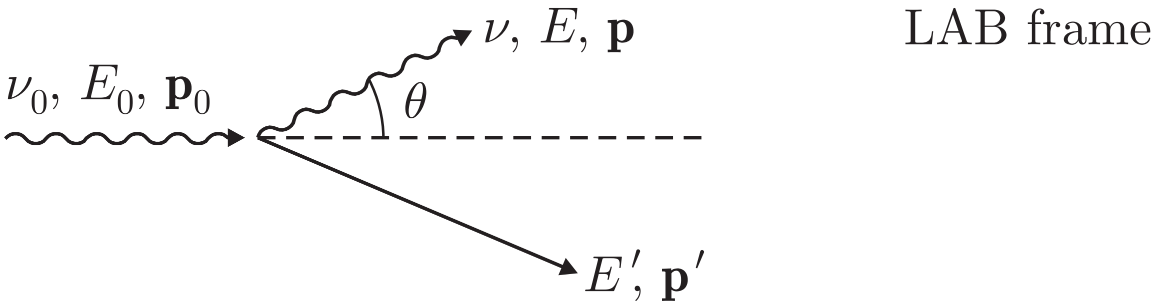
Special Relativity: Lecture 6

1 Compton scattering

$$\gamma + e \to \gamma + e$$

Very important experiment both for confirming the particle nature of light and testing the relativistic mass-velocity relation.



An incident photon of frequency ν_0 is scattered by a stationary electron to give a photon of frequency ν at an angle θ wrt original photon.

Conservation of energy gives

$$E_0 + mc^2 = E + E', \qquad (VI.1)$$

while conservation of 3-momentum gives

$$\mathbf{p}_0 = \mathbf{p} + \mathbf{p}' \qquad \text{or} \quad \mathbf{p}' = \mathbf{p}_0 - \mathbf{p}.$$

Squaring this

$$\mathbf{p}'^2 = \mathbf{p}_0^2 + \mathbf{p}^2 - 2\mathbf{p}_0 \cdot \mathbf{p}$$

and using

$$\begin{aligned} E_0 &= |\mathbf{p}_0 c| \\ E &= |\mathbf{p} c| \\ E'^2 &= (\mathbf{p}' c)^2 + m^2 c^4, \end{aligned}$$

we get

$$E'^2 - m^2 c^4 = E_0^2 + E^2 - 2E_0 E \cos\theta.$$

But from eq.(VI.1) the LHS equals $(E_0 - E)^2 + 2mc^2(E_0 - E)$, thus yielding

$$2mc^2(E_0 - E) = 2E_0 E - 2E_0 E \cos\theta, \qquad (VI.2')$$

or using $E_0 = \dfrac{hc}{\lambda_0}$ etc,

$$\lambda - \lambda_0 = \frac{h}{mc}(1 - \cos\theta). \qquad (VI.2)$$

This is the well known formula for the change in wavelength as a function of angle. The quantity h/mc is known as the Compton wavelength (in this case, of the electron).

From eq.(VI.2')

$$E = \frac{mc^2 E_0}{E_0(1-\cos\theta)+mc^2}\ . \qquad (VI.3)$$

using this with eq.(VI.1), the K.E. of the recoiling electron is given by

$$\begin{aligned} T' &= E' - mc^2 = E_0 - E \\ &= \frac{E_0^2(1-\cos\theta)}{E_0(1-\cos\theta)+mc^2}\ . \end{aligned} \qquad (VI.4)$$

Note that T' is a maximum for $\theta = \pi$ (i.e. photon back scattered) and in this case

$$T'_{\max} = \frac{2E_0^2}{2E_0+mc^2} \qquad \text{and} \qquad E = \frac{mc^2 E_0}{2E_0+mc^2}\ .$$

For $E_0 \gg mc^2$ we get

$$T'_{\max} \to E_0 - \frac{mc^2}{2} \qquad \text{and} \qquad E \to \frac{mc^2}{2}\ ,$$

i.e. the back scattered photon has an energy *independent* of the incident energy.

Some Key Experiments

(i) Cross & Ramsey (*Phys. Rev.* **80** (1950) 929)

Coincidence experiment to check the kinematics of Compton scattering using 2.6 MeV photons.

