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Equilibrum in Tokamaks: Steve Cowley. CCFE.

In this lecture I will introduce you to tokamak equilibrium in my slightly less than rigorous way. I will concentrate on understanding and simple solutions.

Equations of Equilibrium: MAGNETOSTATICS. I will simplify to stationary equilibria $\underset{\sim}{V}=0$.

$\nabla p = \underset{\sim}{J} \times \underset{\sim}{B}$ FORCE BALANCE

$\nabla \times \underset{\sim}{B} = \mu_0 \underset{\sim}{J}$ Ampere's Law $\left(\frac{\partial E}{\partial t} = 0\right)$

$\nabla \cdot \underset{\sim}{B} = 0$ of course.

If the forces are not balanced then:

L = size of plasma

C_s = soundspeed $\approx \sqrt{\frac{T}{m_i}}$

- ACCELERATION OF PLASMA $= \dfrac{\nabla p}{m_i n} \sim \dfrac{1}{L}\dfrac{nT}{m_i n} \sim \dfrac{C_s^2}{L} = a$
- TIME TO ACCELERATE AND HIT WALL AT L $= \tau_{EQ}$ $\quad \dfrac{a\tau^2}{2} = L \Rightarrow \tau_{EQ} \sim \dfrac{L}{C_s} = 2\mu s!$ ON JET

FOR CONFINEMENT LONGER THAN τ_{EQ} MUST BE IN FORCE BALANCE

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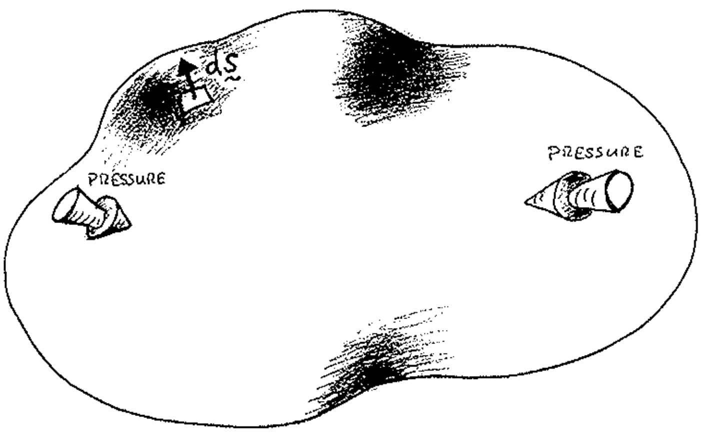
Physics of the forces: Pressure Force. $-\nabla p$: force per unit volume

$$-\int_{\text{VOLUME}} \nabla p \, d^3\underset{\sim}{r} = -\oint_{\text{SURFACE}} p \, d\underset{\sim}{s}$$

$\Rightarrow$ TOTAL FORCE ON A FINITE VOLUME = PRESSURE FORCE ON SURFACE OF VOLUME



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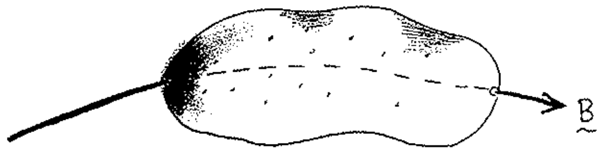
Magnetic Force:-

$$\underset{\sim}{J} \times \underset{\sim}{B} = \frac{\sum_{n=1}^{N} q_n \underset{\sim}{V}_n \times \underset{\sim}{B}}{\text{VOLUME}} = \text{SUM OF MAGNETIC FORCES ON PARTICLES INSIDE VOLUME} \Big/ \text{VOLUME}$$



Fluid mechanicist call this a BODY force because it acts on the volume not the surface.

QUESTION. WHY NO $\underset{\sim}{E}$ FORCE ?

