  is perpendicular to  $\nabla \phi$  and  $\nabla \psi$

$$\Rightarrow \underline{B} = a(r) (\nabla \psi \times \nabla \phi)$$

a) Now use condition that  $\nabla \cdot \underline{B} = 0$

$$\nabla \cdot (a \nabla \psi \times \nabla \phi) = \nabla \psi \times \nabla \phi \cdot \nabla a$$

$$\Rightarrow \underline{B} \cdot \nabla a = 0$$

$$\Rightarrow a = a(\psi, \phi)$$

DEFINE

$$\alpha = \int^{\psi} a(\psi', \phi) d\psi'$$

$$\beta = \phi$$

$$\nabla \alpha = a \nabla \psi + \left( \int^{\psi} \frac{\partial a}{\partial \phi} d\psi' \right) \nabla \phi$$

$$\Rightarrow \underline{B} = \nabla \alpha \times \nabla \beta$$

CLEBSCH  
REPRESENTATION.

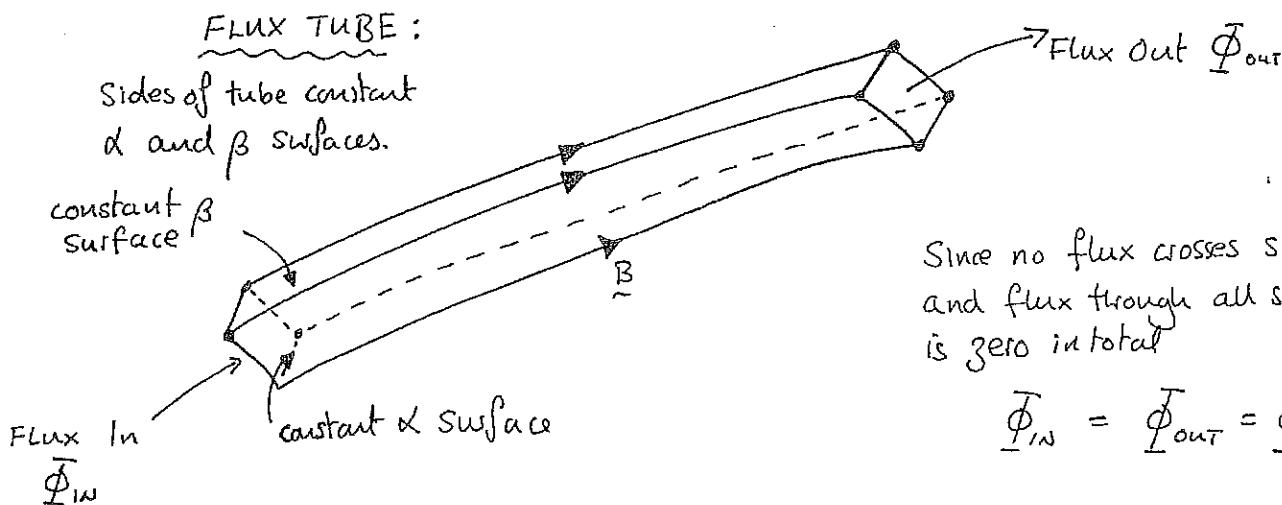
- (v) a) Note Field Lines are then solutions of  $\alpha(r) = \text{constant}$   $\beta(r) = \text{constant}$ .
- b) We can define  $\alpha \& \beta$  locally but they may become multi-valued globally - e.g. if a field line crosses  $S$  many times then it becomes multi-valued (several  $\psi, \phi$  values for same line).
- c) Since  $\nabla \times \alpha \nabla \beta = \nabla \alpha \times \nabla \beta = \underline{B} = \nabla \times \underline{A}$

$$\rightarrow \boxed{\text{VECTOR POTENTIAL}} \quad \boxed{\underline{A} = \alpha \nabla \beta + \nabla \lambda} \quad \lambda = \text{any scalar.}$$

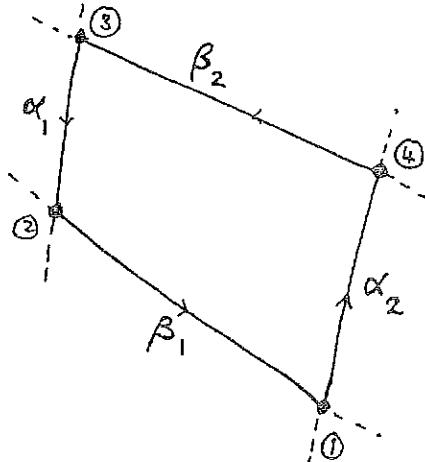
(vi) MAGNETIC FLUX  $\Phi = \int_S \underline{B} \cdot d\underline{s}$  DEFINITION.

$$\text{OVER CLOSED SURFACE } S_c \quad \oint_{S_c} \underline{B} \cdot d\underline{s} = \int_V \nabla \cdot \underline{B} dV = 0$$

EVALUATE AT FIXED TIME



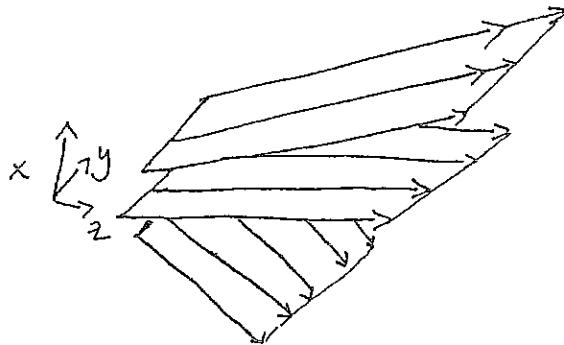
CALCULATING FLUX THROUGH THE END



$$\begin{aligned} \Phi &= \int_S \underline{B} \cdot d\underline{s} = \int_S \nabla \times (\alpha \nabla \beta) \cdot d\underline{s} = \oint_L \alpha \nabla \beta \cdot d\underline{s} \\ &= \int_1 \alpha \nabla \beta \cdot d\underline{s} + \int_2 \alpha \nabla \beta \cdot d\underline{s} + \int_3 \alpha \nabla \beta \cdot d\underline{s} + \int_4 \alpha \nabla \beta \cdot d\underline{s} \\ &\quad \alpha_2(\beta_2 - \beta_1) \quad " \quad 0 \quad " \quad -\alpha_1(\beta_2 - \beta_1) \quad " \\ \int_S \underline{B} \cdot d\underline{s} &= (\alpha_2 - \alpha_1)(\beta_2 - \beta_1) = \Delta \alpha \Delta \beta \end{aligned}$$

so  $d \& R$  /hol pl...

Examples: Sheared Slab  $\tilde{B} = B_0 \left( \hat{z} + \frac{x}{l_s} \hat{y} \right)$   $l_s$  = shear length.



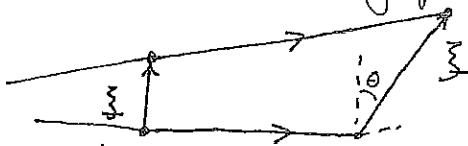
since  $\tilde{B} \cdot \nabla x = 0$  choose  $\beta = x$

$$\tilde{B} = \nabla \alpha \times \nabla x \Rightarrow \begin{cases} \frac{\partial \alpha}{\partial y} = -B_0 \\ \frac{\partial \alpha}{\partial z} = \frac{B_0 x}{l_s} \end{cases}$$

$$\Rightarrow \alpha = -B_0 \left( y - \frac{xz}{l_s} \right)$$

**SHEAR**

Twisting of vector joining two field lines as we move along the lines

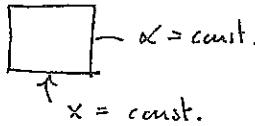


OR Twisting of  $\tilde{B}$  as we move in  $x$ .

in  $\nabla \alpha$  direction

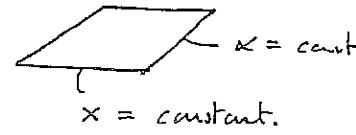
CROSS SECTION OF FLUX TUBE.

$z=0$



$$\cos \theta = \frac{\nabla \alpha \cdot \nabla x}{|\nabla \alpha| |\nabla x|} = -\frac{z}{l_s} \frac{1}{\sqrt{1 + \frac{x^2 + z^2}{l_s^2}}}$$

$z = l_s$



## 222a. Lecture #3. Particle Motion in Constant & Nearly Constant Fields I.

(i) First motion in uniform, constant fields. NONRELATIVISTIC.  
USE VECTOR NOTATION IT HELPS LATER.

$$(ii) m \frac{d\mathbf{v}}{dt} = q \left\{ \mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c} \right\}$$

$\mathbf{B} \& \mathbf{E}$  uniform & constant.

$$(iii) \text{Introduce "}\mathbf{E} \text{ cross } \mathbf{B\text{" velocity: } } \mathbf{v}_E = c \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

not charge dependant.

$$\text{Note: } \mathbf{v}_E \frac{\mathbf{E} \times \mathbf{B}}{c} = -\mathbf{E} + \mathbf{E}_{||} \mathbf{\hat{b}} : \mathbf{E}_{||} = \mathbf{E} \cdot \mathbf{\hat{b}} \quad \mathbf{\hat{b}} = \frac{\mathbf{B}}{B} \text{ UNIT VECTOR.}$$

(iv) Define  $\underline{u}$  as velocity in  $\mathbf{E} \times \mathbf{B}$  frame.  $\mathbf{v} = \underline{u} + \mathbf{v}_E$

$$m \frac{d\underline{u}}{dt} = q \left\{ \mathbf{E}_{||} \mathbf{\hat{b}} + \underline{u} \frac{\mathbf{E} \times \mathbf{B}}{c} \right\}$$

(v) Parallel Component:  $u_{||} = \underline{u} \cdot \mathbf{\hat{b}}$

$$\frac{du_{||}}{dt} = \frac{qE_{||}}{m} \rightarrow \boxed{u_{||} = \frac{qE_{||}}{m}t + u_{||0}}$$

(vi) Perpendicular Component:  $\boxed{\underline{u}_{\perp} = \underline{u} - u_{||} \mathbf{\hat{b}}}$  perpendicular velocity.

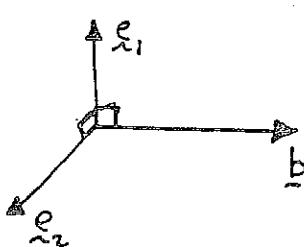
$$\frac{du_{\perp}}{dt} = \Omega (\underline{u}_{\perp} \times \mathbf{\hat{b}}) \quad \Omega = \text{gyrofrequency.}$$

or  
cyclotron frequency.

$$\Rightarrow \boxed{\underline{u}_{\perp} \cdot \frac{d\underline{u}_{\perp}}{dt} = 0} \Rightarrow |\underline{u}_{\perp}| = \text{constant in time.} = u_{\perp}$$

(vii) Define two constant unit vectors  $\underline{e}_1$  &  $\underline{e}_2$  so that

$$\underline{e}_1 \times \underline{e}_2 = \underline{b} \quad \text{right handed basis.}$$



$$\underline{e}_2 \times \underline{b} = \underline{e}_1 \text{ & } \underline{b} \times \underline{e}_1 = \underline{e}_2$$

(viii) gyro-angle  $\phi(t)$   $\underline{u}_\perp = u_\perp (\sin\phi \underline{e}_1 + \cos\phi \underline{e}_2)$

$$\frac{d\phi}{dt} = \omega \Rightarrow \boxed{\phi = \phi_0 + \omega t}$$

$$\frac{dr_\perp}{dt} = \underline{u}_\perp \Rightarrow r_\perp = r_0 + \frac{u_\perp}{\omega} \left\{ -\cos\phi \underline{e}_1 + \sin\phi \underline{e}_2 \right\}$$

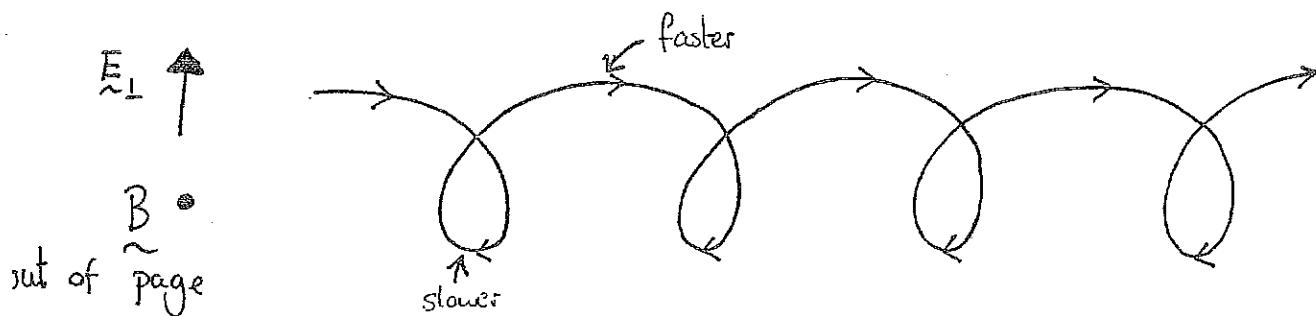
thus.

putting it  
together.

$$\boxed{\underline{r} = \underline{r}_0 + (\underline{v}_E + \underline{u}_\perp \underline{b})t + \frac{qE_\parallel t^2}{m\omega^2} \underline{b} + \rho \left\{ -\cos\phi \underline{e}_1 + \sin\phi \underline{e}_2 \right\}}$$

$$\rho = \text{Larmor Radius or gyration radius} = \frac{u_\perp}{\omega}$$

(ix) PERPENDICULAR MOTION.



$$\frac{\underline{E} \times \underline{B}}{B^2}$$

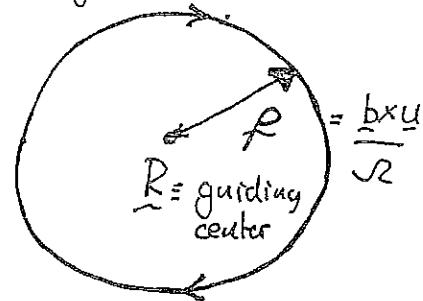
•  $E$  slows down / speeds up particle making it curve more / less in  $B$ .

• Same effect in gravitational field  $g$   $\frac{mc}{q} \frac{g \times B}{B^2}$   
except ions go one way electrons the other.

(x) Guiding Center. define:

$$\underline{r} = \underline{R} + \frac{\underline{b} \times \underline{u}}{\omega}$$

Moving frame:-



$\underline{R}$  is the center of the helix - position of the Guiding Center.

$$\frac{d\underline{R}}{dt} = \underline{v} - \frac{\underline{b}}{\omega} \times \frac{du}{dt} = \underline{v} - \underline{u}_\perp = \underline{u}_{\parallel} \underline{b} + \underline{v}_E$$

$$\boxed{\underline{R}(t) = \underline{R}_0 + (\underline{v}_E + \underline{u}_{\parallel 0} \underline{b}) t + \frac{q E_{\parallel}}{m} \frac{t^2}{2} \underline{b}}$$

(xi) In a mildly inhomogeneous field we need to describe motion perpendicular in 2 parts.

(a) Rapid oscillatory motion around the field line

(b) Secular / smooth motion across the field line

(xii) Inhomogeneity is assumed small so that

$$f \sim \rho |\nabla B| \ll |B| \quad \rho |\nabla E| \ll |E|$$

$\rho$  scale length of field.

- LOWEST ORDER LOCAL MOTION IS SAME AS HOMOGENEOUS CASE.

- HIGHER ORDER MOTION REQUIRES AVERAGING OUT LOWEST ORDER MOTION.

### Multiple Time Scale Methods:

I will show you how to use this important method via <sup>simple</sup> example first.

#### DUFFING'S EQN

$$\frac{d^2x}{dt^2} = -x + x^3$$

nonlinear oscillation.

SMALL AMPLITUDE  
 $\epsilon \ll 1$

$$x = \epsilon x_1(t, \epsilon^2 t) + \epsilon^2 x_2(t, \epsilon^2 t) + \epsilon^3 x_3(t, \epsilon^2 t) \dots$$

we introduce 2 time scales  $t$  and  $\tau = \epsilon^2 t$ .   
 short long   
 treat as separate variables.

$$\frac{d}{dt} \Rightarrow \frac{\partial}{\partial t} + \epsilon^2 \frac{\partial}{\partial \tau} \quad \frac{d^2}{dt^2} \Rightarrow \frac{\partial^2}{\partial t^2} + 2\epsilon^2 \frac{\partial^2}{\partial t \partial \tau} + \epsilon^4 \frac{\partial^2}{\partial \tau^2}$$

$\mathcal{O}(\epsilon)$

$$\frac{d^2x_1}{dt^2} = x_1 \Rightarrow x_1(t, \tau) = A(\tau) \cos t + B(\tau) \sin t$$

A & B to be found below.

$\mathcal{O}(\epsilon^2)$

$$\frac{d^2x_2}{dt^2} = x_2$$

$\mathcal{O}(\epsilon^3)$

$$\frac{d^2x_3}{dt^2} - x_3 = 2 \frac{\partial A}{\partial \tau} \sin t - 2 \frac{\partial B}{\partial \tau} \cos t - A^3 \cos^3 t - B^3 \sin^3 t - 3AB^2 \cos t \sin^2 t - 3AB^2 \sin t \cos^2 t$$

Anhilate  $x_3$  by averaging out.

using  $\int_0^{2\pi} \cos t \left[ \frac{d^2x_3}{dt^2} - x_3 \right] dt = \int_0^{2\pi} \sin t \left[ \frac{d^2x_3}{dt^2} - x_3 \right] dt = 0$

Assume  $x_3$  is periodic over oscillation.

Multiplying by  $\sin t$  and integrating we get.

$$\begin{aligned} \frac{\partial A}{\partial \tau} 2 \int_0^{2\pi} \sin^2 t dt &= A^3 \int_0^{2\pi} \cancel{\sin t \cos^3 t} dt + B^3 \int_0^{2\pi} \sin^4 t dt \\ &\quad + 3AB^2 \int_0^{2\pi} \cancel{\sin^3 t \cos t} dt + 3BA^2 \int_0^{2\pi} \cancel{\sin^2 t \cos^2 t} dt \end{aligned}$$

doing the integrals gives:

$$\frac{dA}{dt} = \frac{3}{8} \left\{ B^3 + BA^2 \right\} \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad \frac{dB}{dt} = -\frac{3}{8} \left\{ A^3 + AB^2 \right\}$$

from multiplying by const we get:-

$$\Rightarrow A^2 + B^2 = \text{constant} = \bar{x}^2$$

$$\frac{dA}{dt} = \frac{3\bar{x}^2}{8} B \quad \frac{dB}{dt} = -\frac{3\bar{x}^2}{8} A$$

$$\Rightarrow A = A_0 \sin \left( \frac{3\bar{x}^2}{8} t + \alpha_0 \right) \quad B = A_0 \cos \left( \frac{3\bar{x}^2}{8} t + \alpha_0 \right)$$

$A_0, \alpha_0$  are constants.

$$X_1 = A_0 \sin \left[ t + \frac{3\bar{x}^2}{8} e^{i\omega t} + \alpha_0 \right]$$

OVER A LONG TIME  $X_1$  ACCUMULATES A LARGE PHASE SHIFT DUE TO NONLINEAR TERM. THIS MEANS  $X_3$  NOW HAS NO SECULAR INCREASE AND IT OSCILLATES LIKE  $\sin 3t$  etc.



## 22a. Lecture #4 Particle Motion in Inhomogeneous Fields II.

Read Hazeltine & Waelbroeck Chapter 2.

- (i) We are not going to do everything at once since the algebra is horrible. Instead we look at the  $E=0$  ( $\frac{\partial B}{\partial t}=0$ ) case.

$$\frac{d\mathbf{v}}{dt} = \mathbf{v}(\mathbf{v} \times \mathbf{B}) \quad \text{---(1)} \quad \mathcal{L}(\mathbf{r}) = \frac{qB(\mathbf{r})}{mc} \quad \mathbf{b} = \frac{\mathbf{B}}{B}$$

(ii) TWO LENGTH SCALES

$$R = \text{gyroradius/Larmor radius} \approx v/\omega$$

$$L = \text{equilibrium } \underline{B} \text{ field variation length} \approx \left( \frac{|\nabla B|}{B} \right)^{-1}$$

TWO TIMESCALES

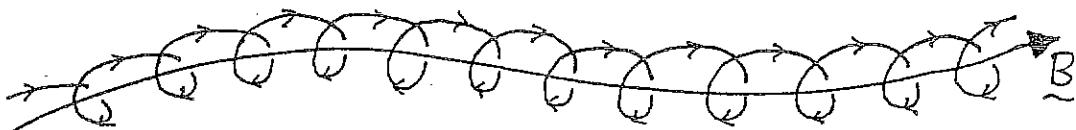
$$\tau_s = \frac{1}{\omega} = \text{Gyroperiod}$$

$$\tau_L = \frac{L}{v} = \text{time to "see } \underline{B} \text{ change".}$$

(iii) FUNDAMENTAL SMALL PARAMETER FOR GUIDING CENTER APPROXIMATION

$$\epsilon = \frac{\tau_s}{\tau_L} = \frac{v}{\omega L} = \frac{R}{L} \ll 1$$

- (iv)  $\underline{B}$  field varies little over a gyroperiod:- motion looks locally like a homogeneous field case.



- a) Small wobbling on each gyration slightly elliptical

SMALL VARIATIONS  
OF  $\underline{B}$  OVER GYROPERIOD  
HAS TWO EFFECTS



- b) long term changes because of accumulation of many small changes. DRIFTS + MIRRORING  
conservation of  $v_{\perp}^2/R$

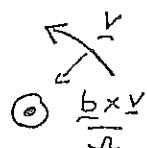
SOLVING FOR LOWEST ORDER MOTION

(v) NOTE  $\underline{v} \cdot \frac{d\underline{v}}{dt} = \frac{d}{dt}(\underline{v}^2/2) = \underline{v} \cdot \underline{v} \times \underline{b} \cdot \underline{s} = 0$

so ENERGY =  $\frac{1}{2} m v^2 = \epsilon = \text{constant of motion.}$

(vi) DEFINE LAMBERT RADIUS VECTOR BY

$$\underline{f} = \frac{\underline{b} \times \underline{v}}{\omega} \quad \longrightarrow \textcircled{2}$$



and the guiding center by

GUIDING CENTER POSITION  $\underline{R}(t) = \underline{R}_0 + f_0(t, t/\tau_L) + \epsilon \underline{f}_p(t, t/\tau_L) + \epsilon \underline{a}_p(t, t/\tau_L)$

$$\frac{\underline{f}}{R} \sim \mathcal{O}(\epsilon) \quad \frac{\epsilon \underline{f}_p}{R} \sim \mathcal{O}(\epsilon^2) \dots \text{etc.}$$

velocity:

$$\underline{v} = \frac{d\underline{R}}{dt} + \frac{df}{dt} + \epsilon \frac{d\underline{f}_p}{dt} \dots \quad \text{--- } \textcircled{4}$$

$\mathcal{O}(\frac{L}{\tau_L}) \quad \mathcal{O}(\epsilon p) \quad \mathcal{O}(\epsilon^2 p \epsilon) \quad R = R_0 + \epsilon R_1 \dots$

SAME SIZE  
ROUGHLY

(vii) LOWEST ORDER MOTION since  $f \cdot \underline{b} = 0$  we have

let  $f = f_0 \{ -\cos \theta : \underline{e}_1 + \sin \theta \underline{e}_2 \} \quad \text{--- } \textcircled{5}$

$$\begin{cases} \underline{e}_1 \times \underline{e}_2 = \underline{b} \\ \underline{b} \times \underline{e}_1 = \underline{e}_2 \\ \underline{e}_2 \times \underline{b} = \underline{e}_1 \end{cases}$$

$$\theta = \theta(t) = \mathcal{O}(\frac{1}{\epsilon}).$$

Substituting  $\textcircled{5} \rightarrow \textcircled{4} \rightarrow \textcircled{2}$  we get to lowest order

$$\frac{\underline{b}}{\omega} \times \left\{ \frac{d\underline{R}_0}{dt} + \frac{1}{\rho_0} \frac{df_0}{dt} f_0 \right\} = f_0 \left( 1 - \frac{1}{\omega} \frac{d\theta_0}{dt} \right) \quad f = f_0 + \text{H.O.T.}$$

$$\theta = \theta_{-1} + \theta_0 + \epsilon \theta_1$$

(viii) Bits that don't depend on the fast timescale must cancel out so :-

$$\text{LOWEST ORDER : } \underline{b} \times \frac{d \underline{R}_o}{dt} = 0 \Rightarrow \boxed{\frac{d \underline{R}_o}{dt} = V_{11} \underline{b}}$$

$\underline{k} \times \underline{f}$  direction:  $\Rightarrow \frac{1}{\rho_o} \frac{d \rho_o}{dt} \sim \mathcal{O}(1)$  so  $\rho = \rho(t/\epsilon_L)$  long time variation only.

$$f \text{ direction: } \Rightarrow \frac{d \theta_o}{dt} = \Omega(t/\epsilon_L) \Rightarrow \boxed{\theta_o = \int^t \Omega dt' + \theta_o.} \quad \begin{matrix} \uparrow \\ \text{very large } \mathcal{O}(\frac{1}{\epsilon}) \end{matrix}$$

(ix) NOTE:  $\underline{V}_L \approx \frac{d \rho_o}{dt} \Rightarrow \boxed{\rho_o = \frac{\underline{V}_L}{\Omega}} \quad \underline{V}_L \approx \underline{V}_L (\sin \theta \underline{e}_1 + \cos \theta \underline{e}_2)$

(x) BUT WE MUST FIND HOW  $\rho$  AND  $\underline{R}_o$  VARY ON THE LONG TIMESCALE TO COMPLETE THE LOWEST ORDER.

