Special Relativity: Lecture 8

1 Covariant electromagnetism

This will not be a complete treatment but a sketch to show the steps necessary to arrive at the covariant equations and to give a flavour of the elegance and power of the resulting formalism. For the full treatment refer to Jackson or Panofsky & Phillips.

Equations:

Charge conservation:

$$\frac{\partial \rho}{\partial t} + \mathrm{div}\,\mathbf{j} = 0. \qquad (VIII.1)$$

Maxwell's equations:

$$\begin{aligned} \mathrm{div}\,\mathbf{B} &= 0\ , \\ \mathrm{div}\,\mathbf{E} &= \frac{\rho}{\epsilon_0}\ , \\ \mathrm{curl}\,\mathbf{E} &= -\frac{\partial B}{\partial t}\ , \\ \mathrm{curl}\,\mathbf{B} &= \mu_0 \mathbf{j} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}\ . \end{aligned} \qquad (VIII.2)$$

Potentials:

$$\begin{aligned} \mathbf{B} &= \mathrm{curl}\,\mathbf{A}, \\ \mathbf{E} &= -\frac{\partial \mathbf{A}}{\partial t} - \mathrm{grad}\,\phi. \end{aligned} \qquad (VIII.3)$$

Lorentz gauge:

$$\mathrm{div}\,\mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0.$$

In this gauge, eqs.(VIII.2) and (VIII.3) give

$$\begin{aligned} \nabla^2 \phi - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} &= -\rho/\epsilon_0 = -\rho c^2 \mu_0\ , \\ \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= -\mu_0 \mathbf{j}\ . \end{aligned} \qquad (VIII.4)$$

Step 1: 4-vector current density j_μ:

$$j_\mu = \rho_0 \frac{\mathrm{d}x_\mu}{\mathrm{d}\tau}\ , \qquad (VIII.5)$$

ρ_0 is the proper charge density i.e. charge density in IF S in which charges are at rest.

$$\text{so} \quad j_\mu = \rho_0 \gamma(\mathbf{v}, ic) = (\mathbf{j}, ic\rho), \qquad \rho = \gamma \rho_0. \qquad (VIII.5')$$

Checks

i) NR limit $\surd$

ii) charge conservation $\partial j_\mu/\partial x_\mu = 0$ i.e.

$$\nabla \cdot \mathbf{j} + \frac{\partial(ic\rho)}{\partial(ict)} = 0 \quad \surd$$

iii) charge is an invariant, i.e.

$$\rho_0 \mathrm{d}V_0 = \gamma\rho_0 \frac{\mathrm{d}V_0}{\gamma} = \rho \mathrm{d}V$$

(NB: Lorentz contraction of distance.)

Step 2: 4-vector potential:

$$A_\mu = (\mathbf{A}, i\phi/c). \qquad (VIII.6)$$

Check Lorentz condition:

$$\frac{\partial A_\mu}{\partial x_\mu} = \nabla \cdot \mathbf{A} + \frac{\partial(i\phi/c)}{\partial(ict)} = \nabla \cdot \mathbf{A} + \frac{1}{c^2}\frac{\partial \phi}{\partial t} = 0. \quad \surd$$

(Strictly we need to show that A_μ *is* a 4-vector.)

Step 3: Wave equation for potentials

Consider $\Box A_\mu$ where $\Box = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}$ is a Lorentz scalar operator

$$\left.\begin{array}{lrl} \text{space part} & \nabla^2 \mathbf{A} - \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} & = -\mu_0 \mathbf{j} \\ \text{fourth comp.} & \frac{i}{c}\left(\nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2}\right) & = -i\rho c\mu_0 \end{array}\right\} \text{using eq.}(VIII.4) \text{ above}$$

now

$$\text{LHS} = -\mu_0(\mathbf{j}, i\rho c),$$

hence

$$\Box A_\mu = -\mu_0 j_\mu. \qquad (VIII.7)$$

This equation can now be integrated (in principle) to give the 4-potential in terms of the source j_μ.

