Problem Set 1

Special Relativity (8 lectures, HT 2001) Dr S. Sarkar

(These are mostly problems set in past finals examinations and cover the entire syllabus. The two starred problems are based on research papers and may be attempted by the enthusiastic.)

1. What is meant by an *inertial frame*?

State the postulates on which Einstein's Special Theory of Relativity is based.

The Lorentz tranformations of co-ordinates (x, y, z, t) in one inertial frame to those (x', y', z', t') in another moving at constant relative velocity v along the common x-axis are:

$$x' = \gamma(x - vt), \quad y' = y, \quad z' = z, \quad t' = \gamma(t - vx/c^2),$$

where $\gamma = (1 - v^2/c^2)^{-1/2}$.

Show that the expression $\mathbf{r}^2 - c^2t^2 = 0$ is invariant under the Lorentz transformation and explain the significance of this result.

An express train passes through a station travelling at a constant relative speed v. On the station platform there are two synchronized clocks, A at the entrance and B at the exit, separated by a distance L. A passenger near the front of the train notes the time shown by the two clocks as she passes them. If both her watch and clock A read zero when she reaches A:

(i) What is the time shown by clock B when she reaches B?

(ii) What time does her own watch show as she reaches B?

Another passenger is so placed near the rear of the train that he is opposite clock A when (according to the synchronized clocks on the train) the first passenger is opposite clock B. What time does he read on clock A? `[84/IIIA/4]`

2. Give the relativistic expressions for the energy E and momentum $\mathbf{p}$ of a particle in terms of its rest mass m and velocity $\mathbf{v}$.

Show how the relativistic formulae for energy and momentum conservation in elastic collisions between two particles reduce to the corresponding non-relativistic formulae in the limit of small velocities. Comment on the case of inelastic collisions. A particle x travelling with momentum $\mathbf{p_x}$ in the laboratory frame is observed to decay into two particles y and z:

$$x \to y + z.$$

The rest masses of x and y are m and $0.7m$ respectively. In one particular decay $p_x = 3mc$ and y is observed to have a momentum of $2.8mc$ in the same direction as $\mathbf{p_x}$. Calculate the velocities of x and y and deduce the rest mass of z. `[86/VA/2]`

3. Show that two successive Lorentz transformations with velocities v_1 and v_2 in the same direction are equivalent to a single Lorentz transformation with velocity $(v_1 + v_2)/(1 + v_1v_2/c^2)$. (Hint: The hyberbolic form of the LT may be useful.)

With the aid of this result, show that if there existed an action that could be propagated relative to an inertial frame with a velocity greater than c, it would be possible to have a situation in which *an effect precedes its cause.*

***4.** In a binary star system, two stars orbit around their common centre-of-mass with period $T = 2\pi/\omega$. When one of the stars is a pulsar (i.e. emits pulses of radiation at a constant rate in its rest frame), this motion is revealed through the time-dependent Doppler shift of the emitted pulses. Take the (constant) orbital velocity of the pulsar to be $v_0(\ll c)$, the semi-major axis of its orbit to be r_0, and the inclination angle of the orbital plane to be i, so the line-of-sight component of the velocity is $v = v_0 \sin i$ and the projected radial velocity towards an observer on Earth is $v_{\rm E}(t') = v\cos(\omega t')$ where t' is measured in the source frame. Let $D(\gg r_0)$ be the distance to the binary system.

Now let us assume, contrary to special relativity, that the speed of light is modified by the motion of the source as $c \to c + kv_{\rm E}$. What limit can be imposed on k from simply requiring that expected sinusoidal curve for the arrival time of the pulses not be grossly disturbed? Show that the value of $v_{\rm E}$ inferred from the observed Doppler curve is

$$v_{\rm E} \simeq \frac{v}{\cos\phi}\cos\left[\omega\left(t + \frac{kDv}{c^2}\cos\omega t\right) + \phi\right],$$

where the phase $\phi \equiv \tan^{-1}(k\omega\, D/c)$. (Hint: The zero of time may be arbitrarily chosen.)