(xi) To find the long timescale evolution we substitute ④  $\rightarrow$  ①

$$\frac{d^2}{dt^2} \left\{ \underline{R}_o + \underline{f} + \epsilon \underline{\dot{f}} \right\} = \Omega \underline{V} \times \underline{b}$$

$$\mathcal{O}(1) \quad \mathcal{O}(\frac{1}{\epsilon}) \quad \mathcal{O}(1) \quad \mathcal{O}(\frac{1}{\epsilon})$$

$$\underline{V} \times \underline{b} \Omega = \underline{V} \cdot \nabla \left( \frac{\underline{b}}{\Omega} \right) \text{ by chain rule.}$$

$$\frac{d}{dt} \underline{f} = \frac{d}{dt} \left( \frac{\underline{b}}{\Omega} \times \underline{V} \right) = \frac{\underline{b}}{\Omega} \times \frac{d \underline{V}}{dt} + \frac{d}{dt} \left( \frac{\underline{b}}{\Omega} \right) \times \underline{V}$$

$$\frac{d^2 \underline{f}}{dt^2} = \frac{d}{dt} \left\{ \underline{b} \times (\underline{V} \times \underline{b}) + \underline{V} \cdot \nabla \left( \frac{\underline{b}}{\Omega} \right) \times \underline{V} \right\} = \Omega (\underline{V} \times \underline{b}) + \underline{V} \cdot \nabla \underline{b} \times (\underline{V} \times \underline{b}) + \underline{b} \times (\underline{V} \times \underline{V} \cdot \nabla \underline{b}) + \Omega \underline{V} \cdot \nabla \underline{b} \times (\underline{V} \times \underline{b}) + \text{H.O.T.}$$

$\mathcal{O}(\frac{1}{\epsilon})$  term cancels and we get:-

$$\frac{d^2 \underline{B}_0}{dt^2} = \frac{d}{dt} (V_{||} \underline{b}) = - \left[ 2 \sqrt{\underline{v}} \cdot \nabla \left( \frac{\underline{b}}{\underline{n}} \right) \times (\underline{v} \times \underline{b}) + \frac{\underline{b}}{\underline{n}} \times (\underline{v} \times \underline{v} \cdot \nabla (\underline{n} \underline{b})) \right. \\ \left. + 2 \underline{v} \times \underline{b} \cdot \nabla \left( \frac{\underline{b}}{\underline{n}} \right) \times \underline{v} \right] - \frac{d^2 \Delta f}{dt^2}$$

↑  
fast timescale

AVERAGE OVER  $\theta$  AND USE THAT

$$\frac{d}{dt} = \sqrt{\underline{n}} \frac{d}{d\theta} + \frac{d}{dt}$$

NOW NOTE:  $\frac{d^2 \Delta f}{dt^2} \approx \underline{n}^2 \frac{d^2 \Delta f}{d\theta^2}$  averages to zero and

↑ slow.

$$\int_0^{2\pi} d\theta \quad \underline{v} \underline{v} = \langle \underline{v} \underline{v} \rangle = V_{||}^2 \underline{b} \underline{b} + \frac{V_{\perp}^2}{2} \left( \underline{\epsilon}_{||} \underline{\epsilon}_{||} + \underline{\epsilon}_{\perp} \underline{\epsilon}_{\perp} \right)$$

$\underline{\epsilon} = \underline{b} \underline{b}$   
is UNIT TENSOR!

After a lot of algebra: (averaging)

$$\frac{d(V_{||} \underline{b})}{dt} = -V_{||}^2 \underline{b} \cdot \nabla \underline{b} - \frac{1}{2} \frac{V_{\perp}^2}{B} (\underline{b} \cdot \nabla B) \underline{b}$$

but since  $\langle \frac{d}{dt} \underline{b} \rangle = V_{||} \underline{b} \cdot \nabla \underline{b}$

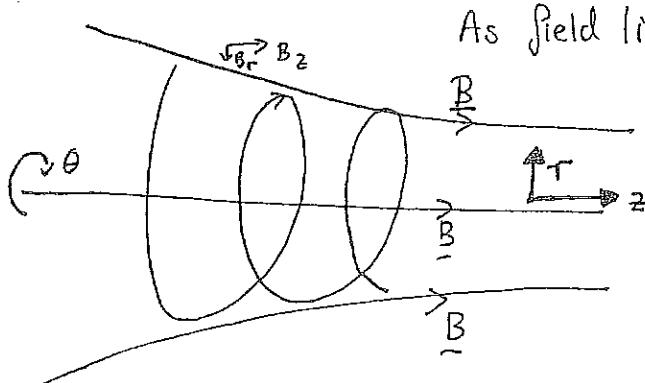
LOWEST ORDER GUIDING CENTER MOTION

$$\frac{d}{dt} V_{||} = -\frac{1}{2} \frac{V_{\perp}^2}{B} \underline{b} \cdot \nabla B$$

"MIRROR FORCE"

$$\frac{d \underline{B}_0}{dt} = V_{||} \underline{b}$$

Guiding center sticks to field line on long timescale (to lowest order)



As field lines converge particle feels average

$$+ \frac{q}{mc} V_{||} B_r = F_z \quad \text{force pushing}$$

away from high  $B$  area. This gives the mirror force.

## ADIABATIC INVARIANT.

Note  $\frac{d}{dt} \frac{V_{||}^2}{2} = -\frac{1}{2} V_{||}^2 \frac{V_{||} \mathbf{B} \cdot \nabla \mathbf{B}}{B} = -\frac{1}{2} \frac{V_{||}^2}{B} \frac{dB}{dt}$

but  $V_{||}^2 = \frac{2E}{m} - V_{\perp}^2 = v^2 - V_{\perp}^2$  and  $\frac{dE}{dt} = 0$

$$\Rightarrow \frac{d}{dt} \left( \frac{V_{||}^2}{B} \right) = 0$$

So

$$\mu = \frac{1}{2} m \frac{V_{||}^2}{B}$$

1st Adiabatic invariant

CONSTANT ON LONG TIMESCALE  
TO LOWEST ORDER.

$$V_{||} = \sqrt{\frac{2}{m} (\varepsilon - \mu B)}$$

Integrates equation for  $V_{||}$ .

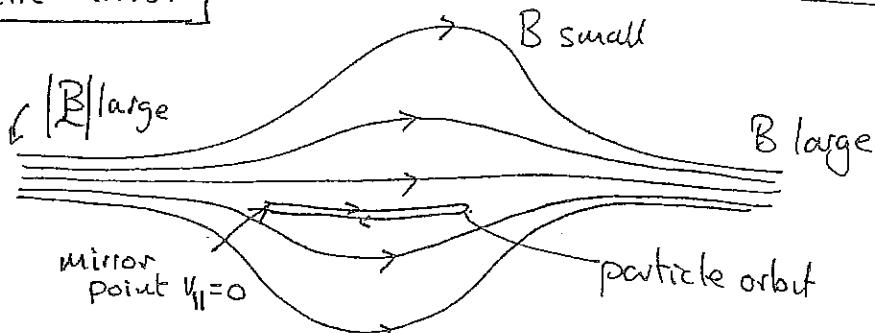
$$\mathbf{r} = \left( \frac{2\mu B}{m} \right)^{1/2} \frac{1}{\mu} (-\cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2)$$

$$\frac{dR_0}{dt} = \sqrt{\frac{2}{m} (\varepsilon - \mu B)} \pm$$

keep:  $\varepsilon$  &  $\mu$  constant on orbit.

$$B = B(r).$$

### Magnetic Mirror



- $V_{||} = 0$  at mirror points where  $\varepsilon = \mu B_{\text{mirror}}$

- $V_{||}$  real when  $B < B_{\text{mirror}}$  oscillates in region where  $B < B_{\text{mirror}}$ .

$$\left( \frac{V_{||}}{V} \right) = \left( \frac{B}{B_{\text{mirror}}} \right)^{1/2}$$

TRAPPED PARTICLES

THOSE THAT MIRROR.  $B_{\text{mirror}} < B_{\text{max}}$



## 222a. Lecture #5. Guiding Center motion. III

$$\left( \frac{dv}{dt} = \omega v \times b \right)$$

Last time we learnt that in a stationary  $B$  field, the particle motion in the GUIDING CENTER APPROXIMATION is

LOWEST ORDER.

$$r = R_0 + f + \epsilon \delta f$$

WHERE

$$\frac{dR_0}{dt} = V_{\parallel} b$$

PARALLEL MOTION  
ONLY

$$m \frac{dV_{\parallel}}{dt} = -\mu \nabla B$$

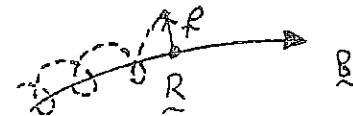
$$f = \frac{b}{\omega} \times v = \text{Larmor radius vector.}$$

$$\mu = \frac{1}{2} m \frac{v_{\perp}^2}{B} = \text{1st Adiabatic invariant.}$$



$\mu = \text{constant to this order}$

$$2 \frac{\mu B}{m} = V_{\perp}^2$$



ALSO  $\epsilon = \frac{1}{2} m v^2 = \text{constant.}$

$$\Rightarrow V_{\parallel} = \sqrt{\frac{2}{m} (\epsilon - \mu B)}$$

Guiding Center Motion.

$$f = \frac{1}{\omega} \sqrt{\frac{2\mu B}{m}} \left\{ -\sin\theta \hat{e}_1 + \cos\theta \hat{e}_2 \right\}$$

Motion of particle about guiding center.

$$\text{Gyro-Angle. } \Theta = \int_{-\infty}^t \omega(t) dt' + \Theta_0 \quad \omega(t) = \omega(r(t)) \approx \omega(R(t))$$

$$\hat{e}_1 \times \hat{e}_2 = \hat{b}, \hat{b} \times \hat{e}_1 = \hat{e}_2, \hat{e}_2 \times \hat{b} = \hat{e}_1$$

(i) In Clebsch Coordinates  $B = \nabla \alpha \times \nabla \psi$   $|B| = B(\alpha, \psi, \ell)$   $\ell = \text{distance along } B$

$$\left( \frac{dl}{dt} \right)_{\alpha, \psi} = V_{\parallel} = \sqrt{\frac{2}{m} (\epsilon - \mu B(\alpha, \psi, \ell))}$$

Integrate keeping  $\alpha, \psi, \epsilon$  and  $\mu$  constant

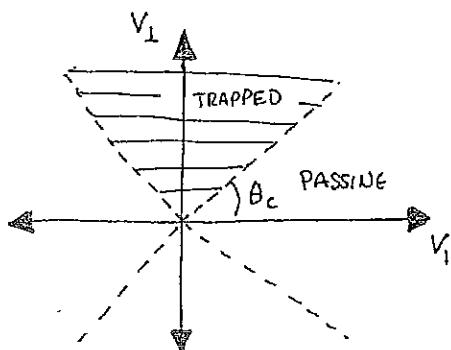
(ii) EXAMPLE:  $B = B_0 + \Delta B \sin\left(\frac{\ell}{L_b}\right)$



(iii) TRAPPED IF FOR SOME  $\ell$  —  $E - \mu B = 0$

$$B(\ell_{\text{BOUNCE}})$$

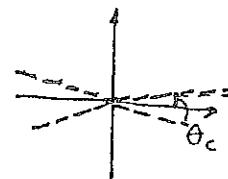
$$\frac{V_1^2}{V^2} = \frac{B(\ell)}{B(\ell_{\text{BOUNCE}})} \geq \frac{B(\ell)}{B_0 + \Delta B} = \sin^2 \theta_c(\ell)$$



- At minimum of the field  $\sin(\frac{\ell}{\ell_b}) \approx -1 + \frac{\Delta \ell^2}{\ell_b^2} \dots$

$$\sin^2 \theta_c \approx \frac{B_0 - \Delta B}{B_0 + \Delta B}$$

- If  $B_0 - \Delta B \ll B_0$  Almost all particles are trapped



- Small angle scattering will make a particle trapped near  $B \approx B_{\text{min}} = B_0 - \Delta B$ . Reduces effective mean free path.

BOUNCE MOTION: "deeply trapped" particles are trapped near  $B = B_{\text{min}}$  and they have  $\Delta \ell \ll \ell_b$  and  $v_{\parallel} \ll v_{\perp}$

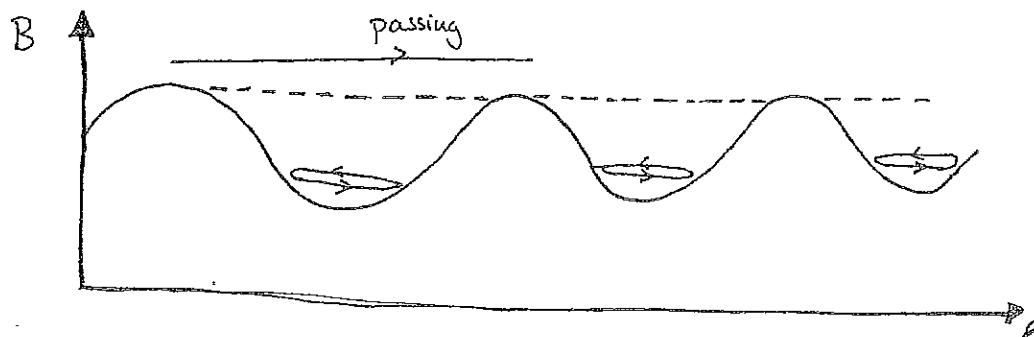
$$\frac{d\ell}{dt} \approx \sqrt{\frac{2}{m} \left[ E - \mu \left( B_0 - \Delta B + \Delta B \frac{(\Delta \ell)^2}{\ell_b^2} \right) \right]}$$

$$\Delta \ell = \Delta \ell_T \sin \omega_B t$$

$$\Delta \ell_T = \ell_b \sqrt{\frac{E - \mu B_{\text{min}}}{\mu \Delta B}}$$

BOUNCE:  
FREQUENCY

$$\omega_B = \sqrt{\frac{2\mu \Delta B}{m \ell_b}}$$



PERPENDICULAR DRIFTS: In the next order motion the particles drift off the field lines. Let's now calculate this drift.

$$\underline{v} = \frac{d}{dt} [R + f + \epsilon \delta f]$$

$$\text{using } f = \frac{\underline{b}}{\lambda} \times \underline{v}$$

$$\text{and } \frac{d\underline{v}}{dt} = \Omega \underline{v} \times \underline{b}$$

$$\frac{d f}{dt} = \frac{\underline{b} \times (\underline{v} \times \underline{b})}{\lambda} + \underline{v} \cdot \nabla \left( \frac{\underline{b}}{\lambda} \right) \times \underline{v}$$

AVERAGE OVER FAST TIME - OVER  $\theta$

$$\langle \epsilon \frac{d}{dt} \delta f \rangle = O(\epsilon^2)$$

$$\frac{dR}{dt} = V_{||} \underline{b} - \langle \underline{v} \cdot \nabla \left( \frac{\underline{b}}{\lambda} \right) \times \underline{v} \rangle \quad \text{using} \quad \langle \underline{v} \underline{v} \rangle = V_{||}^2 \underline{b} \underline{b} + \frac{V_{\perp}^2}{\lambda} \left[ \underline{I} - \underline{b} \underline{b} \right]$$

$$\frac{dR}{dt} = V_{||} \underline{b} + \frac{V_{||}^2}{\lambda} \left( \underline{b} \times \underline{b} \cdot \nabla \underline{b} \right) + \frac{V_{\perp}^2}{\lambda} \left( \underline{b} \times \frac{\nabla B}{B} \right) + \frac{V_{\perp}^2}{\lambda} \left( \underline{b} \cdot \nabla \times \underline{b} \right) \underline{b} \dots \text{H.O.T.}$$

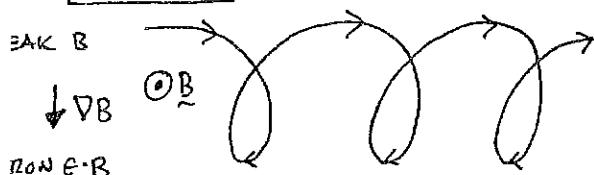
$\mathcal{O}(v)$

CURVATURE  
DRIFT  
 $\mathcal{O}(ev)$

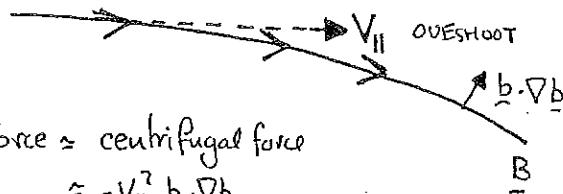
$\nabla B$  DRIFT.

CORRECTION TO  
PARALLEL MOTION.  
 $\mathcal{O}(ev)$

DB Drift



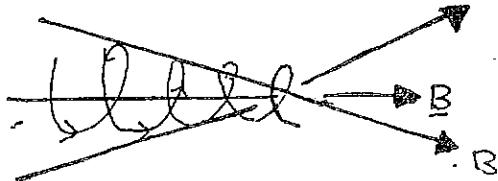
CURVATURE DRIFT



$$F_x \text{ Force} \approx \text{centrifugal force} \\ \approx -V_{||}^2 \underline{b} \cdot \nabla \underline{b}$$

CORRECTION TO PARALLEL MOTION.

Shear in magnetic field makes perpendicular motion project onto parallel direction.





## 22a. Lecture #6. Polarization Drift, Gyrokinetics.

Read: Hazeltine's Waelbroeck, Chapter 2.

- (i) Drift equations with  $E$  &  $B$  fields + time dependence: Summary no derivation

$$\underline{r} = \underline{R} + \frac{\underline{b}}{\Omega} \times \underline{v}_E + \dots \quad \text{defines guiding center position.} \quad \underline{u} = \underline{v} - \underline{U} \quad \begin{matrix} \text{guiding} \\ \text{center} \\ \text{velocity.} \end{matrix}$$

•  $\underline{U} = \frac{d\underline{R}}{dt} = U_{||} \underline{b} + \frac{c \underline{E} \times \underline{B}}{B^2} + \frac{\underline{b}}{\Omega} \times \left\{ \frac{m}{\underline{b}} \nabla B + \frac{V_{||}^2}{B} \underline{b} \cdot \nabla \underline{b} \right. \\ \left. + U_{||} \left[ \frac{\partial \underline{b}}{\partial t} + \underline{v}_E \cdot \nabla \underline{b} \right] \right\}$

LARGE TERMS

MOTION OF FIELD LINES

+  $\frac{\underline{b}}{\Omega} \times \left\{ \frac{\partial \underline{v}_E}{\partial t} + \underline{v}_E \cdot \nabla \underline{v}_E \right\}$

"Polarization Drift"

•  $m \frac{dV_{||}}{dt} = q E_{||} - \mu \nabla_{||} B - m \underline{b} \cdot \frac{d\underline{v}_E}{dt}$

$E_{||} \sim \frac{1}{L} E_{\perp}$

- (ii) Adiabatic Invariant.

$$I = \frac{m}{2} \frac{V_{\perp}^2}{B}$$

holds when fields change slowly compared to the gyration time.

- (iii) 2nd Adiabatic Invariant. When the fields change slowly compared to the bounce time we have a second invariant.

$$J = \oint V_{||} dl \quad \text{integrated around one bounce.}$$

Polarization Drift: When  $\underline{E}$  field varies we get a polarization of plasma.

$$\frac{d\underline{v}}{dt} = \frac{q}{m} \dot{\underline{E}_0} t + \frac{q \underline{v} \times \underline{B}}{mc}$$

simple case:

$$\dot{\underline{E}_0} \cdot \underline{B} = 0$$

$$\underline{E} = \dot{\underline{E}_0} t$$

DRI  
F

$$\underline{v} = \frac{c \dot{\underline{E}_0} \times \underline{B}}{B^2} t +$$

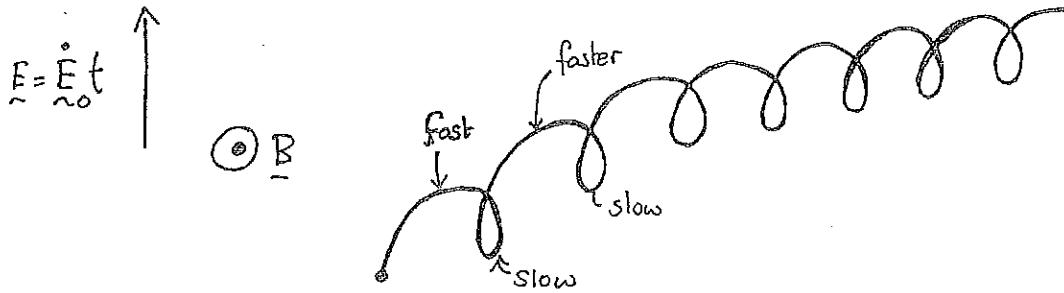
" $E \times B$  drift"

$$\frac{c}{\sqrt{2}} \frac{\dot{\underline{E}_0}}{\underline{B}}$$

Polarization drift

$$\underline{v} = \underline{u} + \underline{U}$$

$$\boxed{\frac{du}{dt} = \sqrt{2} \underline{u} \times \underline{b}}$$



recharge drifts in direction of electric field - Polarizes plasma as a current is set up

$$\text{POLARIZATION CURRENT} = \underline{J} = \sum_{\text{ion, electrons}} q n \underline{v} = \sum_{\text{species}} n m c^2 \frac{\partial \underline{E}_0}{\partial t}$$

$E \times B$  same for both species so it cancels out.

↑ mostly ion part.

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} + \frac{1}{c} \frac{\partial \underline{E}}{\partial t} = \epsilon_{\perp} \frac{\partial \underline{E}}{\partial t}$$

$$\epsilon_{\perp} = 1 + \frac{4\pi n m c^2}{B^2}$$

or

$$\epsilon_{\perp} = 1 + \frac{c^2}{V_A^2}$$

$$V_A^2 = \frac{B^2}{4\pi \rho}$$

↑  
ALFVEN SPEED  
 $n m = m v^2 / e B$

USUALLY  $c^2 \gg V_A^2$  SO SECOND TERM DOMINATES

## SIMPLE GYROKINETICS.

- Turbulence in many plasmas is slow - low frequency - but has a long parallel wavelength and short perpendicular wavelength.
- In many situations where the plasma has a strong  $\underline{B}$  field the fluctuations are electrostatic i.e.

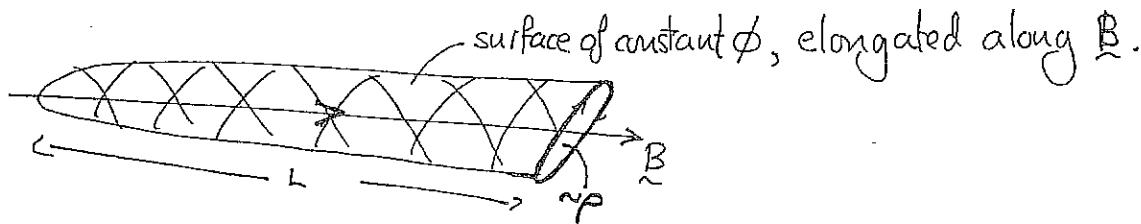
$$\underline{E} = -\nabla\phi(\underline{r}, t) \text{ and } \underline{B} = \underline{B}_0(\underline{r})$$

- A slightly different approximation (than the guiding center approx.) is used to describe these plasmas.

Take  $\frac{q\phi}{T} \approx \frac{f}{L} \sim \omega$   $T \sim \frac{1}{2}mv^2$

**ORDERING:**

$$|\nabla\phi| \sim \frac{\phi}{k_{\perp}\phi} \quad |\underline{b} \cdot \nabla\phi| \sim \frac{\phi}{k_{\parallel}\phi}$$



## GYROKINETIC EQUATIONS OF MOTION FOR GUIDING CENTER.

$$\frac{d\underline{v}}{dt} = -\frac{q}{m} \nabla\phi + \sqrt{2} \underline{v} \times \underline{b}$$

$$\partial\left(\frac{v^2}{2}\right) \quad \partial(\sqrt{2}\underline{v})$$

SIMPLIFY TAKE

$$\underline{B} = \underline{B}_0 \hat{z} \quad B_0 \text{ constant.}$$

not necessary but for our lecture we do it.

TO LOWEST ORDER E FIELD MAKES NO DIFFERENCE.

So Take  $\underline{r} = \underline{R}(t) + \frac{\underline{k}}{\sqrt{2}} \times \underline{v} + \underline{f}(nt, t)$  As before.

"guiding center"       $f(nt, t)$       wiggles

Differentiating.

$$\frac{d\underline{r}}{dt} = \underline{v} = \frac{d\underline{R}}{dt} + \frac{\underline{b}}{\underline{v}_\parallel} \times \frac{d\underline{v}}{dt} + \frac{d}{dt} \underline{\phi}$$

$$\frac{d\underline{R}}{dt} = \underline{V}_{\parallel} \underline{b} + \frac{c \underline{b}}{B_0} \times \nabla \phi + \underbrace{\frac{d}{dt} \underline{\phi}}$$

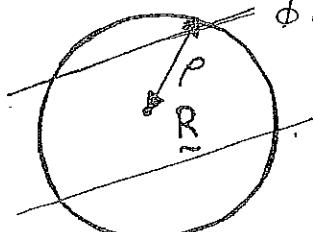
Average this out.

slow time

$$\nabla \phi(\underline{r}, t) = \frac{\partial}{\partial \underline{R}} \phi(R + \rho(\sin \theta \underline{e}_1 + \cos \theta \underline{e}_2), t)$$

DEFINE THE AVERAGE  $\bar{\phi}$   
AVERAGED OVER A RING  
OF RADIUS  $\rho$  ABOUT  $\underline{R}$ .

$$\bar{\phi} = \frac{1}{2\pi} \int_0^{2\pi} d\theta \phi(R + \rho(\sin \theta \underline{e}_1 + \cos \theta \underline{e}_2), t)$$



$\phi$  variations over ring radius are not small as they were  
in guiding center "drift kilometers".

$$\frac{d\underline{R}}{dt} = \bar{\underline{V}}_{\parallel} \underline{b} + \frac{c \underline{b}}{B_0} \times \frac{\partial \bar{\phi}}{\partial \underline{R}}$$

EQUATION OF MOTION FOR THE CENTER OF  
THE RING.

Also

$$\frac{d\bar{\underline{V}}_{\parallel}}{dt} = - \frac{q}{m} \underline{b} \cdot \nabla \bar{\phi}$$

NOTE: When,  $\phi = \phi(z) e^{ik_z r - i\omega t}$  take  $k_z = k_z \underline{e}_z$

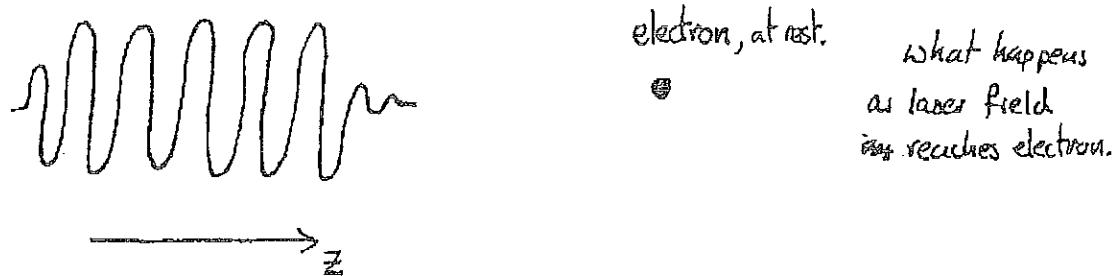
$$\bar{\phi} = \phi(z) \frac{e^{-i\omega t}}{2\pi} \int_0^{2\pi} e^{ik_z \rho \cos \theta} d\theta = \phi(z) e^{-i\omega t + ik_z \rho} J_0(k_z \rho)$$

Average of  
sine wave over  
ring is Bessel  
function

## 222a. Lecture #6 Motion of a Charge Particle in an Electromagnetic Wave.

Modern Lasers can deliver huge powers ( $10^{20} \text{ Wcm}^2$ ) — and the electric field of the EM Wave can accelerate electrons to relativistic energies. To understand the interaction we must understand how electrons move in an EM wave.