Step 4: Fields:

$$\begin{aligned} \mathbf{B} &= \text{curl}\,\mathbf{A}, \\ B_1 &= \partial_2 A_3 - \partial_3 A_2, \quad \text{etc} \\ \mathbf{E} &= -\text{grad}\,\phi - \frac{\partial \mathbf{A}}{\partial t}\,, \\ E_1 &= -\partial_1\phi - \frac{\partial A_1}{\partial t}\,, \quad \text{etc} \end{aligned}$$

We need a 4-vector generalisation of the curl. In 4-space this is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu. \qquad (VIII.8)$$

Here this is the EM field tensor (antisymmetric 2nd rank tensor)

2 Elements of $F_{\mu\nu}$

$$
\begin{aligned}
F_{12} &= \partial_1 A_2 - \partial_2 A_1 = B_3, \\
F_{23} &= \partial_2 A_3 - \partial_3 A_2 = B_1, \\
F_{13} &= \partial_1 A_3 - \partial_3 A_1 = -B_2, \\
F_{14} &= \partial_1 A_4 - \partial_4 A_1 = \frac{i}{c}\frac{\partial \phi}{\partial x_1} - \frac{\partial A_1}{\partial ict} = -\frac{i}{c}\left(-\frac{\partial \phi}{\partial x_1} - \frac{\partial A_1}{\partial t}\right) \\
&= -\frac{iE_1}{c}, \\
F_{24} &= \partial_2 A_4 - \partial_4 A_2 = \frac{i}{c}\frac{\partial \phi}{\partial x_2} - \frac{1}{ic}\frac{\partial A_2}{\partial t} = -\frac{iE_2}{c}, \\
F_{34} &= \partial_3 A_4 - \partial_4 A_3 = \frac{i}{c}\frac{\partial \phi}{\partial x_3} - \frac{i}{ic}\frac{\partial A_3}{\partial t} = -\frac{iE_3}{c}, \\
F_{\mu\nu} &= \begin{pmatrix} 0 & B_3 & -B_2 & -i\frac{E_1}{c} \\ -B_3 & 0 & B_1 & -i\frac{E_2}{c} \\ B_2 & -B_1 & 0 & -i\frac{E_3}{c} \\ i\frac{E_1}{c} & i\frac{E_2}{c} & i\frac{E_3}{c} & 0 \end{pmatrix}.
\end{aligned}
\tag{VIII.9}
$$

3 Transformation of Fields

Remember

$$
\begin{aligned}
A'_\mu &= L_{\mu\nu} A_\nu, \\
\frac{\partial}{\partial x'_\mu} &= L_{\mu\nu}\frac{\partial}{\partial x_\nu} ,
\end{aligned}
$$

so it is plausible that $\partial_\mu A_\nu$ will transform by a product of $L_{\mu\nu}$'s — this of course needs more justification which is given by tensor theory. We state

$$
F'_{\mu\nu} = L_{\mu\sigma} L_{\nu\delta} F_{\sigma\delta} = L_{\mu\sigma} F_{\sigma\delta} L^{\mathrm{T}}_{\delta\nu}. \tag{VIII.10}
$$

Written this way, it is a matrix multiplication. Using this equation we can identify the transformation rules for the fields. The RHS is

$$
\begin{aligned}
&\begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}
\begin{pmatrix} 0 & B_3 & -B_2 & -i\frac{E_1}{c} \\ -B_3 & 0 & B_1 & -i\frac{E_2}{c} \\ B_2 & -B_1 & 0 & -i\frac{E_3}{c} \\ i\frac{E_1}{c} & i\frac{E_2}{c} & i\frac{E_3}{c} & 0 \end{pmatrix}
\begin{pmatrix} \gamma & 0 & 0 & -i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \\
&= \begin{pmatrix} 0 & \gamma(B_3 - \beta\frac{E_2}{c}) & -\gamma(B_2 + \beta\frac{E_3}{c}) & -i\frac{E_1}{c} \\ -\gamma(B_3 - \beta\frac{E_2}{c}) & 0 & B_1 & -i\frac{\gamma(E_2 - c\beta B_3)}{c} \\ \gamma(B_2 + \beta\frac{E_3}{c}) & -B_1 & 0 & -i\frac{\gamma(E_3 + c\beta B_2)}{c} \\ i\frac{E_1}{c} & i\frac{\gamma(E_2 - c\beta B_3)}{c} & i\frac{\gamma(E_3 + \beta c B_2)}{c} & 0 \end{pmatrix}
\end{aligned}
$$

Hence for the 3-fields

$$
\begin{aligned}
B'_1 &= B_1, \\
B'_2 &= \gamma(B_2 + \beta\frac{E_3}{c}), \\
B'_3 &= \gamma(B_3 - \beta\frac{E_2}{c}), \\
E'_1 &= E_2, \\
E'_2 &= \gamma(E_2 - c\beta B_3), \\
E'_3 &= \gamma(E_3 + \beta c B_2).
\end{aligned}
\qquad (VIII.11)
$$