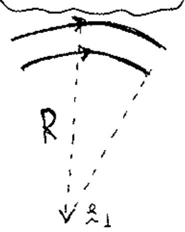
Helpful Physical Interpretation.

$$\mu_0 \underset{\sim}{J} = \nabla \times \underset{\sim}{B} \qquad \underset{\sim}{B} = B\underset{\sim}{b} \leftarrow \text{unit vector.}$$

Perpendicular (to $\underset{\sim}{B}$) derivative.

$$-\underset{\sim}{J} \times \underset{\sim}{B} = \frac{1}{\mu_0}(\nabla \times \underset{\sim}{B}) \times \underset{\sim}{B} = -\underbrace{\nabla\left(\frac{B^2}{2\mu_0}\right)}_{\text{MAGNETIC PRESSURE}} + \underbrace{\frac{\underset{\sim}{B}\cdot\nabla\underset{\sim}{B}}{\mu}}_{\text{MAGNETIC CURVATURE FORCE}} = \underbrace{-\nabla_\perp\left(\frac{B^2}{2\mu_0}\right)}_{\text{Perp. Pressure force}} + \frac{B^2}{\mu_0}\underbrace{\underset{\sim}{b}\cdot\nabla\underset{\sim}{b}}_{\text{field line curvature}}$$

CURVATURE



$$\underset{\sim}{b}\cdot\nabla\underset{\sim}{b} = \frac{\hat{e}_1}{R}$$

radius of curvature.

- MAGNETIC PRESSURE FORCE → TRIES TO MAKE B UNIFORM.
- CURVATURE FORCE → TRIES TO STRAIGHTEN LINES.

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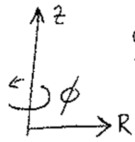
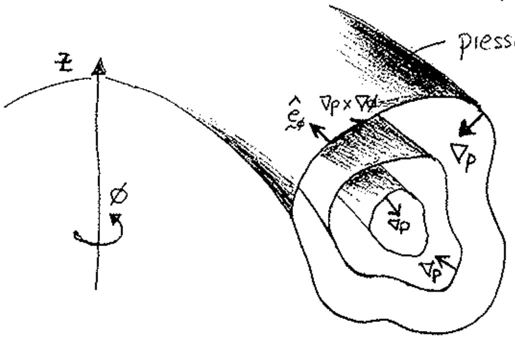
Toroidal Equilibrium:



CYLINDRICAL COORDINATES.

Axisymmetric $\Rightarrow \dfrac{\partial p}{\partial \phi} = 0 \quad p = p(R,z)$

FOR CONFINEMENT WE MUST HAVE NESTED TOROIDAL PRESSURE SURFACES: HIGH PRESSURE IN THE MIDDLE LOW AT THE EDGE.



∇p = perpendicular to pressure surface pointing inward.

$$\nabla\phi = \frac{\hat{e}_\phi}{R}$$

$$\boxed{\nabla p = \underset{\sim}{J} \times \underset{\sim}{B}} \Rightarrow \underset{\sim}{B}\cdot\nabla p = \underset{\sim}{J}\times\underset{\sim}{B}\cdot\underset{\sim}{B} = 0$$

$\therefore$ - $\underset{\sim}{B}$ is perpendicular to ∇p and therefore parallel to p surface.

$$\Rightarrow \underset{\sim}{J}\cdot\nabla p = \underset{\sim}{J}\times\underset{\sim}{B}\cdot\underset{\sim}{J} = 0$$

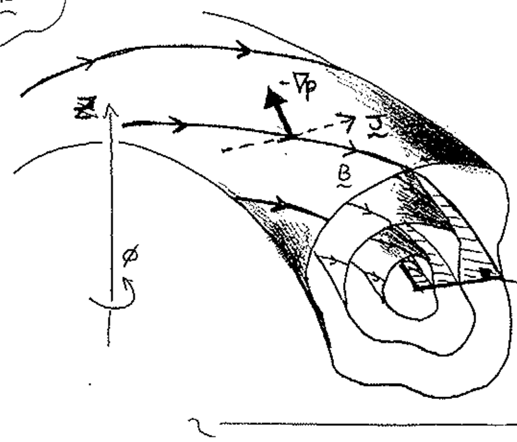
$\therefore$ $\underset{\sim}{J}$ is parallel to the pressure surfaces.