### Plane EM wave (pulse)



$$\underline{E} = E(t - \frac{z}{c}) \hat{x} \quad (1)$$

where  $E(t - \frac{z}{c})$  is an arbitrary function of  $t - \frac{z}{c}$ .

$$\phi = t - \frac{z}{c} \quad \text{"kind of a phase"}$$

$$\frac{\partial \underline{B}}{\partial t} = -c \nabla \times \underline{E} \Rightarrow \frac{\partial \underline{B}}{\partial t} = -c \frac{\partial E}{\partial z} \hat{y} = \frac{dE}{d\phi} \hat{y}$$

$$\text{Thus } \underline{B} = E(t - \frac{z}{c}) \hat{y} \quad (2)$$

The pulse is limited so that for fixed  $z$ ,  $E(t - \frac{z}{c}) \rightarrow 0$  as  $t \rightarrow \pm \infty$ .

We will imagine that in the middle of the pulse  $E = E_0 \sin(\omega t - kz)$   
 $\omega = kc$ .

We will use the vector potential

$$\underline{E} = -\frac{1}{c} \frac{\partial \underline{A}}{\partial t} \quad \underline{B} = \nabla \times \underline{A} \Rightarrow \underline{A} = A(t - \frac{z}{c}) \hat{x} \quad E = -\frac{1}{c} \frac{dA}{d\phi} \quad (3)$$

Nonrelativistic Motion.  $\frac{v}{c} \ll 1$

$$\frac{dp}{dt} = -e \left\{ E + \frac{v \times B}{c} \right\} \quad \rightarrow \quad \boxed{\frac{dv}{dt} = \frac{+e \partial A}{cm} \hat{x}}$$

↓  
Drop  $\partial(v/c)$

using  $v=0$  for  $t \rightarrow -\infty$

$$V_y = V_z = 0 \Rightarrow y = y_0 = \text{constant}$$

$$z = z_0 = \text{constant} \quad \text{- No motion in } z \text{ direction} \therefore \frac{d}{dt} = \frac{d}{d\phi}$$

$$V_x = \frac{eA}{mc} \quad \text{and} \quad x - x_0 = \frac{e}{mc} \int_0^t A dt'$$

$$A_0 = -\frac{eE_0}{\omega}$$

In the middle of the pulse  $V_x = \frac{eA_0}{mc^2} \cos(\omega t - kz)$

we define :-

$$\frac{eA_0}{mc^2} = \frac{V_{osc}}{c}$$

Electron just jiggles up and down in the  $x$  direction. No motion in  $z$  or  $y$ .

Relativistic Motion.

$$\textcircled{4} \quad \frac{d(m\gamma v)}{dt} = -e \left\{ E + \frac{v \times B}{c} \right\} \quad \textcircled{5} \quad \frac{d(m\gamma c^2)}{dt} = -e v \cdot E$$

ENERGY EQUATION is not independent of the other equations but useful nonetheless.

USING  $\textcircled{1}$  AND  $\textcircled{2}$  AND  $\textcircled{3}$  WE HAVE FROM  $\textcircled{4}$

$$\frac{d m\gamma v_y}{dt} = 0 \Rightarrow v_y = 0 \quad \text{Motion only in } x-z \text{ plane.}$$

use initial conditions

From ④

$$\textcircled{6} \quad \frac{d}{dt}(m\gamma V_x) = +\frac{e}{c} \frac{dA}{d\phi} \left(1 - \frac{V_z}{c}\right) \quad \text{BUT} \quad \frac{d\phi}{dt} = 1 - \frac{d^2 z}{dt^2 c} = 1 - \frac{V_z}{c}$$

$$= \frac{e}{c} \frac{dA}{d\phi} \frac{d\phi}{dt} = \frac{e}{c} \frac{dA}{dt}$$

Integrating we get:- (Using the initial conditions)

$$\textcircled{7} \quad \boxed{\gamma V_x = \frac{eA}{mc}}$$

From ③

$$\textcircled{8} \quad \frac{d}{dt}(m\gamma V_z) = \frac{e}{c^2} V_x \frac{dA}{d\phi} = -eV_x B_y$$

From ⑤ the energy equation.

$$\textcircled{9} \quad \frac{d}{dt} m\gamma c^2 = \frac{e}{c} V_x \frac{dA}{d\phi}$$

Subtracting we obtain  $\frac{d}{dt}(m\gamma c^2 - m\gamma V_z) = 0 \Rightarrow \boxed{\gamma \left(1 - \frac{V_z}{c}\right) = 1}$

BUT  $1 - \frac{V_z}{c} = \frac{d\phi}{dt} = \frac{1}{\gamma}$

So from ⑧ use ⑦

$$\frac{d}{dt}(m\gamma V_z) = \frac{e}{c^2} \gamma V_x \cdot \frac{1}{\gamma} \frac{dA}{d\phi} = \frac{e^2}{mc^3} A \frac{dA}{dt}$$

$$\textcircled{10} \quad \boxed{\gamma V_z = \frac{1}{2} \frac{e^2}{mc^3} A^2}$$

Now (finally) we integrate ⑦ and ⑧ to get  $x(t)$  and  $z(t)$

$$\textcircled{9} \quad \int V_x dt = x - x_0 = \int \frac{eA}{mc} \frac{dt}{\gamma} = \int \frac{eA}{mc} d\phi'$$

$$\textcircled{10} \quad \int V_z dt = z - z_0 = \int \frac{1}{2} \frac{e^2 A^2}{m c^3} d\phi'$$

In the middle of the pulse (where  $A = A_0 \cos(\omega t)$ )

$$\textcircled{11} \quad x - x_0 = \frac{eA_0}{mc^2} \frac{\sin(\omega t - kz)}{k}$$

$$\textcircled{12} \quad z - z_0 = \frac{1}{4} \frac{e^2 A_0^2}{m^2 c^3} \left\{ t - \frac{z}{c} + \frac{\sin 2(\omega t - kz)}{2\omega} \right\}$$

we can redefine  $z$  and  $t$  to get rid of  $z_0$  - so just set  $z_0 = 0$

AND  $x_0 = 0$  FOR SIMPLICITY.

Rearranging ⑫ gives.  $\therefore \frac{V_{osc}}{c} = \frac{eA_0}{mc^2}$  again by definition.

$$z = \frac{\frac{1}{4} \left( \frac{V_{osc}}{c} \right)^2}{1 + \frac{1}{4} \left( \frac{V_{osc}}{c} \right)^2} \left\{ ct + \frac{\sin 2(\omega t - kz)}{2k} \right\}$$

This is an implicit relation for  $z(t)$  since  $z$  appears on both sides.

Note:-  $z$  increases secularly with time with

a MEAN velocity:-  $\bar{V}_z = \frac{\frac{1}{4} \left( \frac{V_{osc}}{c} \right)^2}{1 + \frac{1}{4} \left( \frac{V_{osc}}{c} \right)^2} \cdot c$

$$\frac{V_{osc}}{c} \gg 1$$

$$\boxed{\bar{V}_z \rightarrow c}$$

We define a coordinate to take out the mean  $z$  drift

$$z' = z - \bar{v}_z t$$

then

$$z' = \frac{\bar{v}_z}{c} \frac{\sin 2(\omega't - kz')}{2k}, \quad x = \frac{v_{osc}}{c} \frac{\sin(\omega't - kz')}{k}$$

where  $\omega' = \omega(1 - \frac{\bar{v}_z}{c})$  : moving electron sees slower oscillation of E field.

ELECTRON OSCILLATES WITH PERIOD  $\frac{2\pi}{\omega'}$  IN  $x$  DIRECTION

AND WITH PERIOD  $\frac{\pi}{\omega'}$  IN  $z$  DIRECTION.

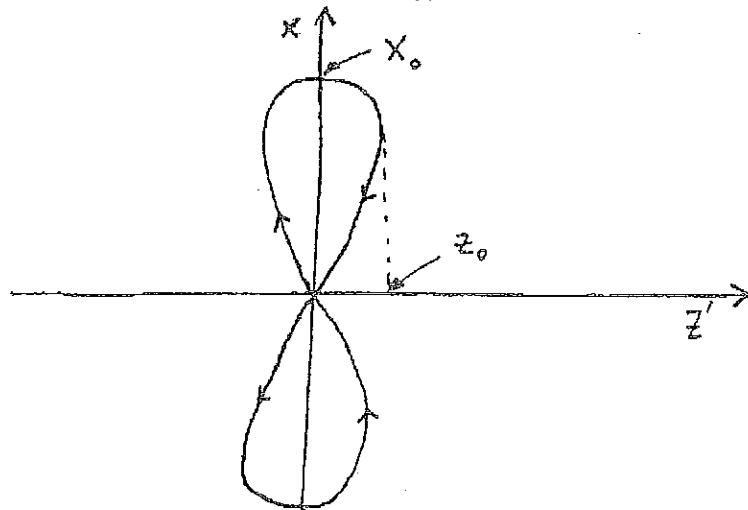
WE CAN EASILY SHOW THAT

$$z' = \pm 2z_0 \left( \frac{x}{x_0} \right) \sqrt{1 - \left( \frac{x}{x_0} \right)^2}$$

$$z_0 = \frac{\bar{v}_z}{2kc}$$

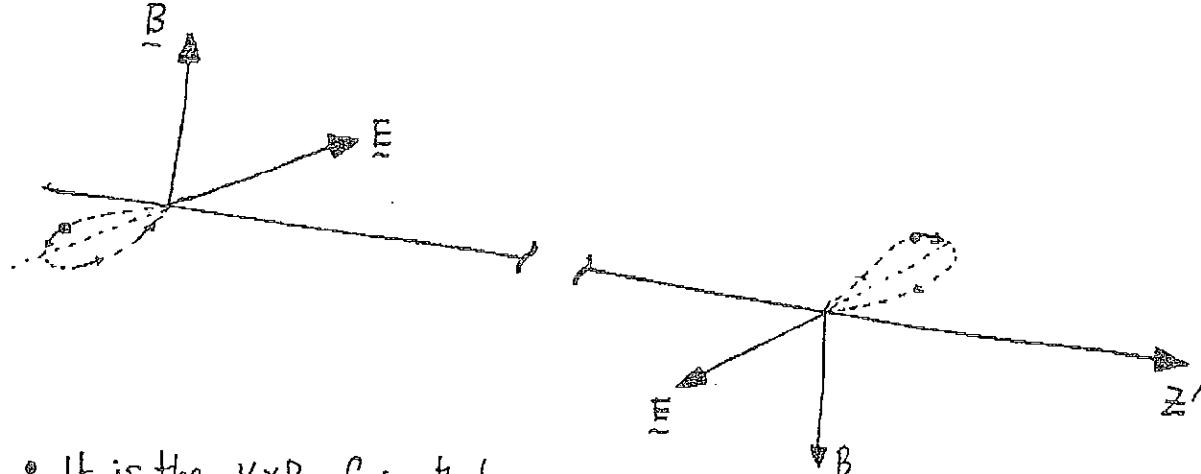
$$x_0 = \frac{v_{osc}}{kc}$$

FIGURE OF EIGHT



6

## Motion of Electron. Snapshots in time



- If it is the  $\underline{v} \times \underline{B}$  forces that bend the electron into its  $z$  motion from the pure  $x$  motion of the nonrelativistic case.

### One final note

Define  $1 - \frac{\vec{V}_z}{c} = \frac{1}{\gamma} \Rightarrow \boxed{\bar{\gamma} = 1 + \frac{1}{4} \left( \frac{V_{ox}}{c} \right)^2}$

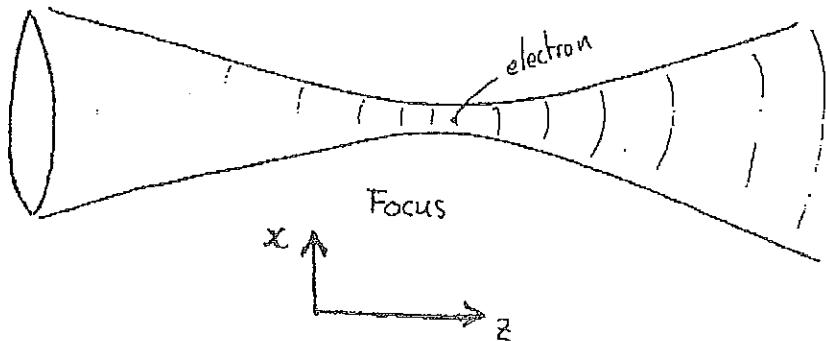
## Lecture #7: The Ponderomotive Force.

I'm back on the 31st of October. Next class Nov. 2nd.

In the last lecture we/you learnt about motion of charged particles in an EM wave. Most waves are not simple plane waves and it is important to find how particles move in a wave that varies (slowly) in space.

Consider the focus of a laser (in 2D)

Lens.



To a good approximation the field of the laser is a plane wave - we will take it to be polarized in the  $\hat{x}$  direction.

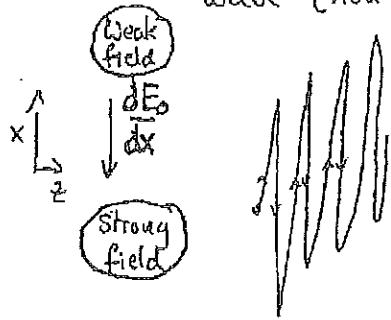
$$\underline{E} = E_0(x, z) \cos(\omega t - kz) \hat{x}$$

To get the actual behavior in the focus region we have to solve Maxwell's equations but we won't do that here.

The variation in the  $x$  and  $z$  direction is typically long compared to the ~~wavelength~~ wavelength so let's simplify things and choose only variation in  $x$ . So

$$\frac{1}{K} \frac{\partial E_0}{\partial x} \ll E_0 \quad \textcircled{1}$$

Thus consider physically the motion of an electron in the wave (non-relativistic)



- when the field pushes electron upwards the electron experiences a stronger field than when it is coming back down.
- Thus it slowly drifts upwards due to the inhomogeneity in  $E$  — this is due to the ponderomotive force.

Remember: if  $E_0$  doesn't vary then

$$X(t) = \tilde{X}(t) = + \frac{e E_0}{m \omega^2} \cos(\omega t - kz) \hat{x} \quad \textcircled{2}$$

note  $|\tilde{v}| \ll c$  by assumption so  $\omega/\tilde{v} \ll c$  and from  $\textcircled{1}$

$$\hat{x} \frac{dE_0}{dx} \ll E_0 \quad \textcircled{3}$$

From our physical picture it is fairly clear that in each oscillation the electron sees only a moderate average force.

Thus we define

$$x(t) = \bar{x}(t) + \tilde{x}(t)$$

↗   ↗

OSCILLATION CENTER  
accelerates slowly but  
is finite

OSCILLATION. FAST  
ACCELERATION BUT  
SMALL AMPLITUDE.

DEFINE AN INTERMEDIATE TIME FOR AVERAGING.

$$\langle A \rangle = \frac{1}{T} \int_0^T A(t') dt'$$

where  $\omega T \gg 1$  i.e.  $T$  is long compared to an oscillation

BUT  $T \frac{d\bar{x}}{dt} \ll \bar{x}$  i.e.  $T$  is short compared to the time for  $\bar{x}$  to change.

WE INSIST THAT

$$\langle \tilde{x} \rangle \rightarrow 0 \quad \text{i.e. oscillation averages to zero.}$$

AND

$$\langle \bar{x} \rangle = \bar{x} \quad \text{i.e. } \bar{x} \text{ is the average position.}$$

Now we expand the equation of motion about the oscillation center.

$$(3) \quad \frac{d^2x}{dt^2} = \frac{d^2\bar{x}}{dt^2} + \frac{d^2\tilde{x}}{dt^2} = -\frac{e}{m} E_0(x) \cos(\omega t - kz)$$

$$\approx -\frac{e}{m} \cos(\omega t - kz) \left\{ E_0(\bar{x}) + \hat{x} \left( \frac{dE_0}{dx} \right)_{x=\bar{x}} \dots \dots \right\}$$

Clearly the terms get smaller in this expansion. Clearly the first term gives no oscillation so

$$\frac{d^2\tilde{x}}{dt^2} \approx -\frac{e}{m} E_0(\bar{x}) \cos(\omega t - kz) \Rightarrow \boxed{\tilde{x} \approx \frac{+e}{m} \frac{E_0(\bar{x})}{\omega^2} \cos(\omega t - kz)}$$

Averaging (3) above we get:

$$\frac{d^2\bar{x}}{dt^2} \approx -\frac{e}{m} \left( \frac{dE_0}{dx} \right)_{x=\bar{x}} \langle \tilde{x} \cos(\omega t - kz) \rangle$$

$$\approx -\frac{e^2}{m^2 \omega^2} E_0(\bar{x}) \frac{dE_0}{d\bar{x}} \langle \cos^2(\omega t - kz) \rangle^{1/2}$$

$$\frac{d^2\bar{x}}{dt^2} \approx -\frac{1}{2} \frac{e^2}{m^2 \omega^2} \frac{d}{d\bar{x}} \frac{E_0^2}{2}$$

WE DEFINE THE PONDEROMOTIVE POTENTIAL  $V_p(\bar{x}) = \frac{1}{4} \frac{e^2 E_0^2}{m \omega^2}$

So ④  $m \frac{d^2 \bar{x}}{dt^2} = -\frac{dV(\bar{x})}{d\bar{x}}$

Mechanical Equation.  
Electron pushed away from high laser intensity regions.

note that  $V_p = \frac{1}{2} m V_{osc}^2$

$V_{osc} = \frac{1}{\sqrt{2}} \frac{e}{m} \frac{E_0}{\omega}$  RMS oscillation velocity.

So we have a simple way to evolve the oscillation center.

EXAMPLE GAUSSIAN FOCUS.

$$E_0 = E_0 e^{-\frac{\bar{x}^2}{x_0^2}}$$

Gaussian beam.

$$V(\bar{x}) = V_0 e^{-\frac{2\bar{x}^2}{x_0^2}}$$

$$V_0 = \frac{1}{2} m V_{oscmax}^2$$

$$V_{oscmax} = \frac{1}{\sqrt{2}} \frac{e}{m} \frac{E_0}{\omega}$$

Equation ④ Integrates to

$$\frac{1}{2} m \left( \frac{d\bar{x}}{dt} \right)^2 = -V_0 e^{-\frac{2\bar{x}^2}{x_0^2}} + \text{constant, total energy.}$$

$V_0 E_0$  constant

$$\frac{d\bar{x}}{dt} = V_{oscmax} \sqrt{E_0 - e^{-\frac{2\bar{x}^2}{x_0^2}}}$$

like a ball on a gaussian hill



Clearly electron accelerates -& if  $\epsilon_0 \sim 1$  the velocity as  $t \rightarrow \infty$  is  $v_{escmax}$ . The electron is expelled from the laser beam.

Using this effect lasers can bore holes in materials pushing the electrons aside - the laser beam only passes through if the laser frequency  $\omega$  exceeds the plasma frequency  $\omega_p = \frac{4\pi ne^2}{m_e}$ . If it pushes enough electrons aside it can propagate.

### HOMEWORK PROBLEM

Consider the Gaussian Beam  $E_0 = E_0 e^{-x^2/x_0^2}$ .

- (i) Calculate the electrostatic potential needed to oppose and cancel the ponderomotive potential.
- (ii) Calculate the electron density distribution needed to produce the potential of part (i). Assume that there is an immovable constant/uniform distribution of ions of density  $n_0$ . Further assume that the electrons have a mean density  $n_0$ .

222a. Lecture #7      Pondromotive Force:  
 (lecture)

(i) In this section, we examine the effect of fast oscillating fields on particles - this is crucially important in LASER PLASMA interaction and MICROWAVE devices.

(ii) Consider an Electromagnetic Wave that varies in space. (not necessarily a vacuum EM wave)

$$\underline{E}(r, t) = E_0(r) \cos(\omega t - k \cdot r)$$

$$\frac{\underline{B}}{t} = -c \nabla \times \underline{E} \Rightarrow \underline{B}(r, t) = -\frac{c}{\omega} \left\{ \nabla \times \underline{E}_0 \sin(\omega t - k \cdot r) - k \times \underline{E}_0 \cos(\omega t - k \cdot r) \right\}$$

LET US SOLVE FOR THE MOTION OF A PARTICLE IN THIS FIELD.

$$(iii) m \frac{d\underline{v}}{dt} = q \left[ \underline{E}(r, t) + \frac{\underline{k} \times \underline{B}(r, t)}{c} \right]$$

TO SOLVE THIS WE ASSUME THAT  $\underline{E}$  &  $\underline{B}$  VARY SLOWLY ENOUGH THAT

$$\underline{v} \cdot \nabla \underline{E} \ll \omega \underline{E} \quad \therefore \frac{\underline{v} \times \underline{B}}{c} \sim \frac{|\underline{v}| |\nabla| |\underline{E}|}{\omega} \ll |\underline{E}|$$

note this means  $\underline{v} \ll \frac{\omega}{|k|}$  = phase velocity.

(iv)

OVER EACH OSCILLATION  $\underline{E}$  VARIES LITTLE. OF COURSE SMALL CHANGES EACH OSCILLATION TO PRODUCE LONG TIMESCALE CHANGES.

TWO TIMESCALES

$t \approx \frac{\lambda}{v}, k \lambda$  long timescale.

$\tau = \theta \left( \frac{1}{\omega} \right)$  short timescale.

$$\epsilon = \frac{v}{\lambda \omega}$$

NOW WE WRITE THE PARTICLE POSITION AS A SLOWLY VARYING (IN TIME) POSITION PLUS AN OSCILLATING TERM.

$$x = \underline{R}(t) + \xi(t, t)$$

↙ oscillation center.

$$\frac{dx}{dt} = \frac{d\underline{R}}{dt} + \frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial t}$$

$\underline{U}(t)$        $\underbrace{\quad}_{\underline{u}(t, t)}$

$$\xi \sim \mathcal{O}(\epsilon \underline{R})$$

$$\frac{\partial \xi}{\partial t} \sim \mathcal{O}\left(\frac{d\underline{R}}{dt}\right)$$

$$\underline{u} \sim \mathcal{O}(\underline{U})$$

$$\frac{\partial \underline{u}}{\partial t} + \frac{\partial \underline{u}}{\partial t} + \frac{d\underline{U}}{dt} = \frac{q}{m} \left[ \underline{E}(\underline{R} + \xi, t) + \frac{(\underline{u} + \underline{U}) \times \underline{B}(\underline{R} + \xi, t)}{c} \right]$$

"BIG"      "SMALLER"      "SMALLER"      "BIG"      "SMALLER"

EXPAND ABOUT THE OSCILLATION CENTER

$$\underline{E} = \underline{E}(\underline{R}, t) + \xi_0 \nabla \underline{E}(\underline{R}, t) \dots \quad \underline{B} = \text{etc.}$$

"BIG"      "SMALLER"

$$\text{EXPAND ALL VARIABLES: } \underline{u} = \underline{u}_0(t, t) + \epsilon \underline{u}_1(t, t), \quad \xi = \xi_0(t, t) + \epsilon \xi_1(t, t) \dots$$

LOWEST ORDER:  $\mathcal{O}(1)$

$$\frac{\partial \underline{u}_0}{\partial t} = \frac{q}{m} \underline{E}_0(\underline{R}) \cos(\omega t - \underline{k} \cdot \underline{R}(t))$$

$$\underline{u}_0 = \frac{q \underline{E}_0(\underline{R})}{m \omega} \sin(\omega t - \underline{k}_0 \cdot \underline{R}(t))$$

oscillation velocity.

$$\xi_0 = -\frac{q}{m \omega^2} \underline{E}_0(\underline{R}) \cos(\omega t - \underline{k} \cdot \underline{R}(t))$$

oscillating position.

FIRST ORDER:  $\mathcal{O}(\epsilon)$

$$\frac{d\tilde{U}_0}{dt} + \frac{\partial \tilde{u}_0}{\partial t} + \frac{\partial \tilde{u}_1}{\partial \tau} = \frac{q}{m} \left[ \tilde{E}_0 \cdot \nabla \tilde{E} + \frac{\tilde{u}_0 \times \tilde{B}(R, \tau) + \tilde{u}_1 \times \tilde{B}(R, \tau)}{c} \right]$$

↑ Averaging, noting that:-  $\int_0^{\frac{2\pi}{\omega}} d\tau \frac{\partial \tilde{u}_0}{\partial t} = \int_0^{\frac{2\pi}{\omega}} d\tau \frac{\partial \tilde{u}_1}{\partial \tau} = \int_0^{\frac{2\pi}{\omega}} d\tau \tilde{B}(R, \tau) = 0$

$$\frac{d\tilde{U}_0}{dt} = - \frac{q^2}{m^2 \omega^2} \left\{ \begin{aligned} & \tilde{E}_0 \cdot \nabla \tilde{E}_0 \overline{\cos^2(\omega \tau - k \cdot R)} + \tilde{E}_0 \times \nabla \times \tilde{E}_0 \overline{\sin^2(\omega \tau - k \cdot R)} \\ & + \tilde{E}_0 \cdot k \tilde{E}_0 \overline{\cos(\omega \tau - k \cdot R) \sin(\omega \tau - k \cdot R)} \quad \text{etc. . . .} \\ & \quad \vdots \\ & \text{OTHER TERMS THAT AVERAGE TO ZERO.} \end{aligned} \right\}$$

Use that  $\tilde{E}_0 \cdot \nabla \tilde{E}_0 + \tilde{E}_0 \times \nabla \times \tilde{E}_0 = \frac{1}{2} \nabla |\tilde{E}_0|^2$

$$\Rightarrow \boxed{m \frac{d\tilde{U}_0}{dt} = -q \nabla \Phi_{\text{PONDEROMOTIVE}}}$$

$$\Phi_{\text{PONDEROMOTIVE}} = \frac{1}{4} \frac{q}{m \omega^2} |\tilde{E}_0|^2 = \frac{1}{2} m \overline{|\tilde{u}_0|^2}$$

NOTE:

- PONDEROMOTIVE FORCE IS INDEPENDENT OF THE SIGN OF  $q$  AND ALWAYS PUSHES AWAY FROM REGIONS OF HIGH WAVE FIELD.

- If  $\tilde{E}_0 = \tilde{E}_0(x, t)$  <sup>slowtime</sup> SAME FORMULA

- $E_{\text{osc}} = \frac{1}{2} m \overline{U_0^2} + \frac{1}{2} m \overline{|\tilde{u}_0|^2}$  CONSTANT IF  $\tilde{E}_0 = \tilde{E}_0(x)$



222a. Lecture #8Kinetic Description of Plasma.

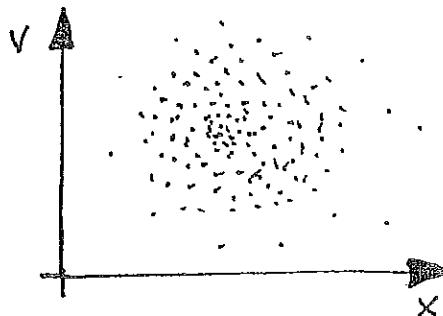
(i) Following every particle would produce an intractable problem we need a STATISTICAL DESCRIPTION.

(ii) CONFIGURATION SPACE / PHASE SPACE.

IDENTIFYING  $\underline{r}, \underline{v}$  for each particle gives position in CONFIGURATION SPACE  $\equiv (\underline{r}, \underline{v})$ .