For the LT considered, v is along Ox so

$$\mathbf{v} \wedge \mathbf{E}_\perp = (0, -vE_3, vE_2) : \mathbf{v} \wedge \mathbf{B}_\perp = (0, -vB_3, vB_2),$$

where $\mathbf{E}_\perp$ and $\mathbf{B}_\perp$ are the components of the fields $\perp$ to the direction of the boost, thus:

$$B'_{||} = B_{||}, \qquad E'_{||} = E_{||}. \qquad (VIII.11')$$

$$\mathbf{B}'_\perp = \gamma(\mathbf{B}_\perp - \mathbf{v} \wedge \mathbf{E}_\perp/c^2), \qquad \mathbf{E}'_\perp = \gamma(\mathbf{E}_\perp + \mathbf{v} \wedge \mathbf{B}_\perp).$$

The division of EM effects into electric and magnetic fields is frame dependent — a pure field in one frame will become mixed in another. The covariant formalism shows the full unification of electricity and magnetism.

4 Plane EM waves

Consider eq.(VIII.7) without sources:

$$\Box A_\mu = 0 \qquad \text{or} \qquad \nabla^2 A_\mu - \frac{1}{c^2}\frac{\partial^2 A_\mu}{\partial t^2} = 0.$$

The solution is

$$A_\mu(x_\mu) = \epsilon_\mu \mathrm{e}^{ik\cdot x}, \qquad (VIII.12)$$

where ϵ_μ is the polarisation 4-vector (constant)

$$k \cdot x = \mathbf{k} \cdot \mathbf{x} - \omega t, \qquad \text{and} \qquad k^2 = \omega^2/c^2.$$

The gauge condition $\partial_\mu A_\mu = 0$ gives

$$k \cdot \epsilon = 0.$$

5 Point charge

Now that we know how the fields transform we can deduce some physics. Consider a point charge situated at the origin of frame S', in which $\mathbf{E}' = (q/4\pi\epsilon_0)(\mathbf{r}'/r'^3)$ and $\mathbf{B}' = 0$. Now transform to S — we need the inverse LT for the fields ($v \to -v$, $E \longleftrightarrow E'$ etc)

$$\begin{aligned}\mathbf{E} &= \frac{q}{4\pi\epsilon_0}\frac{1}{r'^3}\{x', \gamma y', \gamma z'\}\ ,\\ \mathbf{B} &= \frac{q}{4\pi\epsilon_0}\frac{1}{r'^3}\{0, -\frac{v}{c^2}\gamma z', \frac{v}{c^2}\gamma y'\}\ .\end{aligned}$$

At $t = 0$: $\quad x' = \gamma x, \quad y' = y, \quad z' = z$, so

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\mathbf{r}}{r'^3}\ , \qquad \text{and} \qquad \mathbf{B} = \frac{q}{4\pi\epsilon_0}\frac{\gamma\mathbf{v}\wedge\mathbf{r}}{c^2 r'^3}\ , \tag{VIII.13}$$

where $r'^2 = (\gamma x)^2 + y^2 + z^2$.

Limiting cases

a) Non-relativistic: $\quad \gamma \to 1, \quad r'^2 \to r^2$

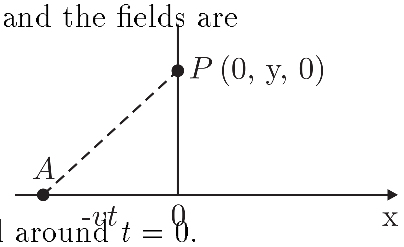
$$\text{then,} \quad \mathbf{E} \to \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3} \quad \text{(Coulomb)}$$

$$\mathbf{B} \to \frac{\mu_0}{4\pi}\frac{q\mathbf{v}\wedge\mathbf{r}}{r^3} \quad \text{(Biot − Savart law)} \quad \text{using } c^2 = 1/\mu_0\epsilon_0$$

b) Extreme relativistic: In this case consider a observer at P $(0, y, 0)$ watching a charge move along the x axis with velocity v. Consider the instant $-t$, then the source charge q is at A so $x' = \gamma vt$, $\quad z' = 0$, $\quad y' = y$, and the fields are

