Problem Set 2

Special Relativity (8 lectures, HT 2001) Dr S. Sarkar

1. Write down the formula for the Lorentz transformation of a space-time 4-vector between an inertial frame Σ and an inertial frame Σ' that is moving along the z-axis with speed $v = \beta c$ relative to Σ. Discuss carefully how the formula describes the effects known as *length contraction* and *time dilation*. What problems in principle would be encountered if an attempt were made to observe directly the length contraction?

An optical medium of refractive index n is stationary in Σ. Write down the energy and momentum components in Σ of a 4-vector describing a photon of frequency ω travelling at an angle θ with respect to the z-axis. Hence or otherwise show that when observed in frame Σ' the frequency is given by

$$\omega' = \omega \frac{1 - n\beta\cos\theta}{\sqrt{1-\beta^2}}$$

Calculate the invariant mass of the photon and comment on its value when $n > 1$. What is perceived by the observers in Σ and Σ' when $n\beta\cos\theta > 1$? `[97/VA/5]`

(Hint: In a medium, the phase velocity $v_{\mathrm{p}} = \omega/k = c/n$.)

2. A system observed in an inertial reference frame consists of free particles with total energies E_1, E_2, ... and momenta $\mathbf{p}_1$, $\mathbf{p}_2$, ... Outline the arguments which lead to the statement that the quantity $(E_1 + E_2 + \cdots)^2 - c^2(\mathbf{p}_1 + \mathbf{p}_2 + \cdots)^2$ is Lorentz invariant. Explain why Lorentz-invariant combinations of observables are of practical importance in relativistic kinematics.

A photon of momentum k moves at angle θ with respect to the trajectory of a highly relativistic proton, of total energy E, with which it collides. Obtain an expression for the threshold value of E for the creation of a neutron through the reaction $\gamma + p \rightarrow n + \pi^+$.

Estimate the minimum proton energy for which such neutron formation could occur through interaction with the 2.7 K blackbody cosmic background radiation. `[96/VA/2]`

3. Give the formulae for a Lorentz transformation of the three components of the momentum and total energy of a particle. Hence, or otherwise, derive the formula for the relativistic Doppler effect for light emitted from a source moving at an arbitrary angle with respect to the line of sight.

A small distant star of radius r is rotating slowly about an axis inclined at an angle δ to the line of sight to the Earth. A point on the surface of the star at an angle θ with respect to the axis of rotation is ejecting material normally from the surface at a relativistic velocity v. Among the material ejected is an excited atomic species which has a mean life τ sufficiently long that a large fraction moves several star radii before decaying ($v\tau\gamma \gg r$, $\gamma = (1 - v^2/c^2)^{-1/2}$). This species radiates at a wavelength λ_0 in its rest frame. Show that the extremes, z_+ and z_- of the wavelength shift (defined by $z = (\lambda - \lambda_0)/\lambda_0$) for this radiation observed on Earth are given by:

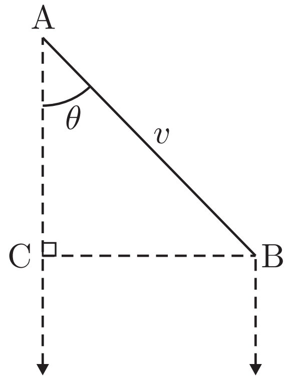
$$z_\pm = \gamma\left[1 - \frac{v}{c}\cos(\theta \pm \delta)\right] - 1$$

Suppose that the radiating material is at rest at the same angle θ on the surface of the star and that $\theta + \delta < 90^0$. What wavelength shifts would be observed on the Earth for radiation emitted when the source is at its closest and its furthest from the Earth, if the star is rotating with frequency f, where $2\pi\, r f = c/20$? `[89/VA/4]`

(Hint: Neglect any motion of the star relative to the Earth.)

4. Show how the Lorentz transformation satisfied by the wave-vector $\mathbf{k}$ and angular frequency ω of an electromagnetic wave can be obtained from the hypothesis that its phase $\mathbf{k} \cdot \mathbf{r} - \omega\, t$ is Lorentz-invariant?

A cloud of matter is ejected with high velocity from a distant galactic nucleus. The cloud moves from A to B in the diagram, emitting photons seen over a period of several years, by an observer on Earth. In the Earth's frame of reference, the cloud's velocity has constant magnitude v and is at an angle θ with the line of sight.