Similarly

$$\text{PHASE SPACE} = (\underline{r}, \underline{p})$$



- each particle a dot in this space
- in any practical situation the number of particles is huge.  $10^{10} - 10^{30}$

(iii) KLIMONTOVITCH DISTRIBUTION:  $\underline{r}_i(t), \underline{v}_i(t)$  position of  $i^{\text{th}}$  particle

DENSITY OF  
ARTICLES IN  
CONFIGURATION SPACE

$$F = \sum_{i=1}^N \delta(\underline{r} - \underline{r}_i(t)) \delta(\underline{v} - \underline{v}_i(t))$$

$$\frac{d\underline{r}_i}{dt} = \underline{v}_i$$

$$\frac{d\underline{v}_i}{dt} = \underline{a}_i$$

- Far too detailed a description for practical use.
- Depends on initial conditions which we don't know.
- Not needed to calculate most important macroscopic results.

$\frac{dF}{dt} = 0$  along a particle orbit

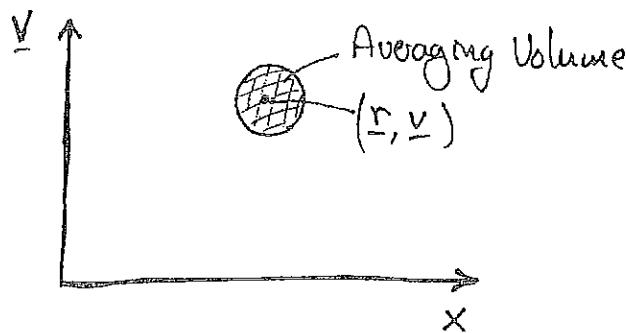
$$\frac{\partial F}{\partial t} + \underline{v} \cdot \nabla F + \underline{a} \cdot \frac{\partial F}{\partial \underline{v}} = 0$$

Klimontovich  
Equation.

$$\underline{a} = \frac{q}{m} (\underline{E} + \underline{v} \times \underline{B}) = \underline{\text{ACCELERATION}}.$$

(iv) COARSE GRAINED AVERAGE.

- We create a smooth distribution by Averaging  $F$  over a small volume around  $\underline{r}, \underline{v}$ . Volume contains many particles but the average itself doesn't change much over the volume.



$$f(\underline{r}, \underline{v}, t) = \int d^3R d^3V K(R, V) F(\underline{r}-R, \underline{v}-V)$$

"Weight Function"  
could be  $\propto e^{-\frac{R^2}{R_0^2} - \frac{V^2}{V_0^2}}$

- Another approach is to treat  $f$  as an ensemble average - I am not going to discuss this until Winter quarter.
- We also split the Electric field into a mean part  $\bar{E}$  and a microscopic fluctuating part,  $\tilde{E}$  where:

$$\nabla \cdot \tilde{E} = 4\pi \bar{\rho} = 4\pi q \int d^3V f(\underline{r}, \underline{v}, t)$$

$$\nabla \times \bar{B} = \frac{4\pi \bar{j}}{c} + \frac{1}{c} \frac{\partial \tilde{E}}{\partial t} \quad \bar{j} = q \int d^3V \underline{v} f(\underline{r}, \underline{v}, t)$$

Averaging the Klimontovich Eq. Yields. "One Particle Kinetic Eqn."

$$\frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{q}{m} \left( \bar{E} + \frac{\underline{v} \times \bar{B}}{c} \right) \cdot \frac{\partial f}{\partial \underline{v}} = \frac{q}{m} \left\langle \left( \bar{E} + \frac{\underline{v} \times \tilde{B}}{c} \right) \cdot \frac{\partial F}{\partial \underline{v}} \right\rangle$$

$$\nabla \cdot \tilde{E} = 4\pi q \int d^3V (F - f) \quad \text{FIELD DUE TO PARTICLE DISCRETENESS.}$$

$\left( \frac{\partial f}{\partial t} \right)_{\text{COLLISIONS}}$

Without collisions this equation is called the Vlasov equation.

- Our purpose isn't to solve this equation or evaluate  $\left(\frac{\partial f}{\partial t}\right)_{\text{collisions}}$  we will do that later (Winter Quarter). Here we look at MOMENTS.

### MOMENT EQUATIONS

(Treating plasma like a fluid)

$$(\text{AVERAGE}) \text{ DENSITY} = \int \overset{\rightarrow}{\text{sum over all velocities.}} d^3v f(r, v, t) = n(r, t)$$

NOTE: Averaged over little volume.

- $n(r, t)$  is much simpler than  $f$  since it is a function of only 4 variables (i.e. 3D space + time) not 6 (i.e. 6D phase space + time)
- $\rho = q n(r, t)$  goes into Maxwell's equations directly.

$$(\text{AVERAGE}) \text{ VELOCITY} = \frac{1}{n} \int d^3v v f(r, v, t) = v(r, t)$$

$$n \cdot \underline{v} = \underline{J}$$

WE WOULD LIKE FLUID EQUATIONS TO EVOLVE THE MOMENTS DIRECTLY - WITHOUT HAVING TO SOLVE FOR  $f(r, v, t)$ . THIS IS NOT POSSIBLE WITHOUT FURTHER SIMPLIFICATION/APPROXIMATION.

TAKING Moments of the Kinetic Equation

$$\int d^3v \left\{ \frac{\partial f}{\partial t} + \underline{v} \cdot \nabla f + \frac{q}{m} \left( \underline{E} + \underline{v} \times \underline{B} \right) \cdot \frac{\partial f}{\partial \underline{v}} \right\} = \left( \frac{\partial f}{\partial t} \right)_c$$

$$\int \nabla \cdot (\underline{v} f) d^3v = \frac{q}{m} \int \frac{\partial}{\partial \underline{v}} \cdot \left[ \left( \underline{E} + \underline{v} \times \underline{B} \right) f \right] d^3v = \frac{q}{m} \left\langle \frac{\partial}{\partial \underline{v}} \cdot \left[ \left( \underline{E} + \underline{v} \times \underline{B} \right) \underline{F} \right] d^3v \right\rangle$$

$$\frac{\partial n}{\partial t} + \nabla \cdot (\underline{v} n) + \int \overset{\circ}{\underline{O}} = 0$$

**CONTINUITY EQUATION**

$$\frac{\partial n}{\partial t} + \nabla \cdot \underline{V} n = 0$$

Expresses conservation of particles.

Now take the velocity moment of the kinetic Eqn.

$$\int d^3v \underline{V} \left\{ \text{Kinetic Equation} \right\}$$

$$\rightarrow \frac{\partial (\underline{V} n)}{\partial t} + \nabla \cdot \underbrace{\int d^3v (\underline{V} \underline{V} f)}_{\text{NEW MOMENT}} - \frac{q n}{m} \left( \underline{E} + \underline{V} \times \underline{\underline{B}} \right) = \text{COLLISIONAL DRAG.}$$

a little algebra:

$$\int d^3v (\underline{V} \underline{V} f) = n \underline{\underline{V} V} + \underbrace{\int d^3v (\underline{V} - \underline{\underline{V}})(\underline{V} - \underline{\underline{V}}) f}_{\text{Pressure Tensor } \underline{\underline{P}} / m}$$

$$\nabla \cdot (n \underline{\underline{V} V}) = \underline{V} \nabla \cdot n \underline{V} + n \underline{\underline{V}} \cdot \nabla \underline{V}$$

some more algebra gives:

MOMENTUM  
EQUATION.

$$mn \left\{ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right\} = - \nabla \cdot \underline{\underline{P}} + \frac{q n}{m} \left( \underline{E} + \underline{V} \times \underline{\underline{B}} \right) + \underline{\underline{\text{DRAG}}}$$

note: Drag is momentum loss - this can happen between ions & electrons but not between ions & ions or electrons & electrons.

Problem: We have equations for  $n$  and  $\underline{V}$  but we now have a new unknown  $\underline{\underline{P}}$  - the system of equations (fluid equations) is not closed. Unfortunately when we write an equation for  $\underline{\underline{P}}$  it involves  $\underline{\underline{P}}$  itself.

## 222a. Lecture #9    Two Fluid Equations.

(i) Usually we write equations for 3 variables/momenta.

- DENSITY:

$$n_s(\mathbf{r}, t) = \int d^3\mathbf{v} f_s(\mathbf{r}, \mathbf{v}, t)$$

$s$  labels the species i.e. electrons or ions.

- VELOCITY:

$$\mathbf{v}_s(\mathbf{r}, t) = \frac{1}{n_s} \int d^3\mathbf{v} \mathbf{v} f_s(\mathbf{r}, \mathbf{v}, t)$$

particle flux  
=  $n_s \mathbf{v}_s$ .

- PRESSURE / ENERGY

$$\frac{3}{2} p_s = \int d^3\mathbf{v} \frac{1}{2} m_s w_s^2 f_s(\mathbf{r}, \mathbf{v}, t)$$

Kinetic Energy Density.

$w_s = v - v_s$   
velocity of particle in frame moving with fluid

The equations for these moments involve the other moments and the "unknown" moments

pressure tensor.

$$\tilde{\mathbf{P}}_s = \int d^3\mathbf{v} m_s \mathbf{w}_s \mathbf{w}_s f_s(\mathbf{r}, \mathbf{v}, t)$$

"Variance of the velocity."

Note:-  $\mathbf{P}_s = \frac{1}{3} \text{Trace}\{\tilde{\mathbf{P}}_s\}$

$$\mathbf{q}_s = \int d^3\mathbf{v} f_s(\mathbf{r}, \mathbf{v}, t) \frac{1}{2} m_s w_s^2 \mathbf{w}_s$$

Heat Flux density.

Temperature is DEFINED by:-  $T_s = \frac{P_s}{n_s}$

so  $\frac{3}{2} T_s$  = mean KE of a particle.

CONTINUITY EQN.

(ii) EQUATIONS:      ①  $\frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underline{v}_s) = 0$

MOMENTUM EQN.

②  $m_s n_s \left\{ \frac{\partial \underline{v}_s}{\partial t} + \underline{v}_s \cdot \nabla \underline{v}_s \right\} = - \nabla \cdot \underline{\underline{P}} + q n_s \left[ E + \frac{\underline{v}_s \times \underline{B}}{c} \right]$

+  $\underline{F}_s$  ← collisional drag.

ENERGY EQN.

③  $\frac{3}{2} \left\{ \frac{\partial p_s}{\partial t} + \nabla \cdot (p_s \underline{v}_s) \right\} + \underline{\underline{P}} : \nabla \underline{v}_s + \nabla \cdot \underline{q}_s = \underline{W}_{s\eta}$

collisional heating.

(iii) Entropy Density  $\equiv S_s = n_s \ln \left[ \frac{T_s^{3/2}}{n_s} \right]$

Entropy Flux  $\equiv S_s \underline{v}_s + \underline{q}_s / T_s = \underline{s}_s$

Generalized Viscosity  $\underline{\underline{\tau}}_s = \underline{\underline{P}}_s - \frac{1}{2} \underline{\underline{P}}_s$

ALTERNATIVE TO ENERGY EQN.

④  $\frac{\partial S_s}{\partial t} + \nabla \cdot \underline{s}_s = \frac{\underline{W}_s}{T_s} - \frac{\underline{\underline{\tau}}_s : \nabla \underline{v}_s}{T_s} - \frac{\underline{q}_s}{T_s} \cdot \frac{\nabla T_s}{T_s}$

ENTROPY PRODUCTION.

(iv) LOCAL THERMODYNAMIC EQUILIBRIUM: COLLISIONAL EQUATIONS

$\gamma$  = collision rate = time for particle velocity to randomize.

$$\lambda_{mfp} = \text{mean free path} = \frac{V}{\gamma}$$

$$V_{th} = \text{thermal velocity} = \sqrt{\frac{kT}{m}}$$

if for any moment A:  $\frac{1}{\gamma} \frac{\partial A}{\partial t} + \lambda_{mfp} \nabla A \ll A$     $\lambda_{mfp} \ll E \ll T$

Then the plasma locally comes to <sup>thermal</sup> equilibrium.

$$f(\mathbf{r}, \mathbf{v}, t) = \left( \frac{m_s}{2\pi T_s(\mathbf{r}, t)} \right)^{3/2} n_s(\mathbf{r}, t) e^{-\frac{m_s (\mathbf{v} - \mathbf{V}_s(\mathbf{r}, t))^2}{2 T_s(\mathbf{r}, t)}} + \text{small terms.}$$

MAXWELLIAN DISTRIBUTION

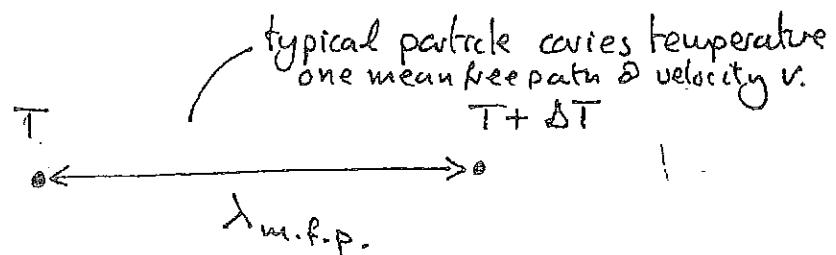
$f_{\max}$

(v) TRANSPORT TERMS: From this  $f$  we find that:

$q_s, \eta_s$  are small

Winter Term  
we will do  
this formally

Estimating  $q_s$



$$\Delta T = \lambda_{m.f.p.} \frac{dT}{dx}$$

HEAT FLUX  $\approx V_{th} \cdot \Delta T = V_{th} \lambda_{m.f.p.} \frac{dT}{dx}$   
 $\ll V_{th} T$

Thermal Conductivity  $\approx V_{th} \lambda_{m.f.p.} = \frac{V_{th}^2}{\sqrt{\nu}} = K$

$q_s \sim K_s \nabla T_s$

(without  $B$  field)

Similar arguments give  $\Pi \approx \mathcal{J} \left( \frac{\lambda_{m.f.p.} P}{L} \right)$

$L \approx$  scale length of  $n_s T_s V_s$ .

Fluid Equations in  
In a Collisional Plasma

Dropping  $\underline{\pi}$  and  $q$  using  
"isotropic pressure"  $\nabla \cdot (\underline{\pi} \underline{P}_s) = \nabla P_s$

$$\left\{ \begin{array}{l} \frac{\partial n_s}{\partial t} + \nabla \cdot (n_s \underline{V}_s) = 0 \\ n_s m_s \left[ \frac{\partial \underline{V}_s}{\partial t} + \underline{V}_s \cdot \nabla \underline{V}_s \right] = - \nabla P_s + q_s n_s \left[ \underline{E} + \frac{\underline{V}_s \times \underline{B}}{c} \right] + \underline{F}_s \\ \frac{\partial}{\partial t} \left( \frac{P_s}{n_s^{5/3}} \right) + \underline{V}_s \cdot \nabla \left( \frac{P_s}{n_s^{5/3}} \right) = 0 \end{array} \right.$$

conservation of entropy

Complete System with Maxwell's Equations.

$$\nabla \cdot \underline{E} = 4\pi \sum_s q_s n_s , \quad \frac{\partial \underline{E}}{\partial t} = c \nabla \times \underline{B} - 4\pi \sum_s q_s n_s \underline{V}_s$$

$$\nabla \cdot \underline{B} = 0 \quad \frac{\partial \underline{B}}{\partial t} = -c \nabla \times \underline{E}$$

SMALL ELECTRON MASS Since  $m_e \sim \frac{m_i}{2000}$  we can often

simplify using this fact. In momentum equation drop inertial (mass) term.

$$-\frac{\nabla P_e}{e n_e} \approx \left[ \underline{E} + \frac{\underline{V}_e \times \underline{B}}{c} \right] - \underline{F}_e$$

GENERALIZED OHM'S LAW.

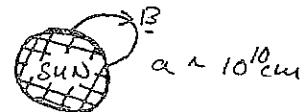
## 222a. Lecture #10. The Magnetohydrodynamics (MHD) Equations.

(i) We are often interested in the macroscopic behaviour of a plasma on relatively long timescales. MHD is often a good approximation to use for such situations - it does vastly simplify the plasmadynamics into the dynamics of ONE FLUID.

(ii) **LET**  $a =$  typical length in the problem.

$v =$  typical velocity in the problem

$$\sim v_i \text{ thermal velocity of the ions} \approx \sqrt{\frac{kT_i}{m_i}}$$



**COLLISIONS**

$$\lambda_{\text{m.f.p.}} = \frac{1}{n \sigma} \propto \frac{T^2}{n e^4}$$

cross section

**SAME FOR IONS AND ELECTRONS**

$$\gamma_e = \text{electron collision rate} = \frac{v_{the}}{\lambda_{\text{m.f.p.}}} \quad \gamma_i = \frac{v_{thi}}{\lambda_{\text{m.f.p.}}} \quad \text{NOTE} \quad \gamma_i = \left(\frac{m_e}{m_i}\right)^{1/2} \gamma_e$$

IONS "COLLIDE" LESS FREQUENTLY THAN ELECTRONS.

(EPP) Justify dropping terms from 2 fluid equations:

FUNDAMENTAL ORDERING ASSUMPTIONS OF MHD

$$\nabla \sim \frac{1}{a} \quad \frac{\partial}{\partial t} \approx \frac{v_i}{a} \approx \frac{v}{a}$$

$$\frac{P}{B^2} = \beta \sim \mathcal{O}(1)$$

NONRELATIVISTIC

$$\frac{v^2}{c^2} \ll 1$$

MAGNETIZED

$$\frac{\rho_i}{a} \ll 1$$

COLLISIONAL

$$\left(\frac{m_i}{m_e}\right)^{1/2} \frac{v}{a \gamma_i} \ll 1$$

Drop viscosities because of high collisionality.  $\frac{U}{P} \sim \frac{v}{a \gamma_i}$

(iv) From last time we write Ohm's law:

$$\underline{E} + \frac{\underline{V}_e \times \underline{B}}{c} = -\frac{\nabla P_e}{n_e e} + \underbrace{\frac{m_e n_e e}{n_e e} (\underline{V}_i - \underline{V}_e)}_{\text{FRICTIONAL DRAG TERM.}}$$

$$\Rightarrow \underline{E} \sim \frac{\underline{V}_e \times \underline{B}}{c}$$

taking  $\underline{V}_e \sim \underline{V}_i$

$$\frac{\nabla P_e}{c n_e e} \sim \mathcal{O}\left(\frac{\rho_i}{a}\right) \ll 1$$

SO HALL TERM IS SMALL.

(v) Displacement current can be ignored:

$$\frac{\frac{1}{c} \frac{\partial \underline{E}}{\partial t}}{\nabla \times \underline{B}} \sim \mathcal{O}\left(\frac{V^2}{c^2}\right)$$

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{j}$$

Ampere's law.

(vi) IONS AND ELECTRONS HAVE ALMOST SAME DENSITY

$$\frac{\Delta n}{n} \sim \frac{n_e - n_i}{n_e} \sim \frac{\nabla \cdot \underline{E}}{n_e e} \sim \frac{1}{a} \frac{V B}{c} \frac{1}{n_e} \sim \left(\frac{B^2}{P}\right) \frac{V^2}{c^2} \frac{\rho_i}{a} \ll 1$$

"QUASI-NEUTRAL"

(vii) IONS AND ELECTRONS HAVE ALMOST THE SAME VELOCITY

$$\underline{j} = n_e (\underline{V}_i - \underline{V}_e) \quad \frac{\Delta V}{V} \sim \frac{j}{n_e} \sim \frac{c B}{n e a} \sim \left(\frac{B^2}{P}\right) \frac{\rho_i}{a} \ll 1$$

(viii) THUS OHMS LAW BECOMES

$$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} = \eta \underline{j}$$

$$\eta = \text{resistivity} = \frac{m_e e}{n e^2}$$

note

$$\frac{\eta j}{V \times B} \sim \left(\frac{m_e}{m_i}\right)^{\frac{1}{2}} \left(\frac{\rho_i}{a}\right)^2 \left(\frac{V_i a}{V_i}\right)$$

↑ ↑ ↑  
SMALL SMALL LARGE

OFTEN VERY SMALL.  
MUST BE QUITE LARGE  
COLLISIONALITY TO BE  
FINITE.

(ix) Entropy / Heat / Equation.

Dropping <sub>electron</sub> thermal conduction means  $n_e \frac{\partial T_e}{\partial t} \gg \nabla \cdot q_e$  heat flow.

and we showed that  $q_e \approx n V_{the}^2 \nabla_i T_e$  only along B heat flow across B is small.

this yields

$$n \frac{\partial T_e}{\partial t} \simeq \left( \frac{m_i}{m_e} \right)^{1/2} \frac{V}{a \gamma_i} \ll 1$$

THIS IS MORE RESTRICTIVE THAN DROPPING T.

(x) Heat Exchange: In the Heat equation  $W_s$  represents the rate of heat exchange between species - clearly by the 2nd law of thermodynamics this is proportional to  $T_e - T_i$ . WE WANT A ONE FLUID DESCRIPTION so WE WANT  $T_i = T_e$ . Thus we want the collisional heat exchange to be LARGE.

$$\frac{T_e - T_i}{t_{eq.}} \gg \frac{\partial T}{\partial t} \sim \frac{V}{a} T$$

$$t_{eq.} = \text{Temperature equilibration time} \approx \gamma_i^{-1} \left( \frac{m_i}{m_e} \right)^{1/2} \quad \left. \begin{array}{l} \text{(I will show} \\ \text{you how to} \\ \text{calculate this} \\ \text{later)} \end{array} \right\}$$

Thus the IONS AND ELECTRONS HAVE EQUAL TEMPERATURES IF

$$\left( \frac{m_i}{m_e} \right)^{1/2} \frac{V}{a \gamma_i} \ll 1 \quad \text{SAME AS ABOVE.}$$

$$T_e = T_i \quad \text{and from before } n_e = n_i$$

(xi) ADDING THE ION AND ELECTRON MOMENTUM EQUATIONS:

$$m_i n \left[ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right] = - \nabla \underbrace{(p_e + p_i)}_P + \rho \underline{E} + \frac{\underline{J} \times \underline{B}}{c}$$

$$\text{but } c \frac{\rho \underline{E}}{\underline{J} \times \underline{B}} \sim \frac{c(\nabla \cdot \underline{E}) \underline{E}}{\underline{J} \underline{B}} \sim \frac{\underline{V}^2}{c^2} \ll 1 \text{ so}$$

so

### MHD EQUATIONS

ONE FLUID:

✓ CONTINUITY

$$\frac{\partial n}{\partial t} + \nabla \cdot (n \underline{V}) = 0$$

EVOLVES  $n$

✓ FARADAY'S LAW

$$\frac{\partial \underline{B}}{\partial t} = -c \nabla \times \underline{E}$$

EVOLVES  $\underline{B}$

✓ OHM'S LAW

$$\underline{E} + \frac{\underline{V} \times \underline{B}}{c} = \eta \underline{J}$$

DETERMINES  $\underline{E}$

✓ AMPERE'S LAW

$$\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J}$$

DETERMINES  $\underline{J}$

✓ MOMENTUM EQN.

$$m_i n \left[ \frac{\partial \underline{V}}{\partial t} + \underline{V} \cdot \nabla \underline{V} \right] = -\nabla p + \frac{\underline{J} \times \underline{B}}{c}$$

EVOLVES  $\underline{V}$

✓ ENERGY / HEAT / ENTROPY EQN.

$$\frac{d}{dt} \left( \frac{P}{n^{5/3}} \right) = 0$$

Evolves  $P$ .

It is common to simply eliminate  $\underline{J}$  and  $\underline{E}$  so that Faraday's Eqn. becomes.

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{V} \times \underline{B}) - \frac{c^2 \nabla \times \eta \nabla \times \underline{B}}{4\pi} = \nabla \times (\underline{V} \times \underline{B}) + \frac{c^2 \eta}{4\pi} \nabla^2 \underline{B}$$

when  $\eta$  is constant

And the  $\underline{J} \times \underline{B}$  force becomes

$$\frac{\underline{J} \times \underline{B}}{c} = \frac{(\nabla \times \underline{B}) \times \underline{B}}{4\pi} = \frac{\underline{B} \cdot \nabla \underline{B} - \nabla B^2/2}{4\pi}$$

222a. Lecture #11FLUX FREEZING.

- (i) Today we talk about just one of the MHD equations.

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{V} \times \underline{B}) + \frac{c^2 \eta}{4\pi} \nabla^2 \underline{B}$$

comes from OHM'S LAW  
 $\underline{E} + \underline{V} \times \underline{B} = \eta \underline{J}$   
 FARADAY'S EQN.  
 $\frac{\partial \underline{B}}{\partial t} = -c \nabla \times \underline{E}$

- (ii) with  $\underline{V} = 0$  we have a diffusion equation  $\underline{B}$  field diffuses

away. FOR EXAMPLE A SOLUTION IS

$$\underline{B} = \left\{ e^{-\gamma t} \sin k z \right\} \hat{x} \quad \text{with} \quad \gamma = \frac{c^2 \eta k^2}{4\pi}$$

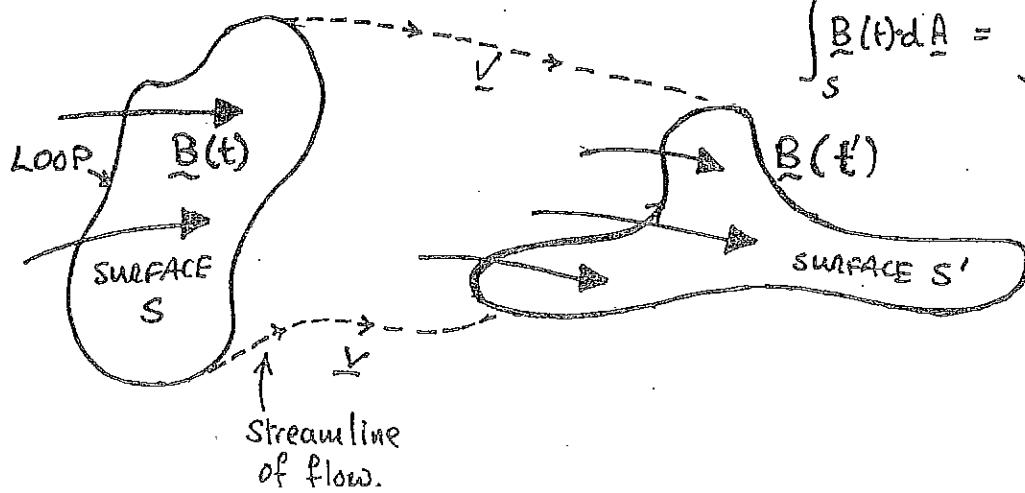
decays away

- (iii) OFTEN  $\eta$  IS VERY SMALL AND WE HAVE

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{V} \times \underline{B}) \quad \text{"IDEAL MHD"}$$

SEVERAL INTERESTING THEOREMS FOLLOW:

- (iv) THEOREM I. The magnetic flux through a surface moving plasma with the fluid (at velocity  $\underline{V}$ ) remains constant.