Consider the binary X-ray system Hercules X-1 in which the pulsar (neutron star) orbits a much larger (red giant) companion quite close to its surface, so that it is eclipsed during every orbit. Show that the midpoint of the eclipses as determined visually from Earth will be offset from that determined from the Doppler curve by just the phase ϕ above. Estimate the magnitude of $Dv\omega/c^2$ given $v_0 \sin i = 169$ km/s, $D \sim 6\times 10^3$ pc and $T = 1.7$ day (1 pc$\simeq$ 3.26 light year). Infer a better limit on k than that obtained earlier given that no discrepancy is observed to within the observational uncertainty in the phase measurement of 0.06 rad? `[K. Brecher, PRL` **39** `(1977) 1051, err 1236]`
(Thanks to Dr Bowler for suggestions which clarified the formulation of this problem)

5. A particle moves in the laboratory system Σ with relativistic velocity v parallel to the x axis of Σ. In the rest system Σ' of the particle, a photon is emitted at an angle θ' to the x' axis of Σ' (which is parallel to the x axis of Σ). Show that in the system Σ the photon appears to be emitted at an angle θ to the x-axis, where

$$\cos\theta' = \frac{\cos\theta - \beta}{1-\beta\cos\theta}, \qquad \beta = \frac{v}{c}.$$

If N_0 photons are emitted isotropically in the particle rest frame show, by considering $\mathrm{d}(\cos\theta')/\mathrm{d}(\cos\theta)$, that in the laboratory system the number of photons $\mathrm{d}N$ emitted into the solid angle $\mathrm{d}\Omega$ is given by

$$\frac{\mathrm{d}N}{\mathrm{d}\Omega} = \frac{N_0}{4\pi}\frac{(1-\beta^2)}{(1-\beta\cos\theta)^2}.$$

Hence or otherwise show that one half of the photons are emitted within an angle θ_0 to the x-axis where $\cos\theta_0 = \beta$. `[88/VA/4]`

6. An accelerator produces a burst of K^+ mesons which (after momentum selection) have a momentum of 10 GeV/c. If they must travel a distance of 100 m from production point to a detector, how many decay before getting there? Unless they have been filtered out, π^+ mesons and protons will predominate over the K^+ mesons in a burst of momentum-selected (positive) secondaries. Find the time intervals (in the Lab) separating the arrival at the detector of the three types of particle, given that $m_{K^+} = 496$ MeV/c^2, $m_{\pi^+} =$ 140 MeV/c^2, $m_p = 940$ MeV/c^2 and $\tau_{K^+} = 1.2 \times 10^{-8}$ s. `[Adapted from Bowler]`

7. A pion of rest mass 140 MeV/c^2 decays into a muon of mass 105 MeV/c^2 and a massless neutrino: $\pi \to \mu + \nu$.

1. Compute the K.E. of the muon in the pion rest frame. (You will also need its momentum and velocity for what follows.)

2. If the pion has total energy 1.4 GeV in the laboratory system and decays in flight, what are the limits on the energies of the muon and the neutrino? (This has practical consequences for high energy neutrino experiments where the beam comes from pion decay in flight.)

3. In Part (2), Are there any limits to the laboratory directions of the muon or neutrino? If so state what they are.

(Hint: All the quantities required can be calculated in terms of m_π and m_μ, and in (2) and (3), γ_μ. So there is no need to put in numbers until the end of your calculation.)

8. For a system of particles with total energy $E_{\rm T}$ and total momentum $p_{\rm T}$, the quantity $E_{\rm T}^2 - p_{\rm T}^2$ is an invariant. Justify and explain the importance of this result and explain the significance of the invariant quantity.

A particle of rest mass m travelling at speed u in the laboratory collides with a stationary particle of equal rest mass and they combine to form a new particle. Calculate the speed v in the laboratory and the rest mass M of the new particle expressing your result in terms of m, u and γ where

$$\gamma = [1/(1 - u^2/c^2)]^{1/2}.$$

The new particle subsequently breaks up into two particles of equal rest mass $\alpha\, m$. Show that the momentum of each particle in the zero momentum (centre-of-mass) frame is

$$p = mc[(1 + \gamma - 2\alpha^2)/2]^{1/2}$$

and hence deduce a greatest value for α. What is the threshold energy for $e^+e^- \to \mu^+\mu^-$ using a beam of positrons on stationary electrons? ($m_e = 0.511$ MeV/c^2, $m_\mu =$ 105.66 MeV/c^2.) `[92/VA/6]`