$$\underset{\sim}{B} = \underbrace{a\nabla p \times \nabla\phi}_{\text{POLOIDAL FIELD}} + \underbrace{b\nabla\phi}_{\text{TOROIDAL FIELD}} \Leftarrow$$

GENERAL VECTOR IN THE SURFACE p = CONSTANT

SOME ALGEBRA: SORRY

$\nabla p = \underset{\sim}{J} \times \underset{\sim}{B}$



$$\nabla\cdot\underset{\sim}{B} = \nabla\cdot(a\nabla p\times\nabla\phi) + \cancel{\nabla\cdot b\nabla\phi} = a\,\cancel{\nabla\cdot(\nabla p\times\nabla\phi)} + \nabla a\cdot\nabla p\times\nabla\phi$$

$$0 = \nabla a\cdot\nabla p\times\nabla\phi \Rightarrow \nabla a \parallel \nabla p \Rightarrow a = a(p)$$

$$\Rightarrow a\nabla p = \nabla\int_0^p a\,dp' \equiv \underset{\text{DEFINE.}}{\nabla\psi} \Rightarrow \begin{cases}\psi = \psi(p)\\ p = p(\psi)\end{cases}$$

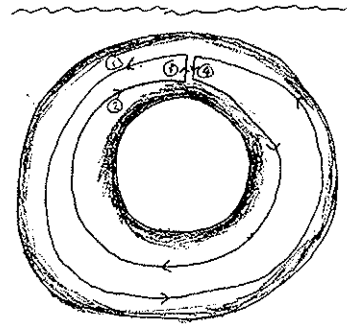
Magnetic flux through this strip $= \psi 2\pi$ "POLOIDAL FLUX"

$$\underset{\sim}{B} = \nabla\psi\times\nabla\phi + b\nabla\phi$$

LABEL SURFACES BY ψ "FLUX SURFACES"

Now we show that $b = F(\psi)$

LOOKING DOWN ON TOKAMAK:



i) Integrate $\underset{\sim}{B}$ around a loop on the flux surface ψ

SEGMENT ① = CIRCLE RADIUS R_1 ON ψ.
SEGMENT ② = CIRCLE RADIUS R_2 ON ψ
③ & ④ CANCEL.

ii) $$\oint \underset{\sim}{B}\cdot d\underset{\sim}{l} = (\underset{\sim}{B}\cdot\underset{\sim}{e}_\phi)_{①} 2\pi R_1 - (\underset{\sim}{B}\cdot\underset{\sim}{e}_\phi)_{②} 2\pi R_2 = 2\pi\left[b_{①} - b_{②}\right]$$

$$= \mu_0\int \underset{\sim}{J}\cdot d\underset{\sim}{S} = \mu_0\int \underset{\sim}{J}\cdot\frac{\nabla\psi}{|\nabla\psi|}\,dS \quad \text{but } \underset{\sim}{J}\cdot\nabla\psi = \underset{\sim}{J}\cdot\nabla p = 0$$

$\Rightarrow$ $b_{①} = b_{②}$ same at all radii on surface

$$\Rightarrow \boxed{b = F(\psi)}.$$

S.C.C.

Getting to the Grad-Shafranov Eq.