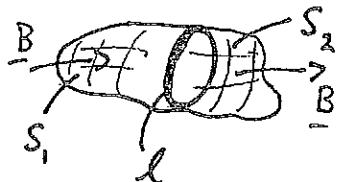
(v)

**Proof.**a) Is an important fact. For a closed surface  $S''$ 

$$\oint_{S''} \underline{B} \cdot d\underline{A} = \int \nabla \cdot \underline{B} dV = 0$$



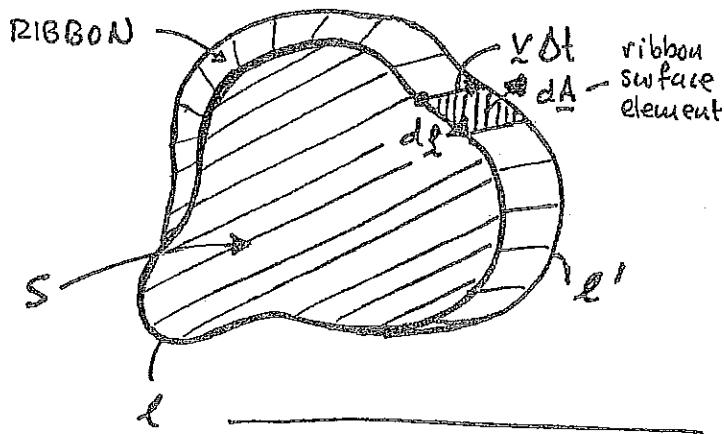
$\Rightarrow$  Take a loop  $\ell$  and span it by 2 surfaces,  $S_1, S_2$ .



$$\oint_{S_1} \underline{B} \cdot d\underline{A} = \oint_{S_2} \underline{B} \cdot d\underline{A}$$

WHEN  $d\underline{A}$  is taken in the same  
"sense"

Flux through any loop spanning  $\ell$  is the same.

b) Take a infinitesimal time  $\Delta t = t - t'$ .

$S$  = surface spanning  $\ell$  at time  $t$ .

$S'$  = surface spanning  $\ell'$  at time  $t'$ .

$\equiv S + \text{ribbon connecting } \ell \text{ to } \ell'$

$$d\underline{A} = \Delta t \times d\underline{l}$$

$$\Phi(t) = \int_S \underline{B}(t) \cdot d\underline{A}$$

$$\Phi(t + \Delta t) = \int_{S'} \underline{B}(t + \Delta t) \cdot d\underline{A} = \int_S \underline{B}(t + \Delta t) \cdot d\underline{A} + \int_{\text{ribbon}} \underline{B}(t + \Delta t) \cdot d\underline{A}$$

Expanding for  $\Delta t \rightarrow 0$

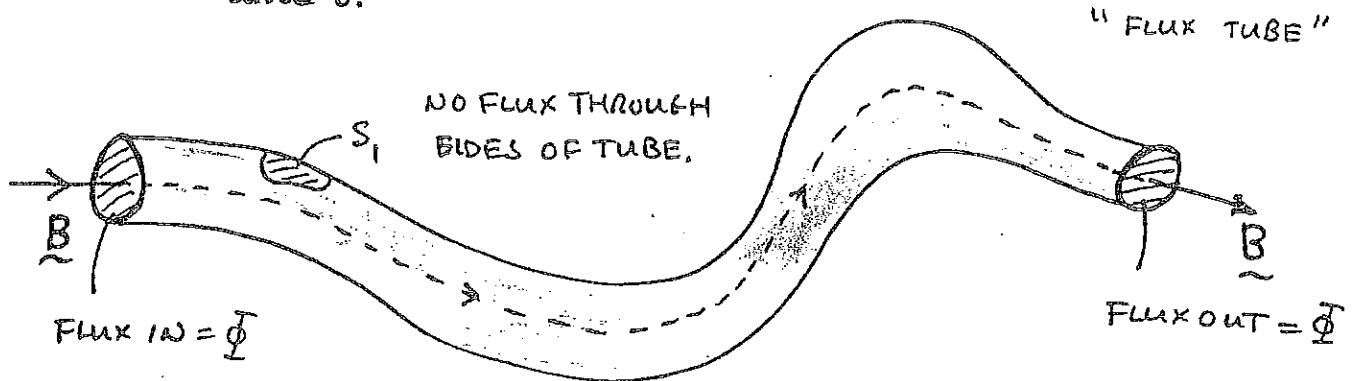
$$= \int_S \underline{B}(t) d\underline{A} + \Delta t \int_S \frac{\partial \underline{B}}{\partial t} \cdot d\underline{A} + \Delta t \int_{\ell} \underline{B} \cdot (\Delta t \times d\underline{l})$$

$$\Rightarrow \frac{d\Phi}{dt} = \int_S \left\{ \frac{\partial \underline{B}}{\partial t} - \nabla \times (\underline{v} \times \underline{B}) \right\} \cdot d\underline{A} = 0 \quad \text{Q.E.D.}$$

(vi) Theorem I is called Flux Freezing, for obvious reasons.

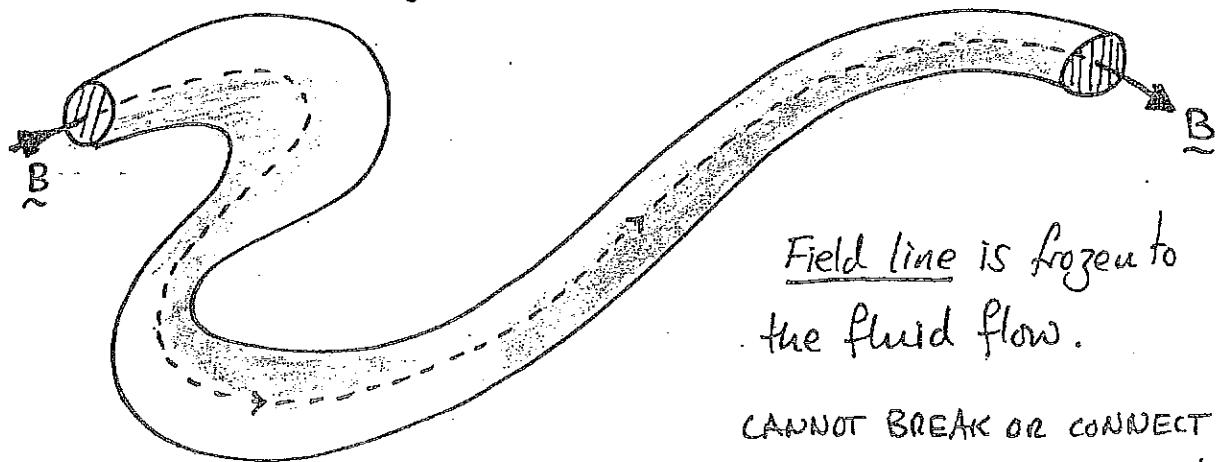
(vii) Theorem II. The magnetic field lines are frozen to the plasma/fluid flow.

a) Consider a tube of plasma surrounding a field line at time  $t$ .



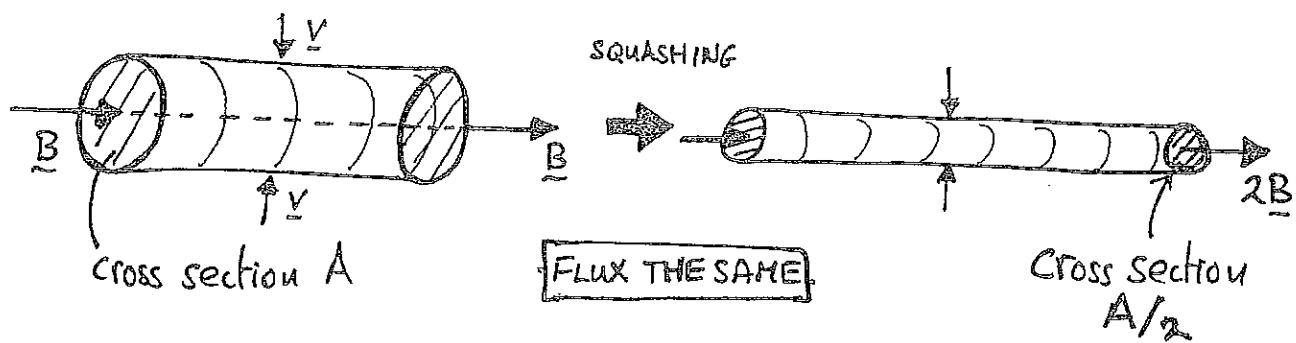
b) Let tube move with velocity  $v$  - i.e. with plasma/fluid.

- At time  $t'$  the flux through  $S_1$  must still be zero - this must be true for all parts of the tube sides.  
the flux through the ends must still be  $\Phi$ .  
 $\Rightarrow$  Field line still goes down tube
- THEOREM I.



4

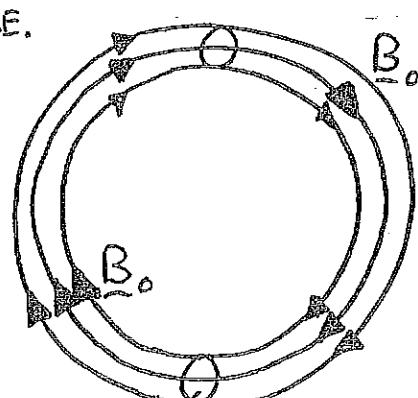
## Amplifying Field by squashing.



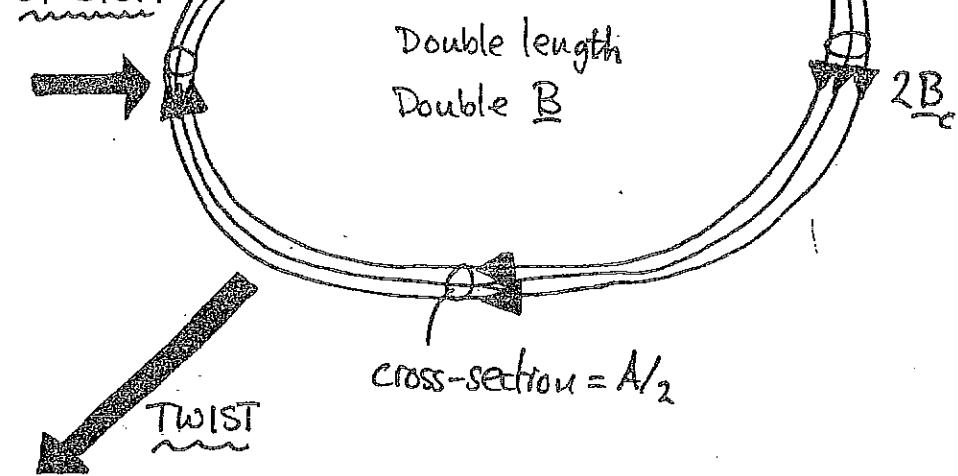
## Zeldovitch's Rope Dynamo:

CIRCULAR FLUX

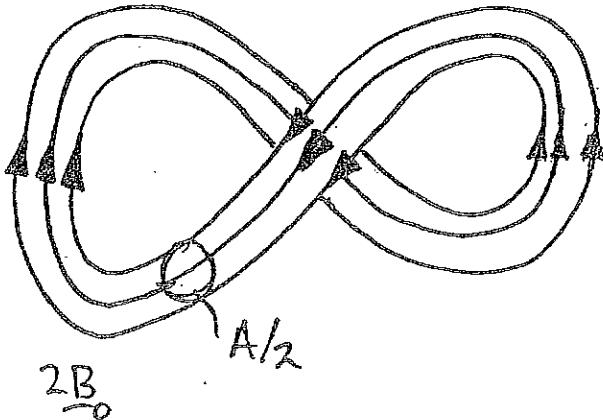
TUBE.



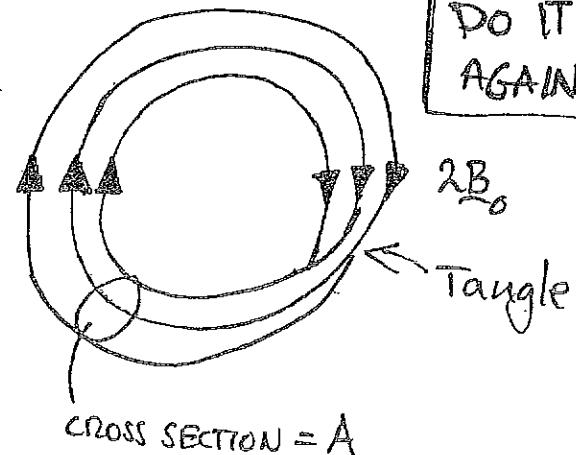
STRETCH



TWIST



FOLD



Nearly the same as the start except  $\underline{B}$  has doubled.  
and there is a tangle.

Physics 222a: Lecture # 12Forces and Equilibrium in MHD.

(i) Last time we discussed the evolution of  $\underline{B}$  — flux freezing etc.

(ii)  $\boxed{\text{FORCE PER UNIT VOLUME ON PLASMA}} = -\nabla p + \frac{\underline{J} \times \underline{B}}{c} + m_{\text{ing}}$

pressure force      magnetic force      gravitational force.

(iv) Net pressure force on a volume  $= - \int_V \nabla p \, d^3r = \oint_S p \, d\underline{A}$  [simple vector identity]

$S \rightarrow \partial V / \partial$   
= pressure forces on surface.

(v)  $\boxed{\text{Magnetic Force Density}} = \frac{\underline{J} \times \underline{B}}{c}$  PERPENDICULAR TO  $\underline{B}$

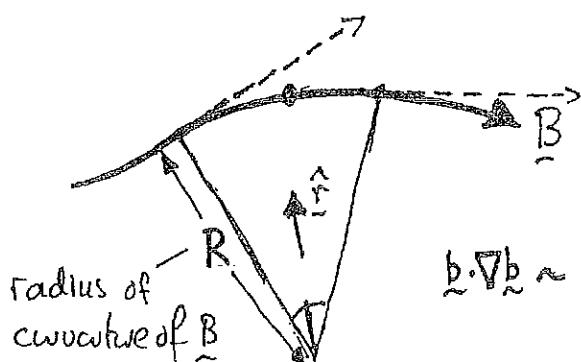
use  $\nabla \times \underline{B} = \frac{4\pi}{c} \underline{J} \rightarrow \frac{\underline{J} \times \underline{B}}{c} = \frac{1}{4\pi} (\nabla \times \underline{B}) \times \underline{B}$

note:  $\nabla \cdot \underline{J} = 0$

$$= -\nabla \frac{B^2}{8\pi} + \frac{\underline{B} \cdot \nabla \underline{B}}{4\pi}$$

MAGNETIC PRESSURE  $= \frac{B^2}{8\pi}$

"MAGNETIC TENSION FORCE"  
"FIELD LINE BENDING."



$$\underline{b} \cdot \nabla \underline{b} \sim -\frac{\underline{k}}{R}$$

"CURVATURE VECTOR"

$$\underline{B} \cdot \nabla \underline{B} = (\underline{B} \cdot \nabla |\underline{B}|) \underline{b} + \underline{B}^2 \underline{b} \cdot \nabla \underline{b}$$

Along  $\underline{B}$   
||

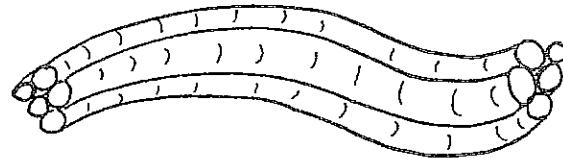
$$\underline{b} \cdot \nabla \frac{\underline{B}^2}{2}$$

perpendicular magnetic pressure.

curvature force.

$$\frac{\underline{J} \times \underline{B}}{c} = \frac{1}{4\pi} \left\{ -\nabla_{\perp} \frac{B^2}{2} - \frac{\underline{k} \cdot \underline{B}^2}{R} \right\}$$

(vi) Loosely I think of the magnetic field acting like a collection of bungee cords that are hard to compress and bend.



"(vii) MHD EQUILIBRIUM.  $\equiv$  FORCE BALANCE NOT THERMAL EQUILIBRIUM"

$\underline{\underline{\nabla p}} = \underline{\underline{J}} \times \underline{\underline{B}} + \rho g$  gives.

$$\underline{\underline{\nabla p}} = \underline{\underline{J}} \times \underline{\underline{B}} + \rho g$$

$\rho = n m_i = \frac{\text{mass}}{\text{density}}$

without gravity.

$$\boxed{\underline{\underline{\nabla p}} = \underline{\underline{J}} \times \underline{\underline{B}}}$$

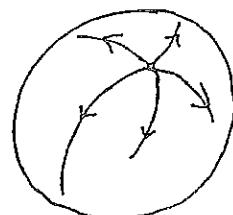
$$\underline{\underline{B}} \cdot \underline{\underline{\nabla p}} = 0$$

$$\underline{\underline{J}} \cdot \underline{\underline{\nabla p}} = 0$$

(viii) FUSION DEVICES MUST HAVE PRESSURE CONTAINMENT (HOT IN THE CENTER).

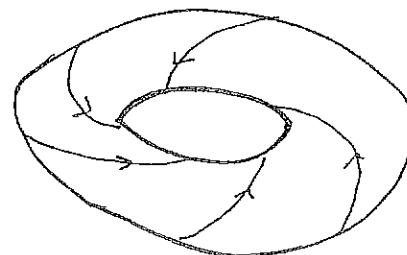
$\Rightarrow p$  - must have closed surfaces.

SPHERICAL SURFACE WON'T WORK AS  $\underline{\underline{B}}$  can't be divergence free.



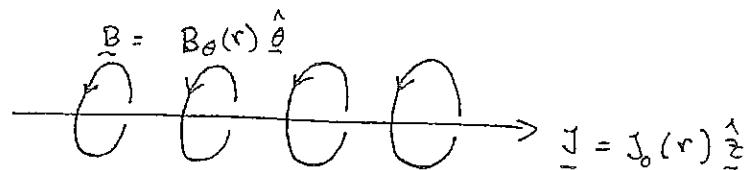
CAN'T "COMB" HAIR EVERYWHERE.

TOROIDAL SURFACE:  
WILL WORK:



## SIMPLE EQUILIBRIA:

(ix) z pinch: z direction of the current.



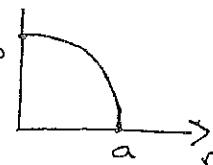
$$\nabla \times \underline{\underline{B}} = \frac{4\pi}{c} J_0(r) \hat{z} \Rightarrow \boxed{\frac{1}{r} \frac{\partial r B_\theta}{\partial r} = \frac{4\pi}{c} J_0(r)}$$

$$\frac{\nabla \times \underline{\underline{B}}}{c} = \frac{1}{4\pi} \left[ \frac{1}{r} \frac{\partial (r B_\theta)}{\partial r} \right] \hat{z} \times B_\theta \hat{z} = -\frac{1}{4\pi r^2} \frac{\partial}{\partial r} \frac{(r B_\theta)^2}{2} \hat{z} = \nabla p$$

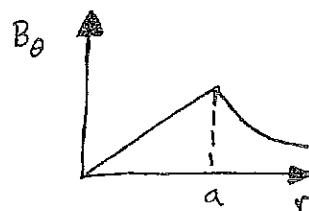
$$\Rightarrow p = p(r) \quad \text{and}$$

$$\boxed{(r B_\theta)^2 = -8\pi \int_0^r dr' r'^2 \frac{dp}{dr'}} \quad \text{EQUILIBRIUM.}$$

$$\text{FOR DEFINITENESS TAKE: } p = p_0 \left(1 - \frac{r^2}{a^2}\right) : r < a \\ = 0 \quad r > a$$



$$B_\theta^2 = 4\pi p_0 \frac{r^2}{a^2} : r < a$$

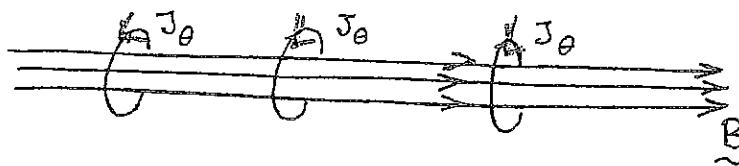


$$B_\theta^2 = 4\pi p_0 \frac{a^2}{r^2} : r > a$$

MAGNETIC FORCES HOLD IN THE PRESSURE: PINCHES. "PINCH EFFECT"

"θ pinch":  $\underline{J} = J_\theta(r) \hat{\underline{\theta}}$ ,  $\underline{B} = B_0(r) \hat{\underline{z}}$

$$-\frac{\partial B_0}{\partial r} = \frac{4\pi}{c} J_\theta(r)$$



$$\frac{\underline{J} \times \underline{B}}{c} = -\frac{1}{4\pi} \frac{\partial B_0}{\partial r} \hat{\underline{z}} \times B_0 \hat{\underline{z}} = -\frac{1}{4\pi} \frac{\partial}{\partial r} \left( \frac{B_0^2}{2} \right) \hat{\underline{r}}$$

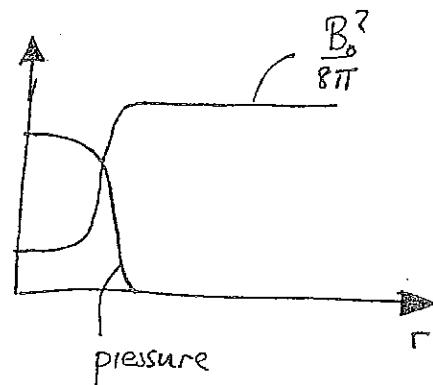
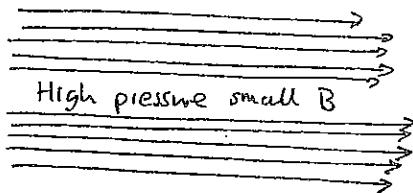
JUST MAGNETIC PRESSURE

$$= + \frac{\partial P}{\partial r} \quad \Rightarrow$$

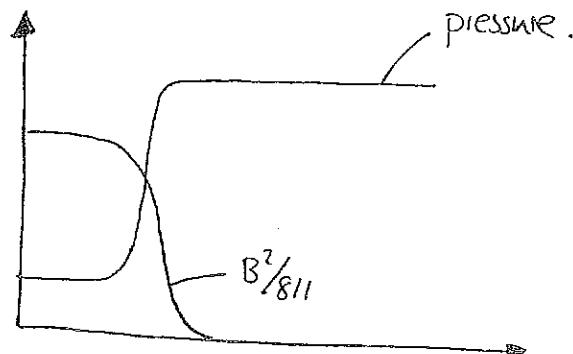
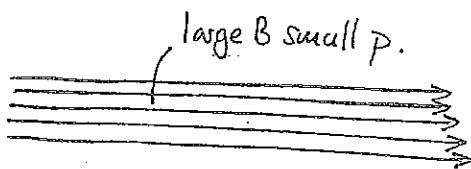
$$\frac{\partial}{\partial r} \left\{ \frac{B_0^2}{8\pi} + P \right\} = 0$$

$$\frac{B_0^2}{8\pi} + P = \text{constant}$$

PRESSURE CONFINEMENT:



FLUX TUBE "PRESSURE EVACUATION"



if  $T = \text{constant}$  the  $n \propto p$  and flux tube is less dense  $\Rightarrow$  buoyant in gravitational field.

## 222a Lecture #13 MHD Waves.

(i) Today we look at the waves in a homogeneous plasma obeying the MHD equations. (NO RESISTIVITY)

(ii) Let  $\underline{B} = B_0 \underline{b} + \delta \underline{B}$  :  $B_0$  &  $\underline{b}$  constant and  $|\underline{b}| = 1$ .  $|\delta \underline{B}| \ll B_0$

$$p = p_0 + \delta p : \quad p_0 \text{ a constant} \quad \delta p \ll p_0$$

$$\rho = \rho_0 + \delta \rho : \quad \rho_0 \text{ a constant} \quad \delta \rho \ll \rho_0$$

$$\underline{v} = \delta \underline{v} : \quad (\underline{v}_0 = 0 \text{ no equilibrium flow})$$

$$= \frac{\partial \xi}{\partial t} \quad \xi \equiv \text{displacement of plasma} = \xi(\underline{r}, t)$$

(iii) LINEARIZE EQUATIONS

keep only terms linear in small quantities  $\delta \underline{B}$  etc.

(?) PRESSURE EQN:  
From.  $\frac{P}{\rho} = \text{constant}$

$$\frac{\partial p}{\partial t} + \underline{v} \cdot \nabla p = -\gamma p \nabla \cdot \underline{v} \Rightarrow \delta p = -\gamma p_0 \nabla \cdot \xi$$

$$\delta p = -\gamma p_0 \nabla \cdot \xi$$

pressure perturbation comes from compression

B EQUATION:

$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B}) \Rightarrow$$

$$\delta \underline{B} = \underline{B}_0 \nabla \xi_{\perp} - \underline{B}_0 (\nabla \cdot \xi_{\parallel})$$

$$\xi = \xi_{\perp} + \xi_{\parallel} \underline{b}$$

↑ perpendicular displacement

MOMENTUM EQN.

$$\rho \left[ \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} \right] = -\nabla p + \frac{\underline{J} \times \underline{B}}{c}$$

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = -\nabla \delta p + \left( \nabla \times \delta \underline{B} \right) \times \underline{B}_0$$

• Notice  $\delta p$  is not needed.