The points B and C represent the space coordinates of space-time points on the paths of photons seen at different times by the observer. Derive an expression, in terms of v and θ, for the apparent transverse velocity of the cloud which would be deduced by the observer from the difference in the arrival times of photons with the paths shown and from the measured spatial separation of B and C?

Find the condition neccessary for the space-time points at B and C on the photon paths shown in the figure to have space-like separation and comment briefly on its significance given that the transverse velocity deduced by the observer is 2.1c.

Determine the velocity v under the assumption that $\theta = 0.3$ radians. If photons with a line spectrum were emitted by the cloud and observed on Earth, by what factor would their wavelengths be blue-shifted? `[94/VA/2]`

5. Give the Lorentz transformation for energy E and momentum $\mathbf{p}$, when the relative velocity of the two frames of reference is in the z direction. Show that $E^2 - \mathbf{p}^2 c^2$ is an invariant and discuss the importance of this result.

A π^0 of rest mass m travels along the z axis with velocity $\beta\, c$ is the laboratory andundergoes decay to two photons. Obtain expressions for the energy and frequency of the photons in the π^0 rest frame and for the maximum and minimum photon energies measured in the laboratory frame.

Derive expressions for the laboratory energy and frequency of a photon
(i) emitted perpendicular to the z axis in the π^0 rest frame
(ii) observed perpendicular to the z axis in the laboratory. `[90/VA/4]`

6. `(The first part of this question gives the answer to the first part of the previous question, so has been deleted!)`

Show that if the energy and momentum are conserved in a collision process according to an observer in one inertial frame, then they are conserved according to observers in all inertial frames.

Find the minimum total energy (in MeV) that a pion must have in order to initiate the reaction

$$\pi^+ + p \to \Delta^{++}$$

when the target is a proton at rest. Suppose now that the proton is confined in a volume with dimensions of order 1 fm (e.g. in a nucleus):

(a) What would be a typical value of its momentum

(b) What would be the minimum total pion energy needed to initiate the above reaction if this momentum is directed towards the incident pion?

($M_{\Delta^{++}} = 1232$ MeV/c^2.) `[85/IIIA/4]`

7. By considering the conservation of energy and momentum show that a free electron cannot absorb a photon. Why can absorption occur if the electron is initially bound?

A certain narrow nuclear γ-transition is examined in both emission and absorption. It is found that the peak absorption energy E_{a} differs slightly from the peak emission energy E_{e}. Find an approximate expression for the difference ΔE.

Discuss whether the effect would be observable for the γ-transition to the ground state of ^{198}Hg from the 0.411 MeV state, which has a lifetime of 2.2×10^{-11} s.

Discuss briefly why it is sometimes possible for transitions to occur for which $\Delta E \approx 0$. `[97/VA/5]`

***8.** The cosmic microwave background radiation — believed to be a relic of the Big Bang — is observed to have a blackbody spectrum, i.e. the number of photons per unit volume with (angular) frequency ω in the range $\mathrm{d}\omega$ and moving in any direction (the solid angle $\mathrm{d}\Omega$ integrated to 4π) is

$$n(\omega)\mathrm{d}\omega = \frac{1}{\pi^2 c^3} \frac{\omega^2 \mathrm{d}\omega}{\exp(\hbar\omega/kT_0) - 1}$$

where the present temperature is $T_0 = 2.728 \pm 0.002$ K. If there exists a reference frame in which this radiation is exactly isotropic over the sky, show that in another frame moving relative to this frame with velocity v ($= \beta c$), the radiation will still have the Planck spectrum, but with a temperature

$$T(\theta) = T_0 \frac{\sqrt{1-\beta^2}}{1 - \beta\cos\theta}$$

where θ is the angle betweeen the direction of motion and the line-of-sight.

Expand this to show that the leading term in v/c is a dipole anisotropy. If the maximum change in the temperature we observe is $\Delta T = 3.355$ mK, with what velocity are we moving with respect to the reference frame in which the radiation is isotropic? Does the existence of such a special reference frame conflict with the principle of special relativity? `[P.J.E. Peebles & D.H. Wilkinson, PR` **`174`** `(1968) 2168]`