GENERAL FORM OF $\underset{\sim}{B}$

$$\boxed{\underset{\sim}{B} = \nabla\psi\times\nabla\phi + F(\psi)\nabla\phi}$$

CURRENT lies in surface $\Rightarrow \underset{\sim}{J} = c\nabla\psi\times\nabla\phi + d\nabla\phi = \nabla\times\underset{\sim}{B}$

$$\underset{\sim}{J} = \frac{1}{\mu_0}\left\{F'\nabla\psi\times\nabla\phi - R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)\nabla\phi\right\}$$

μ_0 Poloidal Current $= \nabla\times \underset{\text{TOROIDAL FIELD}}{F\nabla\phi} = F'\nabla\psi\times\nabla\phi \qquad F' = \frac{dF}{d\psi}$

μ_0 Toroidal current $= \mu_0 d\nabla\phi = \nabla\times(\nabla\psi\times\nabla\phi) \Rightarrow \mu_0\frac{d}{R^2} = \nabla\phi\cdot\nabla\times(\nabla\psi\times\nabla\phi) = \nabla\cdot[(\nabla\psi\times\nabla\phi)\times\nabla\phi] = -\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right)$

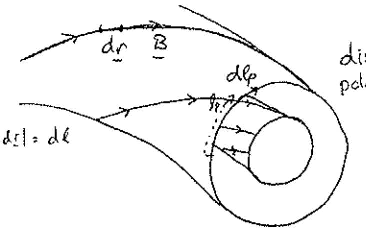
only these pairs.

$$\nabla p = \frac{dp}{d\psi}\nabla\psi = \underset{\sim}{J}\times\underset{\sim}{B} = \frac{1}{\mu_0}\left\{F'\nabla\psi\times\nabla\phi - R^2\nabla\cdot\frac{\nabla\psi}{R^2}\nabla\phi\right\}\times\left\{\nabla\psi\times\nabla\phi + F\nabla\phi\right\}$$

$$\boxed{R^2\nabla\cdot\left(\frac{\nabla\psi}{R^2}\right) = -\left\{\mu_0\frac{dp}{d\psi}R^2 + F\frac{dF}{d\psi}\right\} = R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right) + \frac{\partial^2\psi}{\partial z^2}}$$

- Solve this equation for $\psi = \psi(R,z)$ with e.g. $p = p(\psi)$ and $F = F(\psi)$ given + Boundary condition [e.g. ψ on a closed curve "~~[illegible]~~ boundary"]

q profile: "safety factor"



distance along poloidal plane

poloidal field

$$\frac{dl_p}{dl} = \frac{B_p}{B} = \frac{|\nabla\psi|}{RB}$$

q = NUMBER OF TIMES A FIELD LINE GOES AROUND TOROIDALLY FOR ONCE ROUND POLOIDALLY.

$$d\underset{\sim}{r}\cdot\nabla\phi = d\phi$$

$$\frac{d\phi}{dl} = \frac{\underset{\sim}{B}\cdot\nabla\phi}{B} = \frac{F}{R^2B}$$

Equation for a field line :-
$$\frac{d\underset{\sim}{r}}{|d\underset{\sim}{r}|} = \underset{\sim}{b} = \frac{\underset{\sim}{B}}{B} = \frac{d\underset{\sim}{r}}{dl}$$

$$\frac{d\phi}{dl_p} = \frac{d\phi}{dl}\cdot\frac{dl}{dl_p} = \frac{F}{R^2B}\cdot\frac{RB}{|\nabla\psi|} \;\Rightarrow\; \frac{d\phi}{dl_p} = \frac{F}{R|\nabla\psi|}$$

Equation for the field line in ϕ and l_p.

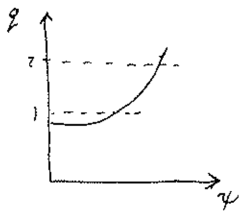
Integrate :- along flux surface
$$\phi = \int_0^{l_p} dl_p' \frac{F(\psi)}{R|\nabla\psi|}$$

ψ stays constant during integration.
Think of R as a function of l_p and ψ.

Now all the way around
$$\phi_{TOTAL} = 2\pi q = F(\psi)\oint \frac{dl_p}{R|\nabla\psi|}$$

$$q(\psi) = \frac{F(\psi)}{2\pi}\oint\frac{dl_p}{R|\nabla\psi|}$$

Typical q profile



To solve the Grad-Shafranov you can specify $p(\psi)$ & $F(\psi)$ or $p(\psi)$ & $q(\psi)$ or .....