After some Algebra we can deduce equations for  $\xi$

parallel Forces: ①  $\rho_0 \frac{\partial^2 \xi_{\parallel}}{\partial t^2} = +\gamma p_0 \underline{b} \cdot \nabla \underline{b} \cdot \nabla \xi_{\parallel} + \gamma p_0 \underline{b} \cdot \nabla (\nabla \cdot \xi_{\perp})$

NO MAGNETIC FORCE ALONG  $\underline{b}$

COMPRESSION OF  $p_0$  ALONG  $\underline{B}$

COMPRESSION OF  $p_0$  ACROSS  $\underline{B}$

perpendicular Forces: ②  $\rho_0 \frac{\partial^2 \xi_{\perp}}{\partial t^2} = \nabla_{\perp} \left\{ \delta p_0 \underline{b} \cdot \nabla \xi_{\parallel} + \left( \gamma p_0 + \frac{B_0^2}{4\pi} \right) \nabla \cdot \xi_{\perp} \right\} + \frac{B_0^2}{4\pi} \underline{b} \cdot \nabla (\underline{b} \cdot \nabla \xi_{\perp})$

COMPRESSION OF  $p_0$  ALONG  $\underline{b}$

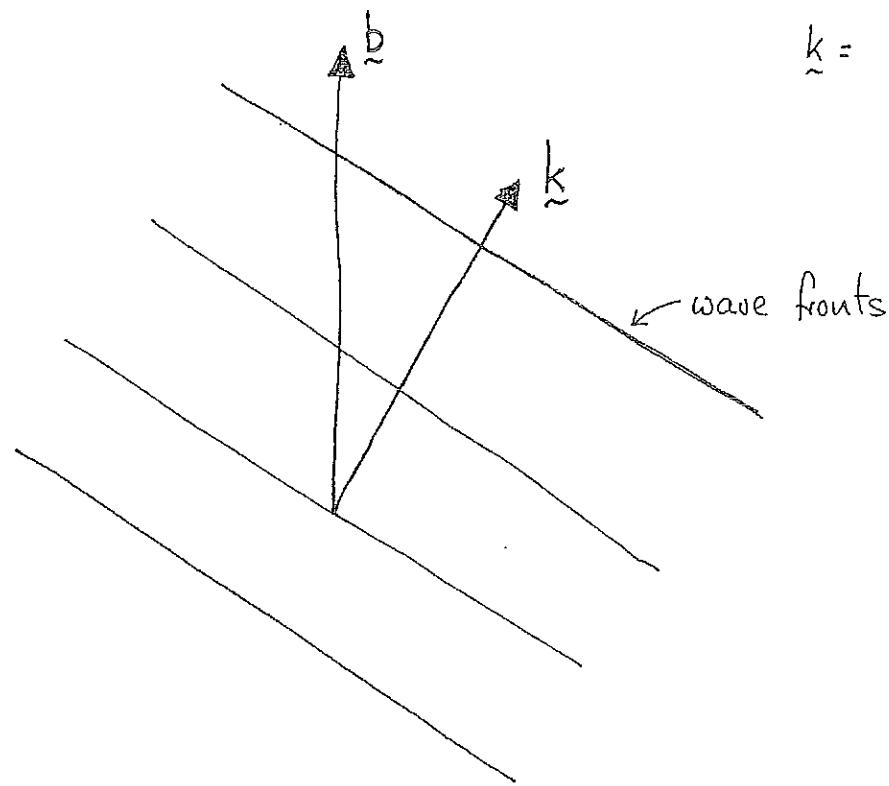
COMPRESSION OF TOTAL PRESSURE ACROSS FIELD

FIELD LINE BENDING FORCE

(iv) Take the wave form

$$\xi = \hat{\xi} e^{ik \cdot \vec{r} - i\omega t}$$

$$\underline{k} = \underline{k}_{\perp} + k_{\parallel} \underline{b}$$



(v) Start with special cases: (generalize shortly)

### PARALLEL PROPAGATION

$$\nabla_{\perp} \equiv 0 \equiv i \underline{k}_{\perp}$$

$$\underline{b} \cdot \nabla = i k_{\parallel} \equiv \nabla_{\parallel}$$

a) Parallel Force  $\Rightarrow (\omega^2 - k_{\parallel}^2 c_s^2) \hat{\xi}_{\parallel} = 0$

$$\rho_0 \frac{\partial^2 \hat{\xi}_{\parallel}}{\partial t^2} = \gamma \rho_0 \nabla_{\parallel}^2 \hat{\xi}_{\parallel}$$

$$c_s^2 = \frac{\gamma \rho_0}{\rho_0} \quad c_s = \text{SOUND SPEED}$$

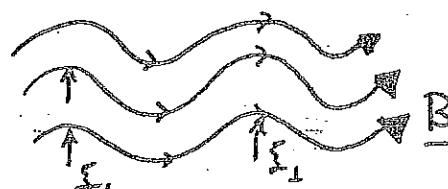
SOUND WAVES

$B$  field unpermeated

$$\omega = \pm k_{\parallel} c_s$$

### b) Perpendicular Forces

$$\rho_0 \frac{\partial^2 \hat{\xi}_{\perp}}{\partial t^2} = \frac{B_0^2}{4\pi} \nabla_{\parallel}^2 \hat{\xi}_{\perp}$$



$$(\omega^2 - k_{\parallel} V_A^2) \hat{\xi}_{\perp} = 0$$

$$V_A^2 = \frac{B_0^2}{4\pi \rho_0} \quad V_A = \text{Alfvén Speed.}$$

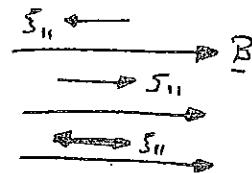
$$\omega = \pm k_{\parallel} V_A$$

SHEAR ALFVÉN  
WAVES

TWO POLARIZATIONS OF SHEAR ALFVÉN WAVES.

$$(vi) \text{ PERPENDICULAR PROPAGATION: } \nabla_{\perp} = i \underline{k}_{\perp} \quad \underline{\underline{\delta}} \cdot \nabla = 0$$

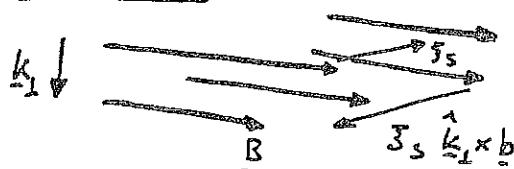
a) Parallel Force:  $\omega^2 \xi_{\parallel} = 0$  (sound wave)



b) Perpendicular Force  $\omega^2 \hat{\xi}_{\perp} = k_{\perp} (\underline{k}_{\perp} \cdot \xi_{\perp}) (c_s^2 + v_A^2)$

$\therefore$  either :-  $\omega^2 = 0 \quad \hat{\xi}_{\perp} = \xi_s (\hat{k}_{\perp} \times \underline{b}) \leftarrow$  (Shear Alfvén wave.)  
or : -  $\omega^2 = k_{\perp}^2 (c_s^2 + v_A^2) \quad \hat{\xi}_{\perp} = \xi_c \hat{k}_{\perp}$

$\omega^2 = 0$  mode

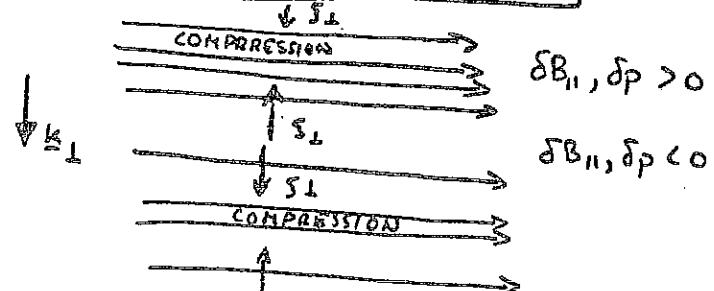


MOVES STRAIGHT FIELD LINES  
AROUND WITHOUT BENDING  
OR COMPRESSION.

"INTERCHANGE"

MAGNETO-ACOUSTIC WAVE.

$\omega^2 = k_{\perp}^2 (c_s^2 + v_A^2)$  Mode



Like a sound wave only it  
compresses pressure and field.

### (vi) OBLIQUE PROPAGATION - THE GENERAL CASE:

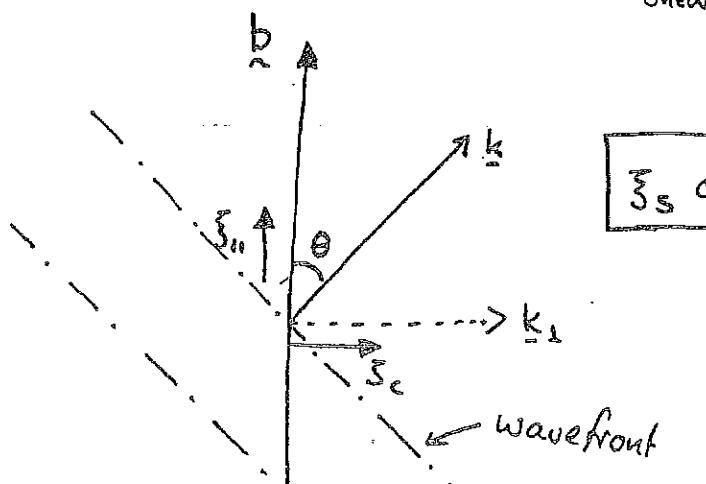
Decompose  $\xi$  into:

$$\xi = \xi_{\parallel} \underline{b} + \xi_s \hat{k}_{\perp} \times \underline{b} + \xi_c \hat{k}_{\perp}$$

"Shear part"

"compressional part"  
note:  $\nabla \cdot [\xi_s (\hat{k}_{\perp} \times \underline{b})]$

$$= i \underline{k} \cdot \hat{k}_{\perp} \times \underline{b} \xi_s = 0$$



$\xi_s$  out of paper

$$k \cdot b = |k| \cos \theta$$

4

Parallel Force Eqn. Becomes (see ①)

$$(\omega^2 - k_{\parallel}^2 c_s^2) \xi_{\parallel} = c_s^2 k_{\parallel} k_{\perp} \xi_c \quad - ③ \text{ b directed}$$

perpendicular compression drives sound waves along  $\mathbf{b}$

Perpendicular Force Becomes (see ②)

$$(\omega^2 - k_{\parallel}^2 V_A^2) \xi_s = 0 \quad - ④ \quad \hat{k}_{\perp} \times \hat{\mathbf{b}} \text{ direction}$$

and

$$(\omega^2 - k_{\parallel}^2 V_A^2 - k_{\perp}^2 (c_s^2 + V_A^2)) \xi_c = c_s^2 k_{\parallel} k_{\perp} \xi_{\parallel} \quad - ⑤ \quad \hat{k}_{\perp} \text{ dir.}$$

parallel compression produces force across  $\mathbf{B}$

3 WAVES

④  $\Rightarrow$  SHEAR ALFVEN WAVES  
"INTERMEDIATE WAVE"

$$\omega^2 = k_{\parallel}^2 V_A^2$$

$$\xi = \xi_s (\hat{k}_{\perp} \times \hat{\mathbf{b}})$$

$$\frac{\omega^2}{k^2} = \cos^2 \theta V_A^2$$



③ & ⑤ Yield

$$\frac{\omega^4}{k^4} - \frac{\omega^2}{k^2} (c_s^2 + V_A^2) + c_s^2 V_A \cos^2 \theta = 0$$

$$\frac{\omega^2}{k^2} = \frac{1}{2} (c_s^2 + V_A^2) \pm \frac{1}{2} \sqrt{(c_s^2 + V_A^2)^2 - 4 c_s^2 V_A^2 \cos^2 \theta}$$

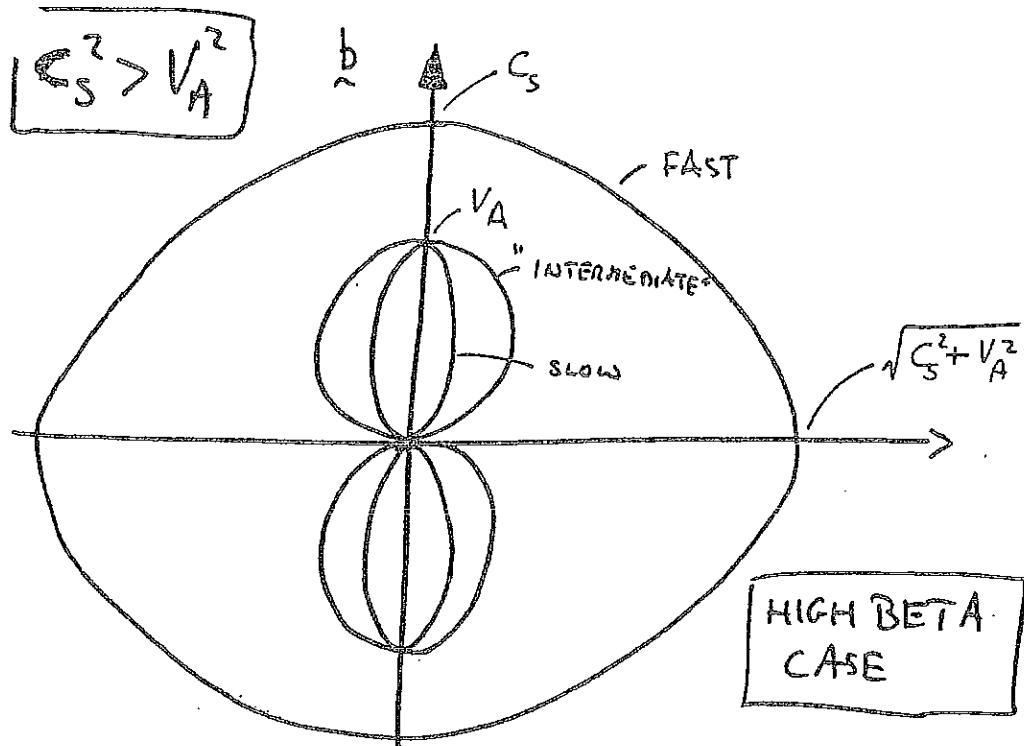
PLUS SIGN = FAST WAVE }

MINUS SIGN = SLOW WAVE }

Easy to show that:

$$\left( \frac{\omega}{k} \right)_{\text{FAST WAVE}} > \left( \frac{\omega}{k} \right)_{\text{SHEAR ALFVEN WAVE}} > \left( \frac{\omega}{k} \right)_{\text{SLOW WAVE}}$$

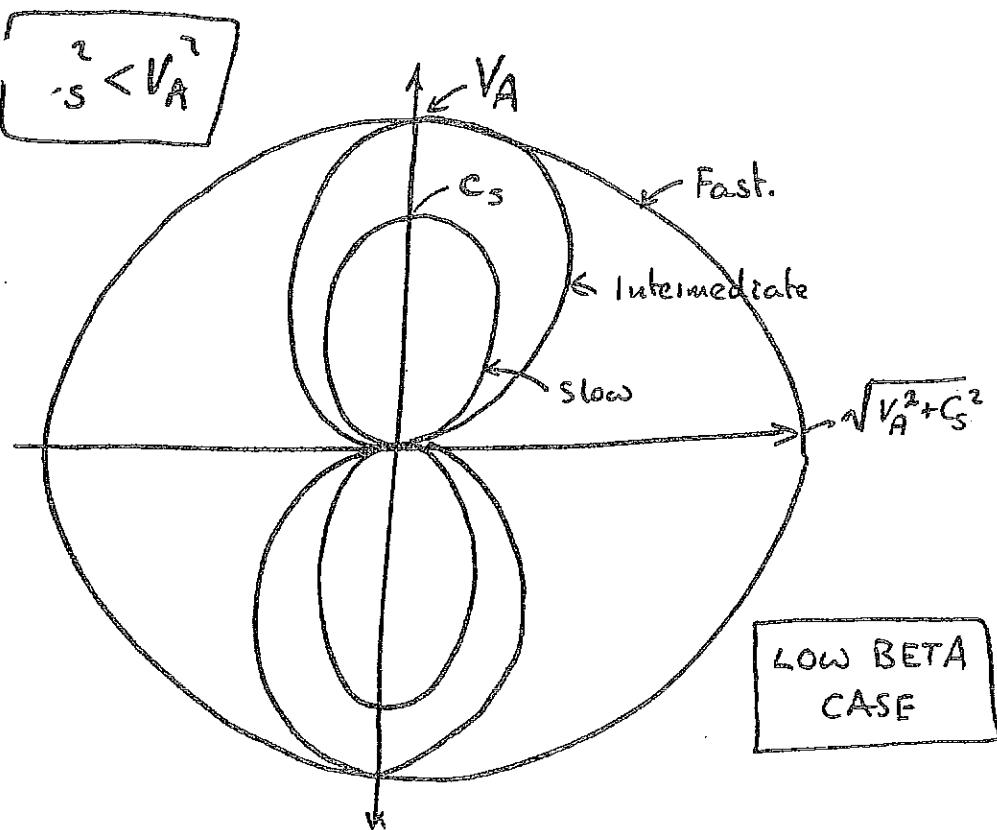
WAVES ARE OFTEN REPRESENTED AS POLAR PLOTS OF  $\frac{\omega}{k}$



$$c_s^2 > v_A^2$$

note  $\frac{c_s^2}{v_A^2} = 4\pi \frac{\delta P_0}{B_0^2} = \delta \beta$

$$\beta = \frac{4\pi \frac{P_0}{B_0^2}}{v_A^2}$$



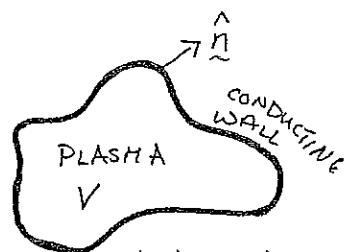


# Physics 222a. Lecture #13: Lagrangian MHD and Energy Principle

(i) MHD has a conserved energy:-

$$E = \int_V \left\{ \frac{\rho v^2}{2} + \frac{B^2}{8\pi} + \frac{P}{\gamma - 1} \right\} d^3\Sigma$$

KINETIC ENERGY      MAGNETIC ENERGY      PRESSURE ENERGY

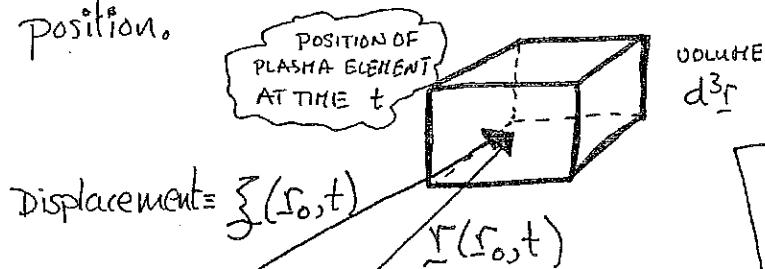


Take simple boundary conditions - no flow out of walls  $v \cdot n$

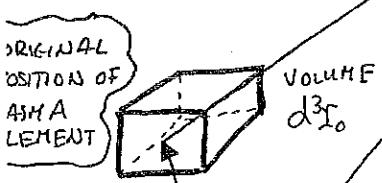
## Homework Q.1.

Using the MHD Equations show that  $\frac{dE}{dt} = 0$  with the simple boundary conditions given above.

(ii) LAGRANGIAN MHD: a convenient way to look at stability is using LAGRANGIAN variables. We write everything in terms of the displacement vector  $\xi(r_0, t)$  of each piece of plasma from its original position.



Displacement =  $\xi(r_0, t)$



$$\begin{aligned} \underline{r} &= \underline{r}_0 + \xi(r_0, t) \\ \text{or: } \underline{r} &= \underline{r}(r_0, t) \\ \text{or: } \underline{r}_0 &= \underline{r}_0(r, t) \end{aligned}$$

: PLAMA VELOCITY =  $v(r_0, t) = \left( \frac{\partial \underline{r}}{\partial t} \right)_{r_0} = \left( \frac{\partial \xi}{\partial t} \right)_{r_0}$

:  $\nabla_{r_0} \xi$  = gradient with respect to  $\underline{r}_0$ .

:  $\nabla_{r_0} \underline{r} = \text{"strain matrix"} = \underline{I} + \nabla_{r_0} \xi$

note  $\nabla_{r_0} \underline{r} \cdot \nabla \underline{r}_0 = \underline{I}$  by chain rule.

$$\begin{aligned} \underline{I} &= \text{unit, "identity matrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

(iii) All quantities can be thought of as either functions of  $\xi$  or  $\xi_0$  and  $t$ . For example density in the displaced box can be written as

$$\rho = \rho_E(\xi, t) \quad \text{or} \quad \rho = \rho_L(\xi_0, t)$$

so the functions are related by

$$\rho_E(\xi_0 + \xi(\xi_0, t), t) = \rho_L(\xi_0, t)$$

For small displacements :- ie.  $\xi \ll L \equiv |\nabla^{\text{displ}}|^{-1}$

$$\rho_L(\xi_0, t) \approx \rho_E(\xi_0, t) + \xi \cdot \nabla \rho_E \dots \dots \dots$$

(iv) We can integrate the equations of MHD using the Lagrangian displacement.

Density - Conservation of Mass defines  $\rho_0(\xi_0) = \rho_L(\xi_0, 0)$

$$\therefore \text{Mass in piece of plasma} = dm = \rho_0(\xi_0) d^3 \xi_0 = \rho_L(\xi_0, t) d^3 \xi$$

BUT from the rules of partial differentiation

$$d^3 \xi = J d^3 \xi_0$$

where  $J = \text{JACOBIAN} = |\nabla_0 \xi| \equiv \text{DETERMINANT OF STRAIN MATRIX}$

(v) THUS

$$\text{① } \rho_L(\xi_0, t) = \frac{\rho_0(\xi_0)}{J(\xi_0, t)}$$

gives density in terms of  $\xi(\xi_0, t)$

Steve Cowley.

i) PRESSURE - conservation of Entropy recall  $\frac{d}{dt} \left( \frac{P}{\rho^\gamma} \right) = \left( \frac{\partial}{\partial t} \left( \frac{P}{\rho^\gamma} \right) \right)_{r_0} = 0$

$\frac{P}{\rho^\gamma}$  = constant for element of plasma.

$$\rightarrow P_L(\xi_0, t) = \frac{P_0(\xi_0)}{\gamma^\gamma} \quad \text{--- (2)}$$

(ii) Acceleration  $\left( \frac{\partial}{\partial t} \right)_r + \underline{v} \cdot \nabla = \left( \frac{\partial}{\partial t} \right)_{\xi_0}$

$$\rightarrow \frac{\partial \underline{v}}{\partial t} + \underline{v} \cdot \nabla \underline{v} = \frac{\partial^2 \xi}{\partial t^2}$$

(vii) MAGNETIC FIELD - FLUX CONSERVATION

Homework Q.2. From  $\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{v} \times \underline{B})$  show that

Reference Horace Lamb, Hydrodynamics, 1879

$$\underline{B}_L(\xi_0, t) = \frac{B_0(\xi_0) \cdot \nabla \xi_0}{\gamma} \quad \text{--- (3)}$$

this is called "Lundquist's identity" by some.

or If you like you can prove this geometrically.

(viii) We have "integrated" all the equations of MHD except the force<sup>or momentum</sup> equation: The force equation yields an equation for  $\xi$  the displacement.

三

(ix) Substituting ①, ② & ③ into the force equation we obtain

$$\frac{P_0}{J} \frac{\partial^2 \xi}{\partial t^2} = - \nabla \xi \cdot \nabla \left\{ P_0 J^{-\delta} + \frac{(B_0 \cdot \nabla \xi)^2}{J^2 8\pi} \right\} + \frac{B_0 \cdot \nabla}{J} \left( \frac{B_0 \cdot \nabla \xi}{J} \right)$$

This is the equation of motion for a sort of elastic medium where the right hand side is a force dependant on the displacement  $\underline{F}(\xi_0, \xi, t)$ .

(X) The ENERGY is also easily written as:

$$E = \int_V \left\{ \frac{\rho_0}{2} \left( \frac{\partial \xi}{\partial t} \right)^2 + \frac{(B_0 \cdot \nabla \xi)^2}{\gamma 8\pi} + \frac{\rho_0 J^{(1-\gamma)}}{\gamma - 1} \right\} d^3 r$$

KINETIC ENERGY                            POTENTIAL ENERGY                            1

AN ASIDE FOR THE THEORETICALLY Minded.

The equation of motion can be derived from an action principle

$$S(\xi) = \int_{t_0}^t dt \int d^3r_0 L \quad \text{with} \quad \frac{\delta S}{\delta \xi} = 0$$

variation  
with end  
points fixed.

where

$$L = \frac{P_0}{2} \left( \frac{\partial \xi}{\partial t} \right)^2 - \frac{(B_0 \cdot \nabla \xi)^2}{\gamma 8\pi} - \frac{P_0 J^{1-\gamma}}{\gamma - 1}$$

"Lagrangian density"

(xi) For small displacements ( $\xi$ ) we can expand the energy to quadratic order to get.

$$E_0 = \int d^3r_0 \left\{ \frac{p_0}{\gamma - 1} + \frac{B_0^2}{8\pi} \right\}$$

$$E_1 = \int d^3r_0 \xi \cdot \left\{ \nabla p_0 - \frac{\underline{J}_0 \times \underline{B}_0}{c} \right\}$$

$$\underline{J}_0 = \frac{\nabla_0 \times \underline{B}_0}{c} \text{ of course}$$

NOT THE JACOBIAN!

$$E_2 = \int d^3r_0 \left\{ \frac{p_0}{2} \left( \frac{\partial \xi}{\partial t} \right)^2 \right\} + \delta W(\xi, \dot{\xi})$$

"called delta W" Formulas to be given

If we assume the system starts in an equilibrium

then  $\nabla_0 p_0 - \underline{J}_0 \times \underline{B}_0 = 0$  and therefore neglecting higher orders

$E_2 = \text{constant}$

since  $E_0 = \text{constant}$  and  $E = \text{constant}$ .

(xii) **FORMS FOR  $\delta W$**  equivalent with integrations by parts

$$\delta W = \frac{1}{2} \int_V d^3r_0 \left[ |Q + \hat{n} \cdot \xi (\underline{J}_0 \times \hat{n})|^2 + \gamma p_0 |\nabla \cdot \xi|^2 - 2 (\underline{J}_0 \times \hat{n}) \cdot \underline{B}_0 \cdot \nabla \hat{n} |\hat{n} \cdot \xi|^2 \right]$$

where  $Q = \nabla \times (\xi \times \underline{B}_0)$        $\hat{n} = \frac{\nabla_0 p_0}{|\nabla_0 p_0|}$        $K = \underline{b} \cdot \nabla_0 \underline{b}_0$

	<b>BENDING ENERGY</b>	<b>MAGNETIC COMPRESSION</b>	<b>PRESSURE COMPRESSION</b>
--	-----------------------	-----------------------------	-----------------------------

or  $\delta W = \frac{1}{2} \int d^3r_0 \left[ |Q_{\perp}|^2 + B_0^2 |\nabla \cdot \xi_{\perp} + 2 \xi_{\perp} \cdot K|^2 + \gamma p_0 |\nabla \cdot \xi|^2 - 2 \xi_{\perp} \cdot \nabla p_0 \cdot K \cdot \xi_{\perp} \right]$

"KINK DRIVE"

$Q_{\perp}^2 = \underline{J}_0 \cdot \underline{B}_0 \cdot \xi_{\perp} \times \underline{B}_0 \cdot Q$

(xiii) The force equation may be linearized. (some algebra.

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = F(\xi) \quad \text{--- (4)}$$

$$F(\xi) = (\nabla_0 \times B_0) \times Q + (\nabla_0 \times Q) \times B_0 + \nabla_0 (\xi \cdot \nabla_0 P_0 + \gamma P_0 \nabla_0 \cdot \xi)$$

(xiv) We differentiate

$$\frac{dE_2}{dt} = \int \rho_0 \frac{\partial \xi}{\partial t} \cdot \frac{\partial^2 \xi}{\partial t^2} d^3 r_0 + \delta W \left( \frac{\partial \xi}{\partial t}, \xi \right) + \delta W \left( \xi, \frac{\partial \xi}{\partial t} \right)$$

using (4) we obtain

$$\int_V \rho_0 \frac{\partial \xi}{\partial t} \cdot F(\xi) d^3 r_0 = - \left\{ \delta W \left( \xi, \frac{\partial \xi}{\partial t} \right) + \delta W \left( \frac{\partial \xi}{\partial t}, \xi \right) \right\}$$

THIS MUST BE TRUE AT  $t=0$  WHEN I CAN CHOOSE  $\xi$  AND  $\frac{\partial \xi}{\partial t}$   
ARBITRARILY HENCE FOR ALL  $\eta$  AND  $\xi$

$$\int \rho_0 \eta \cdot F(\xi) d^3 r_0 = - \left\{ \delta W (\xi, \eta) + \delta W (\eta, \xi) \right\}$$

clearly then  $F(\xi)$  is what we call "SELF ADJOINT"

$$\int \eta \cdot F(\xi) d^3 r_0 = \int \eta \cdot F(\xi) d^3 r_0$$

$$\frac{1}{2} \int d^3 r_0 \xi \cdot F(\xi) = - \delta W (\xi, \xi)$$

## Lecture #16: Stability and the Energy Principle.

(i) Last time we looked at the conservation of Energy in MHD.