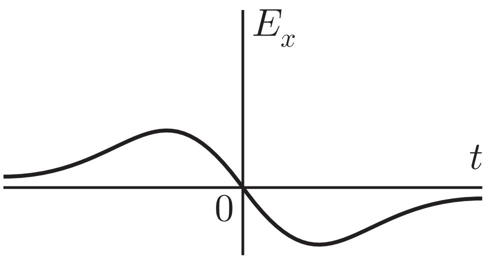
$$\mathbf{E} = \tfrac{q}{4\pi\epsilon_0}\{\gamma vt, \gamma y, 0\}/(\gamma^2 v^2 t^2 + y^2)^{3/2}\ ,$$

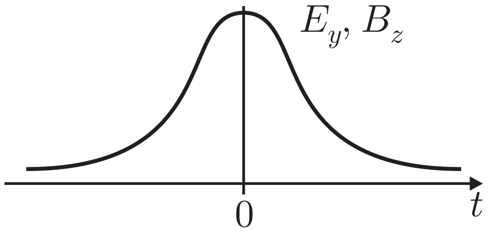
$$\mathbf{B} = \tfrac{q}{4\pi\epsilon_0}\tfrac{v}{c^2}\{0, 0, \gamma y\}/(\gamma^2 v^2 t^2 + y^2)^{3/2}\ ,$$



For large v, E_x is small, E_y and B_z are strongly peaked around $t = 0$.

$$\mathbf{E}\wedge\mathbf{B} \approx \ldots(\gamma^2 y^2, 0, 0).$$





The effect is that of a pulse of EM radiation. This fact can be extended into a useful approximation to deal with the interactions of relativistic electrons.

6 Lorentz force

To avoid confusion write the 4-force as $\mathcal{F}_\mu$. Consider a charge q at rest in IF S' in which there is an electric field $\mathbf{E}'$. Then we know from the definition of electric field that there will be a force on the charge

$$\mathbf{f}' = q\mathbf{E}'. \qquad (VIII.14)$$

In the rest frame $\mathcal{F}'_\mu = (\mathbf{f}', 0)$ so the LHS has an obvious covariant form. What about the RHS? The charge q is an invariant, but $\mathbf{E}'$ is part of $F_{\mu\nu}$ which is not a 4-vector. To get from $F_{\mu\nu}$ to a 4-vector we must contract (multiply) $F_{\mu\nu}$ by a 4-vector — then we have an equation like

$$\mathcal{F}'_\mu = qF'_{\mu\nu}X'_\nu. \qquad (VIII.15)$$

So the question is what to choose for X'_μ?

Look at the structure of $F_{\mu\nu}$ — to pick out only the $\mathbf{E}$ field components X'_μ should have the form $(\mathbf{O}, ia_4)$ i.e. fourth component only. With an eye to the answer we note that there *is* such a 4-vector in the rest-frame, namely 4-velocity $V_\mu = \gamma(\mathbf{v}, ic)$, so $V'_\mu = (0, ic)$ in S'. With this choice eq.(VIII.15) becomes

$$\mathcal{F}_\mu = qF_{\mu\nu}V_\nu, \qquad (VIII.16)$$

(where we have dropped the $'$ since the equation is covariant). This is the covariant form of the Lorentz force equation. We have not shown that it is unique but one can appeal to simplicity, and ultimately of course to experiment.

Consider the components of eq.(VIII.16) in a frame S in which the charge has velocity $\mathbf{v}$.

$$\begin{array}{rrcl} \text{LHS :} & \mathcal{F}_\mu & = & \gamma\left(\mathbf{f}, \frac{i}{c}\mathbf{v}\cdot\mathbf{f}\right) \\ \text{RHS :} & \text{space part} & = & q\gamma(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}) \\ & \text{4th comp.} & = & i\gamma q(\mathbf{v}\cdot\mathbf{E})/c \end{array}$$

so equating them separately gives:

$$\begin{aligned} \mathbf{f} &= q(\mathbf{E} + \mathbf{v}\wedge\mathbf{B}), \\ \mathbf{v}\cdot\mathbf{f} &= q(\mathbf{v}\cdot\mathbf{E}). \end{aligned} \qquad (VIII.17)$$

Therefore this familiar form for the Lorentz force is a consequence of special relativity and is covariant. Historically it was the other way around — Maxwell found the unified theory of elctromagnetism first and provided the motivation for development of special relativity.