We discovered the quadratically conserved energy for small displacements from equilibrium

we also had the force equation for small displacements

$$\rho_0 \frac{\partial^2 \xi}{\partial t^2} = E(\xi)$$

we showed the self-adjointness of the operator  $\hat{E}$  i.e.

$$\int d^3\tau_0 \eta \cdot F(\xi) = \int d^3\tau_0 \xi \cdot F(\eta)$$

PLEASE NOTE: Mistake/Typo last notes  $\equiv p_0$  in this formula.

It then followed that :-

$$\delta W = -\frac{1}{2} \int d^3 \xi_0 \, \xi \cdot F(\xi)$$

(ii) To have a linear instability the displacement must be able to grow indefinitely in time. We now show two important consequences of the energy formulation.

(iii) If  $\delta W > 0$  for all  $\xi$  then system is stable.

- we say  $\delta W > 0$  for all  $\xi$  is sufficient for stability

Proof. If  $\delta W > 0$  then

$$0 < \int d^3 r_0 \rho_0 \left( \frac{\partial \xi}{\partial t} \right)^2 = E_2 - \delta W(\xi, \xi) < E_2$$

the kinetic energy is bounded and cannot grow

(iv) If for some function  $\eta(r_0)$ ,  $\delta W(\eta, \eta) < 0$  then the system is unstable.

- we say  $\delta W > 0$  for all  $\xi$  is also necessary for stability.

Proof: Define  $\delta$  by  $\delta W(\eta, \eta) = -\gamma^2 \underbrace{\int d^3 r_0 \rho_0 \eta^2}_{K(\eta, \eta)} < 0$

Now by differentiating we get

$$\frac{1}{2} \frac{d^2}{dt^2} K(\xi, \xi) = K\left(\frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t}\right) - \delta W(\xi, \xi) \quad \text{--- (1)}$$

Now choose initial conditions  $\xi(r_0, t=0) = \eta(r_0)$

and  $\left(\frac{\partial \xi}{\partial t}\right)_{t=0} = \gamma \eta$  then  $E_2 = 0$  and  $\delta W = -K$  for all time

By Schwartz's inequality

$$\left[ \frac{d}{dt} K(\xi, \xi) \right]^2 \leq 4 K(\xi, \xi) K\left( \frac{\partial \xi}{\partial t}, \frac{\partial \xi}{\partial t} \right) \quad \text{--- (2)}$$

defining a new variable

$$y(t) = \ln \left( \frac{K(\xi, \xi)}{K(\eta, \eta)} \right)$$

$y(0) = 0$
$\left( \frac{dy}{dt} \right)_0 = 2\gamma$

then (1) becomes (using (2))

$$\frac{d^2 y}{dt^2} \geq 0 \Rightarrow \frac{dy}{dt} - 2\gamma \geq 0 \Rightarrow y \geq 2\gamma t$$

$$\text{or } K(\xi, \xi) \geq -\delta w(\eta, \eta) e^{2\gamma t}$$

THUS:  $\xi$  grows at least as fast as  $e^{\gamma t}$

Called unstable by Liapunoff's definition. QED

(v) You may now take the proofs as given and appreciate the simple consequences.

- $\delta w(\xi, \xi) > 0$  for all  $\xi$  is a necessary and sufficient condition for stability.

(vi) Thus if you can guess a simple  $\xi$  for which  $\delta w < 0$  then you have proved the system is unstable.

(vii) One can normalize  $\xi$  in any way that does not restrict its functional form and minimize  $\delta w$  to find out if it has a minimum below zero - if it doesn't it's stable, etc.

square integrable

If you have normal modes then we can write  $\xi = \hat{\xi}_n e^{\lambda n t}$

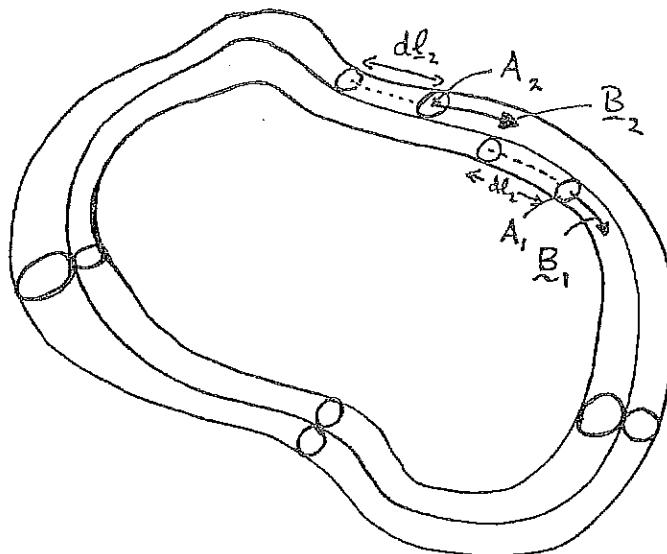
$$\rho \gamma_n^2 \hat{\xi}_n = F(\hat{\xi}_n)$$

All the usual results follow i.e.  $\gamma_n^2$  is real

orthogonality  $\int \rho_0 \hat{\xi}_m^* \hat{\xi}_n = 0$   
if  $\gamma_m \neq \gamma_n$

## Interchange Instability.

- Consider a plasma with closed (loops) of field.



TWO FLUX TUBES  
ADJACENT TO EACH OTHER.

- We Interchange these two flux tubes moving no other field lines in the system and ask if the potential energy goes up or down.

OF COURSE WE PRESERVE THE MAGNETIC FLUX AND THE ENTROPY OF EACH TUBE DURING THE INTERCHANGE.

- We treat all quantities as roughly constant across the tubes

$$V_1 = \text{volume of tube 1} = \int A_1 dl, \quad A_1 = A_1(l_1)$$

$$B_1 = B_1(l_1)$$

But since  $\Psi_1 = \text{magnetic flux of 1} = B_1 A_1 = \text{constant along tube}$

$$V_1 = \Psi_1 \int \frac{dl}{B_1} \quad \text{and obviously} \quad V_2 = \Psi_2 \int \frac{dl_2}{B_2}$$

### CHANGE IN MAGNETIC ENERGY:

for one tube:

$$\int \frac{B^2}{8\pi} dV = \left( \frac{\Psi}{8\pi} \right)^2 \int \frac{dl}{A} \quad \text{note } \Psi \text{ does not change when we move tube.}$$

After a little calculation  
CALCULATE ENERGY BEFORE AND  
AFTER INTERCHANGE,

$$\Delta W_{\text{MAGNETIC}} = - \left[ \left( \frac{\Psi_2}{8\pi} \right)^2 - \left( \frac{\Psi_1}{8\pi} \right)^2 \right] \left[ \int \frac{dl_2}{A_2} - \int \frac{dl_1}{A_1} \right]$$

### CHANGE IN PRESSURE ENERGY:

pressure constant in tube because  $B \cdot \nabla p = 0$   
(from  $I \times B = \nabla p$ )

for one tube:

$$\int \frac{P}{r-1} dV = \frac{(PV)^\gamma}{\gamma-1} V^{\gamma-1}$$

Now conservation of Entropy gives  $PV^\gamma = \text{constant}$  during rates change.

after some algebra

$$\Delta W_p = \frac{(P_2 V_2^\gamma - P_1 V_1^\gamma)}{V^\gamma} (V_2 - V_1)$$

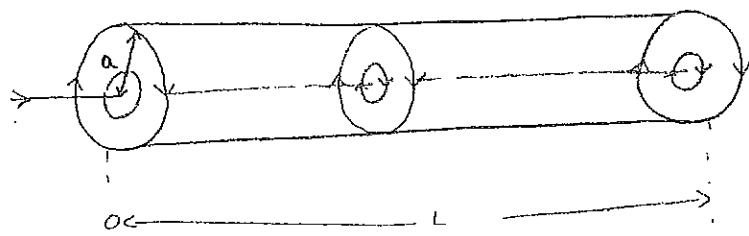
We can now test to see if system is stable to interchange stable field line in Point 1 + n



222a. Lecture #17: stability of the z pinch.

(i) Remember the z pinch has  $\underline{J} = J(r) \hat{z}$  current in z direction

$$\Rightarrow \underline{B} = B_\theta(r) \hat{\ell}_\theta$$



take z pinch surrounded  
by a conducting wall at  $r=a$   
and periodic in z over L

EQUILIBRIUM  $\nabla p = \underline{J} \times \underline{B}$

$$\frac{dp}{dr} + \frac{B_\theta}{r} \frac{d}{dr}(rB_\theta) = 0$$

note  $\underline{J} \cdot \underline{B} = 0$

(ii) We calculate stability by finding the minimum of  $\delta W$ ,  
and if  $\delta W_{\min} < 0$  it is unstable and if  $\delta W_{\min} > 0$  it is stable.

Take:-  $\delta W = \frac{1}{2} \int d^3x \left\{ |Q_\perp|^2 + B_0 |\nabla \cdot \xi_\perp + 2\xi_\perp \cdot K|^2 + \gamma p_0 |\nabla \cdot \xi_\parallel|^2 - 2\xi_\parallel \nabla p_0 (K \cdot \xi_\perp) - \frac{\underline{J} \cdot \underline{B}}{B^2} (\xi \times \underline{B}) \cdot Q \right\}$

curvature term. "kink term" zero for z pinch

$$Q_\perp = \nabla \times (\xi \times \underline{B}) \quad K = \underline{b} \cdot \nabla \underline{b} \quad \text{"curvature"}$$

- Only term which can be negative (destabilizing) is the curvature term.

$$z \text{ pinch } K = -\frac{\underline{n}}{r}$$

$$\nabla p_0 = \frac{1}{r} \frac{dp}{dr}$$

(iii) We can take a Fourier mode in  $\theta$  and  $z$  because system/equilibrium is symmetric in those directions. So

$$\textcircled{1} \quad \hat{\xi}_\perp(r) = \hat{\xi}_r(r) e^{i(m\theta + kz)} + \text{complex conjugate}$$

$$Q_1 = \frac{im}{r} B_\theta(r) \hat{\xi}_\perp(r) e^{i(m\theta + kz)} + \text{c.c.}$$

$m = \text{integer}$   
 $n = \text{integer}$   
 $k = 2n\pi \frac{L}{L}$   
 to be periodic in  $z$

(iv) Substituting (1) into  $\delta W$  we note that for instance :

$$\begin{aligned} \frac{1}{2} \int d^3r \left( -2 \hat{\xi}_\perp \cdot \nabla p_0 \cdot \underline{k} \cdot \hat{\xi}_\perp \right) &= \frac{1}{2} \int_0^a dr \int_0^{2\pi} d\theta \int_0^L dz \ 2 \hat{\xi}_r^2 \frac{dp}{dr} \\ &= \frac{1}{2} \int_0^a r dr \int_0^{2\pi} d\theta \int_0^L dz \frac{dp}{dr} \left[ 2 \hat{\xi}_r^2 e^{i2(m\theta + kz)} + 2(\hat{\xi}_r \hat{\xi}_r^*) + 2 \hat{\xi}_r^2 e^{-i2(m\theta + kz)} \right] \\ &\quad \xrightarrow{\substack{\theta \text{ is } z \text{ integration} \\ \text{make terms zero}}} \\ &= 2\pi L \int_0^a r dr \left| \hat{\xi}_r \right|^2 \frac{2dp}{dr}, \end{aligned}$$

Doing the same thing to all terms we get (after some Algebra)

(iv)

$$\begin{aligned} \delta W_f &= 2\pi L \int_0^a r dr \left\{ \frac{m^2 B_\theta^2}{r^2} \left[ |\hat{\xi}_r|^2 + |\hat{\xi}_z|^2 \right] + B_\theta^2 \left| \frac{d}{dr} \left( \frac{\hat{\xi}_r}{r} \right) + ik \hat{\xi}_z \right|^2 \right. \\ &\quad \left. + \gamma p_0 \left| \frac{1}{r} \frac{d}{dr} (r \hat{\xi}_r) + ik \hat{\xi}_z + \frac{i m}{r} \hat{\xi}_\theta \right|^2 + \frac{2 |\hat{\xi}_r|^2}{r} \frac{dp}{dr} \right\} \end{aligned}$$

(V)  $m \neq 0$  in this case  $\xi_\theta$  only appears in one term and we may minimize this term independantly of all other terms. The minimum of this term is clearly zero since it is positive - thus we choose  $\xi_\theta$  so that

$$\frac{1}{r} \frac{d(r\xi_r)}{dr} + ik\xi_z + \frac{im}{r} \xi_\theta = 0$$

NOTE IF  $m$  IS ZERO WE CANNOT DO THIS.

$$(Vi) \text{ Now } ② - \delta W_f = 2\pi L \int_0^a r dr \left\{ \frac{m^2 B_\theta^2}{r^2} \left[ |\xi_r|^2 + |\xi_z|^2 \right] + B_\theta^2 \left| r \frac{d}{dr} \left( \frac{\xi_r}{r} \right) + ik\xi_z \right|^2 + \frac{2|\xi_r|^2}{r} \frac{dP}{dr} \right\}$$

$\xi_z$  only appears algebraically i.e. it is not differentiated.

so we may find the minimum  $\delta W$  for a given  $\xi_r$  by

minimizing the integrand (at each  $r$ ) with respect to  $\xi_z$ . Thus

the derivative of the integrand w.r.t.  $\xi_z$  must vanish at the minimum.

This gives.

$$\frac{m^2 B_\theta^2}{r^2} i \xi_z + k B_\theta^2 \left( r \frac{d}{dr} \left( \frac{\xi_r}{r} \right) + ik\xi_z \right) = 0$$

as usual.

$$\xi_z = \frac{ikr^3}{m^2 + k^2 r^2} \left( \frac{\xi_r}{r} \right)' \quad | \quad ③$$

$$A' \equiv \frac{dA}{dr}$$

(vii) Now substituting ③ into ② we obtain.

$$\delta W = 2\pi L \int_0^a r dr \left\{ \frac{1}{r^2} (2rp' + m^2 B_\theta^2) |\zeta_r|^2 + \frac{m^2 r^2 B_\theta^2}{m^2 + k^2 r^2} \left( \frac{\zeta_r}{r} \right)'^2 \right\}$$

Positive definite.

Since  $k$  only appears in the 2nd term which is stabilizing we can see that the most unstable modes have  $k \rightarrow \infty$  and

$$\delta W_{\min} = 2\pi L \int_0^a r dr \left\{ \frac{1}{r^2} (2rp' + m^2 B_\theta^2) |\zeta_r|^2 \right\}$$

So we get instability if  $2rp' + m^2 B_\theta^2 < 0$  for some  $r_1 < r < r_2$

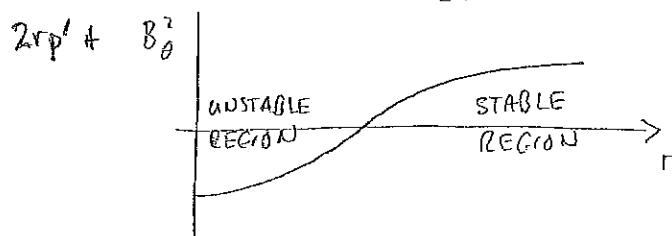
and we localize  $\zeta_r$  in this region of instability

using equilibrium relation we can write

$$(viii) \quad \frac{r^2}{B_\theta} \frac{d}{dr} \left( \frac{B_\theta}{r} \right) > \frac{1}{2} (m^2 - 4) \quad \text{criterion for instability.}$$

- CLEARLY  $m=1$  IS MOST UNSTABLE.

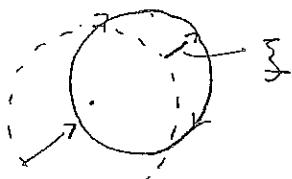
- USUALLY  $\frac{d}{dr} \left( \frac{B_\theta}{r} \right) < 0$  so  $m \geq 2$  are stable



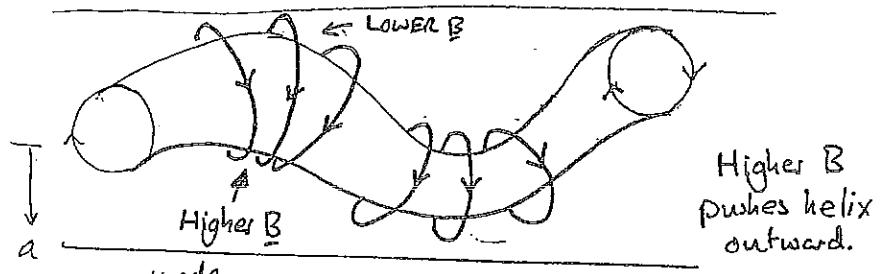
(ix)  $m=1$  look like rigid shifts. for  $m=1 k \rightarrow \infty$

$$\xi_r = \xi_r(r) \cos(\theta + kz) \quad \xi_z \approx 0$$

at fixed  $z$ .



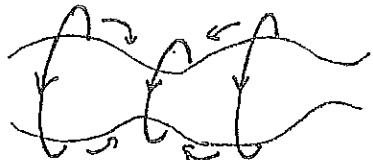
field lines stay circular.  
Helical distortion.



- People call this a kink mode sometimes but others say a kink must be driven by  $\underline{B} \cdot \underline{B}$  term in  $\delta W$ .

(x)  $M=0$   $\delta W_{m=0} = 2\pi L \int_0^a r dr \left\{ B_0^2 \left| \frac{r}{dr} \frac{d\xi_r}{dr} + ik\xi_z \right|^2 + \gamma p_0 \left| \frac{1}{r} \frac{d(r\xi_r)}{dr} + ik\xi_z \right|^2 + 2 \left| \xi_r \right|^2 \frac{dp}{dr} \right\}$

"SAUSAGE MODE"



squeezing like currents attract.

$$+ 2 \left| \xi_r \right|^2 \frac{dp}{dr} \}$$

again we can minimize this with respect to  $\xi_z$  because it appears algebraically - we get after some algebra

$$\delta W = 2\pi L \int_0^a \frac{dr}{r} \left[ \frac{4\gamma p B_0^2}{\gamma p_0 + B_0^2} + 2rp' \right] \left| \xi_r \right|^2$$

Instability if we get

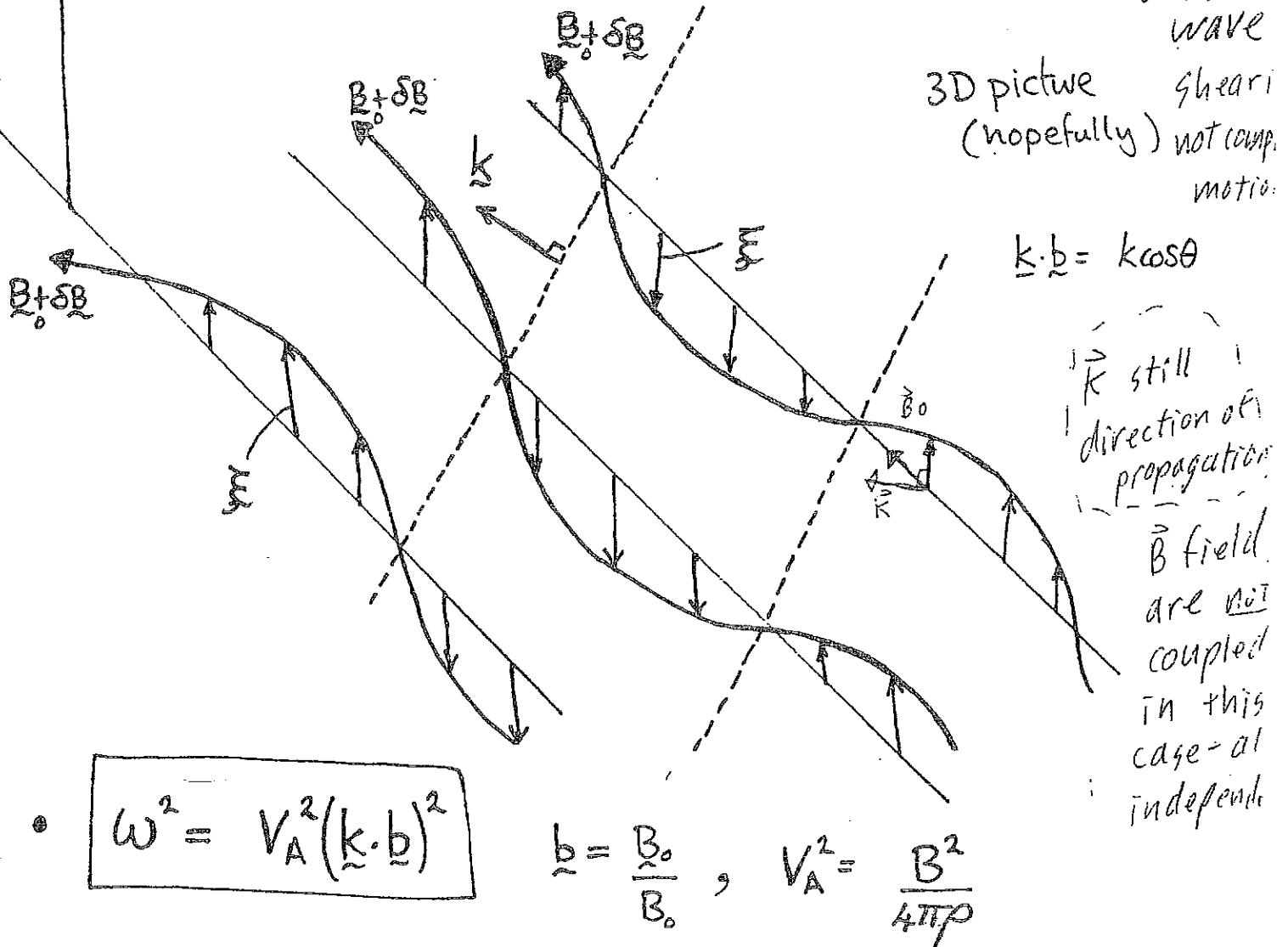
$$-\frac{rp'}{p} > \frac{2\gamma B_0^2}{\gamma p_0 + B_0^2}$$

CANNOT HAVE VERY STEEP PRESSURE PROFILE. Really interchange like.



# E<sub>x</sub> EFT/MHD. Waves #1. Alfvén Wave.

- shear  
Alfvén  
wave  
shear  
(hopefully) not comp.  
motio:



- Polarization  $\vec{\xi} = \xi_0 (\vec{B} \times \vec{k})$  perpendicular to both  $\vec{B}$  and  $\vec{k}$ .  
TRANSVERSE.

- Forces are all due to Field line bending  $\frac{B_0 \cdot \nabla B_0 \cdot \nabla \xi}{4\pi}$

- $\delta p = 0, \quad \delta B_{||} = 0$

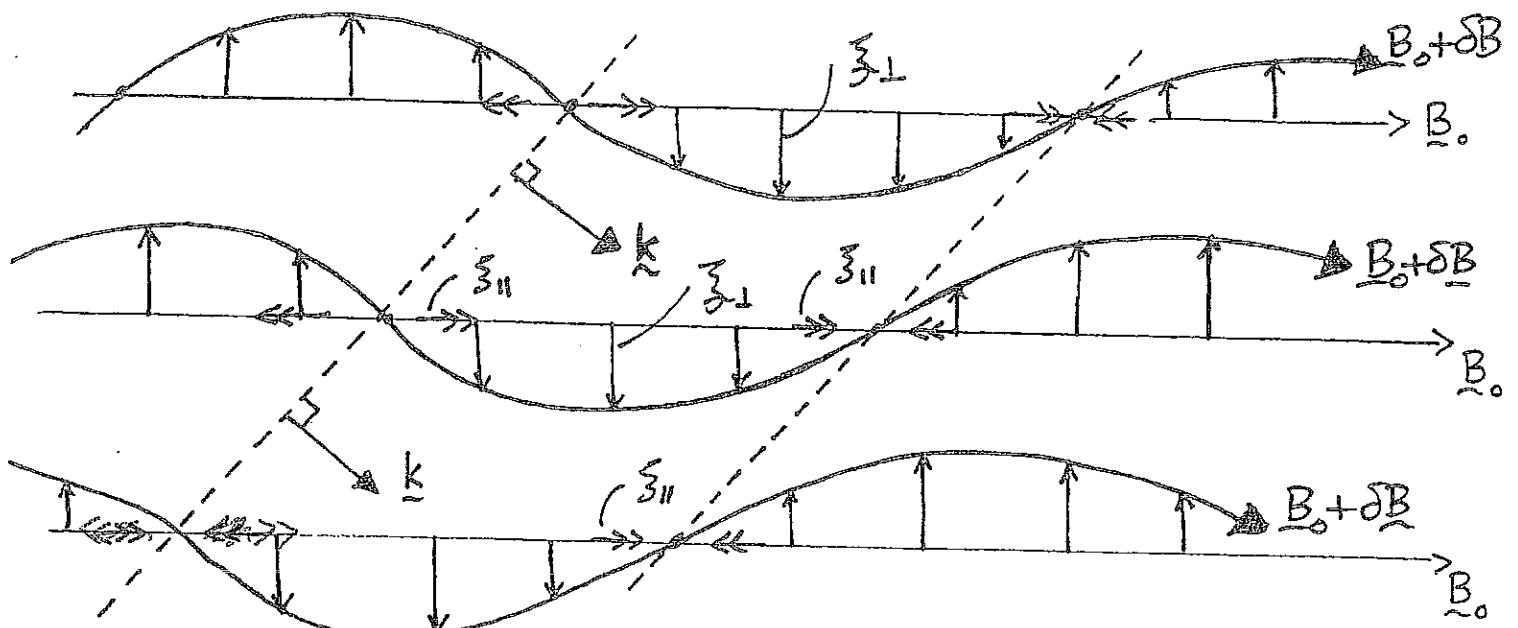
$$\rho \frac{\partial^2 \vec{\xi}}{\partial t^2} = \perp \frac{1}{4\pi} \vec{B} \cdot \nabla \vec{B} \cdot \nabla \vec{\xi}$$

- Also called intermediate wave.

- Dispersion relation also  $\omega^2 = k^2 V_A^2 \cos^2 \theta$  or  $\omega^2 = k_{||}^2 V_A^2$

# MHD Waves: #2. Fast Wave.

2D picture



$$c_s^2 = \frac{\gamma p_0}{\rho}$$

field line  
bending  
& compression

## Dispersion relation.

$$V_{\text{phase}}^2 = \frac{\omega^2}{k^2} = \frac{c_s^2 + V_A^2}{2} + \frac{1}{2} \sqrt{(c_s^2 + V_A^2)^2 - 4 c_s^2 V_A^2 \cos^2 \theta}$$

$$c_s^2 = \frac{\gamma p_0}{\rho}$$

- Polarization - in the plane of  $\underline{B}_0$  and  $\underline{k}$ . Neither fully transverse or longitudinal.

$$\xi = \xi_{\perp} \hat{k}_{\perp} + \xi_{\parallel} \hat{b}$$

- $\delta p$  and  $\delta B_{\parallel}$  same sign ,  $\xi_{\perp}$  and  $\xi_{\parallel}$  same sign. squashes/compresses both  $p$  and  $\underline{B}$ . AND bends field lines.

- If either  $\theta \sim \pi/2$  or  $c_s^2 \ll V_A^2$  or  $V_A^2 \ll c_s^2$

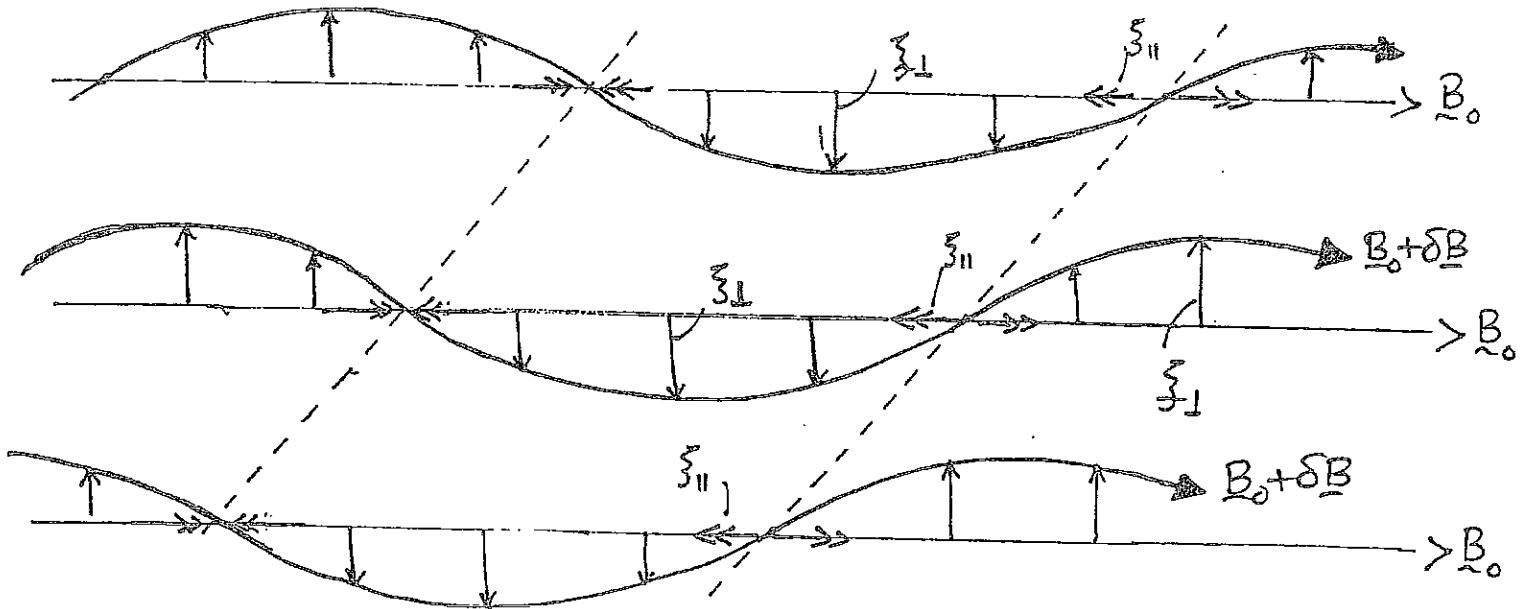
then  $\omega^2 \approx k^2 (c_s^2 + V_A^2)$

Magnetosonic speed.

$\theta = 0$   
 $\omega^2 = k^2 c_s^2, c_s > V_A$   
 $\omega^2 = k^2 V_A^2, c_s < V_A$

# MHD Waves: #3 Slow Wave.

2D picture.



- Dispersion Relation

$$\frac{\omega^2}{k^2} = \frac{c_s^2 + V_A^2}{2} - \frac{1}{2} \sqrt{(c_s^2 + V_A^2)^2 - 4V_A^2 c_s^2 \cos^2 \theta}$$

- Polarization. again in plane of  $B_0$  and  $k$  neither transverse or longitudinal fully.

- $\xi_{\parallel}$  and  $\xi_{\perp}$  opposite signs

- $\delta p$  lowered by parallel expansion where  $\delta B_{\parallel}$  goes up.

- If either  $\theta \sim \pi/2$  or  $c_s^2 \ll V_A^2$  or  $V_A^2 \ll c_s^2$

$$\frac{\omega^2}{k^2} \approx \frac{c_s^2 V_A^2}{c_s^2 + V_A^2} \cos^2 \theta$$

- $\theta = 0$   $\omega^2 = k^2 V_A^2$  if  $c_s^2 > V_A^2$

- $\omega^2 = k^2 c_s^2$  if  $V_A^2 > c_s^2$  SOUND WAVE

stability:

Consider a stationary plasma where

$$0 = -\nabla p_0 + \underline{J}_0 \times \underline{B}_0 + \rho_0 g$$

~~C~~

basic  
questin.  
"are we at  
top or bottom  
of a hill?"

i.e. it is in equilibrium. Now we linearize  
force balance

$$p_0 \rightarrow p_0 + \delta p$$

$$\underline{B}_0 \rightarrow \underline{B}_0 + \delta \underline{B}$$

$$p_0 \rightarrow p_0 + \delta p_0$$

$$\underline{J}_0 \rightarrow \underline{J}_0 + \delta \underline{J}$$

$$\underline{\nu} = \delta \underline{\nu} = \frac{\partial \xi}{\partial t} \quad \xi = \text{plasma displacement}$$

- All perturbed quantities can be written in terms of  $\xi$  keep only 1st order in  $\xi$

$$\frac{\partial \delta p}{\partial t} + \underline{\nu} \cdot \nabla p + \gamma p \nabla \cdot \underline{\nu} = 0 \Rightarrow \delta p = -[\underline{B} \cdot \nabla p_0 + \gamma p_0 \nabla \cdot \underline{J}]$$

$$\delta \underline{B} = \nabla \times (\xi \times \underline{B}_0) = -\underline{B}_0 \nabla \cdot \xi + \underline{B}_0 \nabla \xi - \xi \nabla \underline{B}_0$$

$$\delta p = -\xi \cdot \nabla p_0 - p_0 \nabla \cdot \xi$$

## Equation of Motion for Displacement

$$\left( \rho_0 \frac{\partial^2 \vec{\xi}}{\partial t^2} \right) = F(\vec{\xi}) = -\nabla p_0 + \frac{2 \vec{J} \times \vec{B}_0}{C} + \frac{\vec{J}_0 \times \vec{B}_0}{C} + \vec{s} \cdot \vec{g}$$

Force Operator.

$$F(\vec{\xi}) = \nabla \left[ \vec{\xi} \cdot \nabla p_0 + \gamma \rho_0 \nabla \cdot \vec{\xi} \right] + \frac{\vec{\xi}_0 \times \left[ \nabla \times (\vec{\xi} \times \vec{B}_0) \right]}{4\pi C} \\ + \nabla \times \left( \nabla \times (\vec{\xi} \times \vec{B}_0) \right) \times \frac{\vec{B}_0}{4\pi} \\ - (\vec{\xi} \cdot \nabla p_0 + \rho_0 \nabla \cdot \vec{\xi}) g$$

Obviously  $F(0) = 0$ .

On the computer we can simply try a displacement if it grows then the system is unstable. This is inelegant and prone to difficulties.

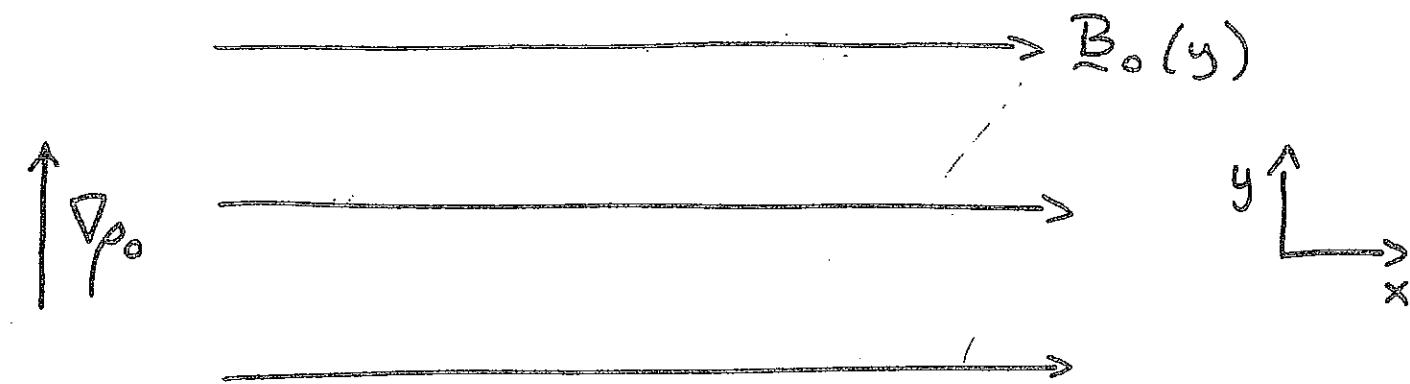
$$\frac{2}{J_t} \int d^3 r \left[ \frac{\rho_0}{2} \left( \frac{\partial \vec{\xi}}{\partial t} \right)^2 - \frac{1}{2} \vec{\xi} \cdot \vec{F}(\vec{\xi}) \right] = 0 \Leftrightarrow \int \frac{\rho_0}{2} \frac{\partial}{\partial t} \left( \frac{\partial \vec{\xi}}{\partial t} \right)^2 = \frac{1}{2} \frac{\partial}{\partial t} \int \vec{\xi} \cdot \vec{F}(\vec{\xi}) d^3 r$$

Look at energy:

$$\rho_0 \int \frac{\partial^2 \vec{\xi}}{\partial t^2} \cdot \frac{\partial \vec{\xi}}{\partial t} = \int \vec{\xi} \cdot \vec{F}(\vec{\xi}) d^3 r$$

# Magnetized Rayleigh Taylor: Instability

mercury  
water



$$\frac{\partial}{\partial y} \left( p_0 + \frac{B_0^2}{8\pi} \right) = -p_0 g \quad \text{Equilibrium.}$$

Take:  $\xi = \left[ \hat{\xi}_x(y) \hat{x} + \hat{\xi}_y(y) \hat{y} \right] e^{ik_x x - i\omega t}$

Also Take: Incompressible plasma  $\cancel{k_x} \rightarrow \infty$

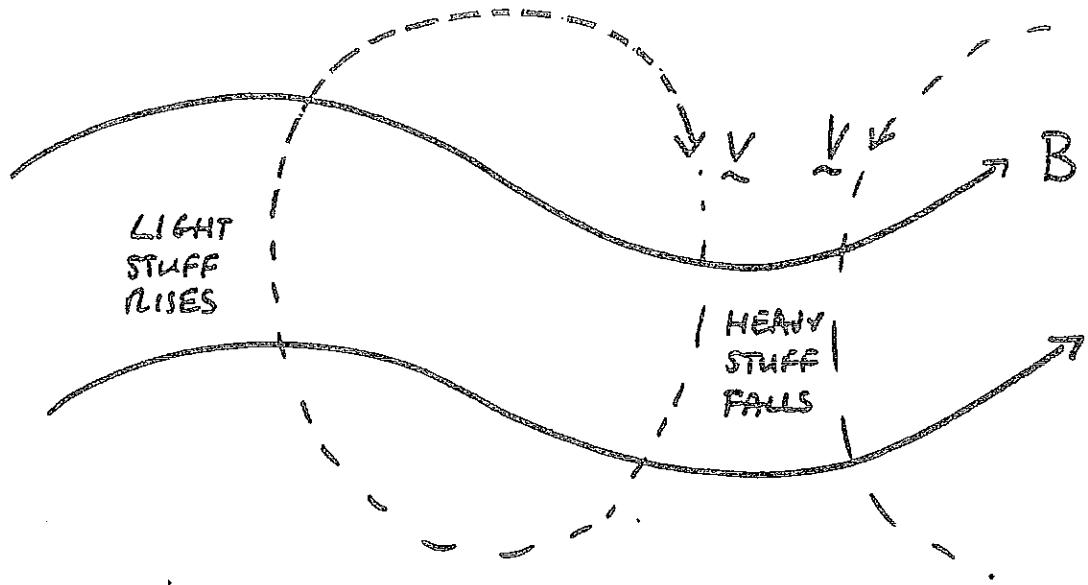
$$\text{and } \frac{\partial p_0}{\partial y} = 0 \Rightarrow \frac{\partial \xi_y}{\partial y} = -i k_x \xi_x$$

$p_0 = \text{constant}$

After Lots of algebra we get

$$\omega^2 \left( \xi_y - \frac{1}{k_x} \frac{\partial \xi_y}{\partial y} \right) - k_x^2 V_A^2 \xi_y = - \xi_y g \frac{dp_0}{p_0 dy}$$

displace up      displace side      perfect differential       $\beta$  effect



LOCAL DISERSION RELATION  $\frac{d}{dy} \rightarrow ik_y$

$$\omega^2 = \frac{k_x^2 V_A^2 - g}{l + \frac{k_y^2}{k_x^2}}$$

$$\frac{l}{L_p} = \frac{1}{\rho} \frac{dp}{dy}$$

density scale length

If  $\frac{g}{L_p} > k_x^2 V_A^2$  unstable

$$\begin{aligned} \omega^2 &< 0 \text{ ad} \\ \omega &= \pm i\gamma \end{aligned}$$

"Instability criterion"

3-D dimension allows



use shear to stabilize



# 222a Final lecture #18 Instability in Disks.

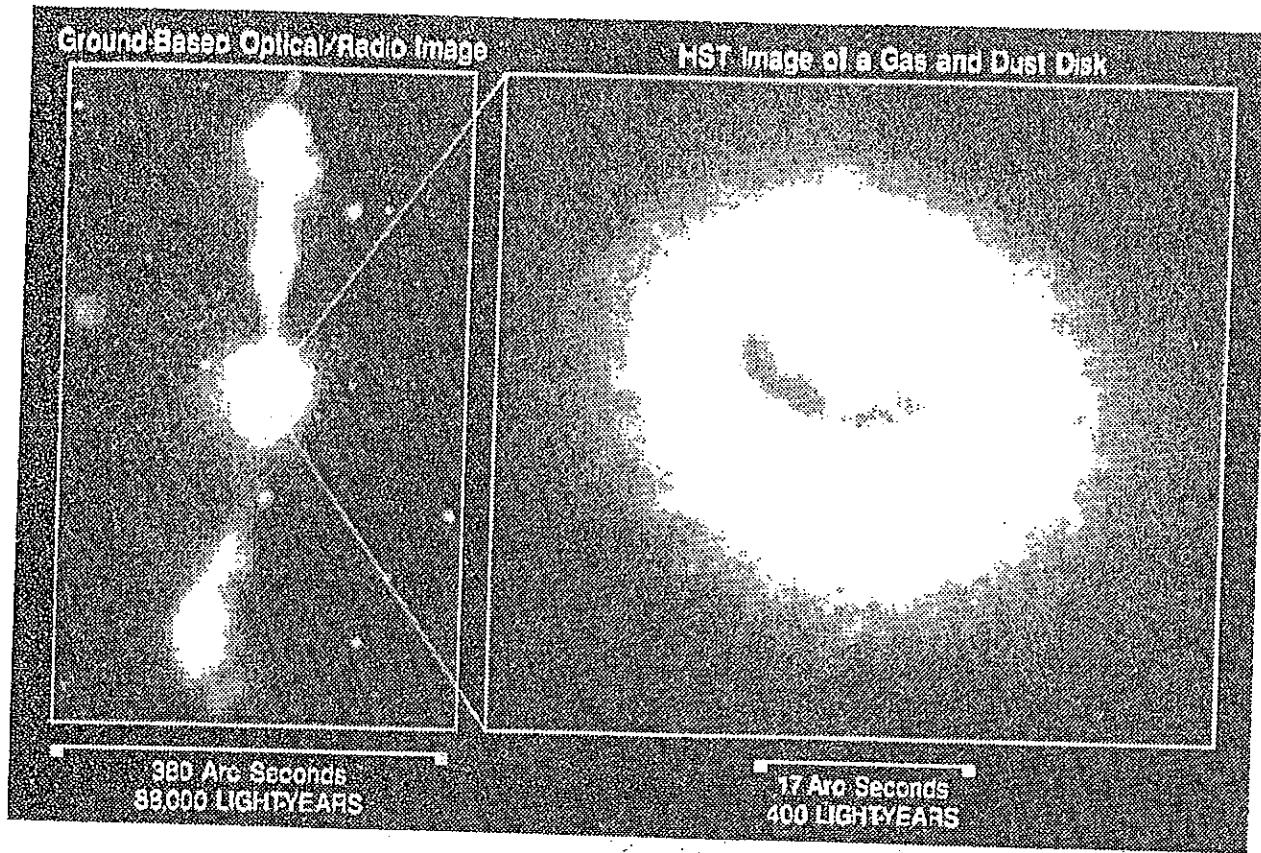


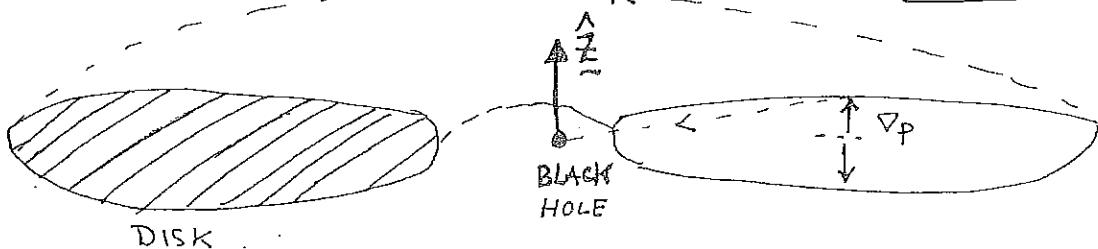
Image of core of galaxy NGC 4261, shows jets emanating from a core. The "Accretion Disk" is the disk like object seen in the Hubble space telescope image. We presume the glow is from hot gas falling into the central object which is probably a black hole.

(i) Keplerian Disk. Gas moves in circles  $\underline{v} = \Omega(R) R \hat{\underline{z}} \phi$

$\rho \underline{v} \cdot \nabla \underline{v}$  = centrifugal force      Gravitational Force on Gas,  $M = BH$  mass.

$$-\rho \Omega^2(R) R \hat{\underline{z}} = -\rho \frac{GM}{R^2} \hat{\underline{z}} \rightarrow$$

$$\Omega^2 = \frac{GM}{R^3}$$



Vertical Force Balance : Thin disk.

$$\frac{\partial p}{\partial z} - \frac{GM\rho(\hat{z}\cdot\hat{R})}{R^2} = -\frac{GM\rho}{R^3}z = -\rho c_s^2 z$$

$$P = nT \quad \Rightarrow P = \rho_0 \exp\left(-\frac{c_s^2 z^2}{2c_s^2}\right) \leftarrow \text{isothermal (in } z\text{)}$$

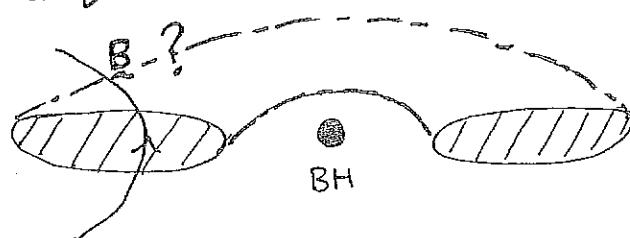
$$c_s^2 = (T/m_p)$$

$$\text{SCALE HEIGHT} \sim \sqrt{\frac{c_s^2}{2g}} \sim \sqrt{\frac{c_s^2 R^2}{V_\phi^2}} \sim H$$

$H \ll R$  (Thin disk) when  $V_\phi \ll c_s$  supersonic flow

- Typically we have supersonic flow in the disk.

- The only way we seem to be able to explain the brightness of these disks is to say gravitational energy of the gas is converted to radiation as gas falls in.
- To fall into the B.H. the gas must lose angular momentum this can only happen by some kind of viscosity which gives the angular momentum of the infalling gas to some gas further out.
- Unfortunately the ordinary viscosity of these hot gasses is very small and to get the gas to fall in we have to invoke some kind "turbulent viscosity" or some kind of magnetic torque.



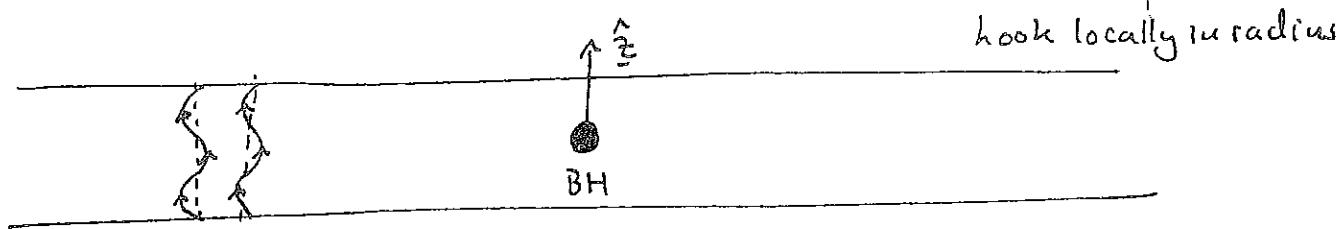
(iv) The fashionable view is that when magnetized the disk is unstable to radial motions which become turbulent and give rise to radial flux of angular momentum outward which allows infall.

(v) Today we look at a simple <sup>MHD</sup> instability of the disk that gives rise to turbulence - it was first found by Velikov in 1959 but is commonly called the Balbus-Hawley instability.  
(we cannot use  $\delta\omega$  because it is for non-flowing plasmas)

(vi) Simple field configuration:  $\underline{B} = B_0 \hat{z}$   
KEPLERIAN DISK:  $\underline{V} = \Omega(R) R \hat{\epsilon}_\phi$

$$\delta \underline{V} = \frac{\partial \xi}{\partial t} \quad \text{take } \xi_z = 0 \quad \text{for simplicity}$$

and  $\xi = \xi^A e^{ikz}$  "ignore" dependence on  $r$ .



$$\frac{\partial \underline{B}}{\partial t} = \nabla \times (\underline{V} \times \underline{B}) \Rightarrow \begin{cases} \delta B_R = ik B_0 \xi_R & , \delta B_z = 0 \\ \frac{\partial \delta B_\phi}{\partial t} = \delta B_R \frac{d\Omega}{dr} + ik B_0 \frac{\partial \xi_\phi}{\partial t} & \end{cases}$$

$\delta\rho = 0 \Rightarrow$  gravitational force unperturbed.

$$\frac{\partial V_R}{\partial t} + (\underline{V} \cdot \nabla \underline{V})_R = \frac{(\underline{B}_0 \cdot \nabla \delta \underline{B})_R}{4\pi \rho_0}, \quad \frac{\partial V_\phi}{\partial t} + (\underline{V} \cdot \nabla \underline{V})_\phi = \frac{(\underline{B}_0 \cdot \nabla \delta \underline{B})_\phi}{4\pi \rho_0}$$

4

After some algebra

$$\frac{\partial^2 \hat{\xi}_R}{\partial t^2} - 2\Omega \frac{\partial \hat{\xi}_\phi}{\partial t} = \frac{ikB}{4\pi\rho_0} \delta B_R$$

$$\frac{\partial^2 \hat{\xi}_\phi}{\partial t^2} + \frac{\partial \hat{\xi}_R}{\partial t} \frac{k^2}{2\Omega} = \frac{ikB}{4\pi\rho} \delta B_\phi$$

$$k^2 = \frac{1}{R^3} \frac{d(R^3 \Omega^2)}{dR}$$

 $k$  = "epicycle frequency" $k^2 > 0$  stable without  $B$ Substituting for  $\delta B_R$  &  $\delta B_\phi$  we get

$$\frac{ikB}{4\pi\rho} \frac{\partial \delta B_\phi}{\partial t} + \frac{k^2}{4\pi\rho} \frac{\partial \hat{\xi}_\phi}{\partial t} = \frac{ikB}{4\pi\rho} \frac{\partial \delta B_R}{\partial t} + \frac{ikB}{4\pi\rho} \frac{d\Omega}{dR} \frac{\partial \hat{\xi}_R}{\partial t}$$

$$\ddot{\hat{\xi}}_R - 2\Omega \dot{\hat{\xi}}_\phi = -\omega_A^2 \hat{\xi}_R$$

$$\ddot{\hat{\xi}}_\phi + 2\Omega \dot{\hat{\xi}}_R = -\left[\omega_A^2 + \frac{d\Omega^2}{dR}\right] \hat{\xi}_\phi$$

$$\omega_A^2 = \frac{k^2 B^2}{4\pi\rho}$$

Take  $\hat{\xi}_\phi = \hat{\xi}_\phi e^{-i\omega t}$        $\hat{\xi}_R = \hat{\xi}_R e^{-i\omega t}$

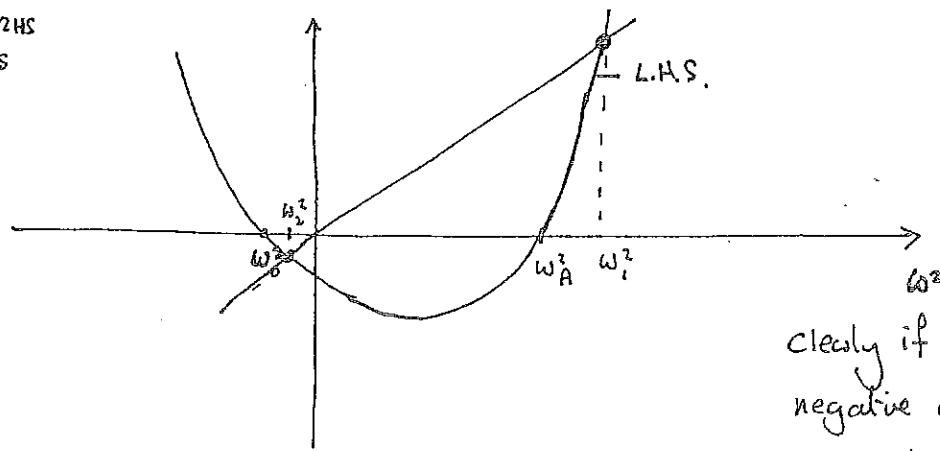
$$\begin{bmatrix} \omega_A^2 - \omega^2, & 2\Omega i\omega \\ -2\Omega i\omega, & \left(\omega_A^2 + \frac{d\Omega^2}{dR} - \omega^2\right) \end{bmatrix} \begin{bmatrix} \hat{\xi}_R \\ \hat{\xi}_\phi \end{bmatrix} = 0$$

 $-6\Omega^2$ 

$$(\omega_A^2 - \omega^2) \left( \omega_A^2 + \frac{d\Omega^2}{dR} - \omega^2 \right) = 4\Omega^2 \omega^2$$

note:  $\frac{d\Omega^2}{dR} < 0$

for keplerian disk



$$\omega_0^2 = \omega_A^2 + \frac{d\Omega^2}{dR}$$

clearly if  $\omega_0^2 < 0$  one root ( $\omega_1^2$ ) is negative and  $\omega = \pm i|\omega_1|$  is growing

Thus Instability when  $-\frac{d\Omega^2}{dR} > \omega_A^2$  Always possible for small enough  $k$ .