(ii) Stovall *et al* (*Nuovo Cim.* **5B** (1971) 79)

Measured recoil electron energy from Compton backscattered photons of incident energy 1–3 MeV using a Ge(Li) detector. The position of the "Compton Edge" was determined to within 2 keV. In our notation they measured $T'_{\max}$ as a function of E_0.

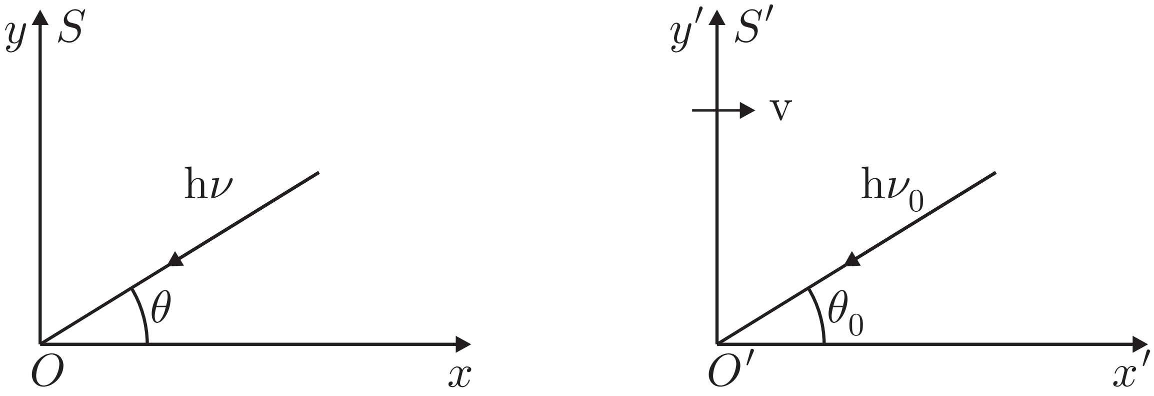
	E_0	$T'_{\max}$	$T'_{\max}$ (calculated)
All energies in keV:	1368.53	1152.4 ± 2.0	1153.2
	2614.46	2382.2 ± 2.0	2381.7
	2753.92	2521.6 ± 2.0	2520.1

Agreement was found with SR to within 0.1%. (The result disagreed by 15 standard deviations from the alternative Abraham theory.)

2 Doppler effect

WARNING! Take care with directions of motion of both frames and photons.

Consider standard IFs S and S'. A light source (e.g. a star) at *rest in* S' emits light of frequency ν_0 at an angle θ_0 (w.r.t. $O'x'$).



What are the frequency ν and angle θ in frame S?
Consider the momentum 4-vector for the photon:

$$\begin{aligned} \text{In} \quad S: \quad p &= \left(-\frac{h\nu}{c}\cos\theta,\ -\frac{h\nu}{c}\sin\theta,\ 0,\ \frac{ih\nu}{c}\right) \\ \text{In} \quad S': \quad p' &= \left(-\frac{h\nu_0}{c}\cos\theta_0,\ -\frac{h\nu_0}{c}\sin\theta_0,\ 0,\ \frac{ih\nu_0}{c}\right). \end{aligned}$$

To get from $S' \to S$, apply the inverse LT:

$$\begin{aligned} -\frac{h\nu}{c}\cos\theta &= \gamma\left(-\frac{h\nu_0}{c}\cos\theta_0 + \beta\frac{h\nu_0}{c}\right) \\ \frac{h\nu}{c}\sin\theta &= \frac{h\nu_0}{c}\sin\theta_0 \\ \frac{h\nu}{c} &= \gamma\left[\frac{h\nu_0}{c} + \beta\left(-\frac{h\nu_0}{c}\cos\theta_0\right)\right] \end{aligned}$$

The last equation gives

$$\nu = \nu_0\gamma(1-\beta\cos\theta_0), \qquad (VI.5)$$

which is the relativistic Doppler formula.

For observational astronomy, eq.(VI.5) is not so useful because both ν_0 and θ_0 refer to the star's frame, not that of the observer. Apply instead the direct LT $S \to S'$ to the photon energy:

$$\begin{aligned} \frac{h\nu_0}{c} &= \gamma\left(\frac{h\nu}{c} + \beta\frac{h\nu}{c}\cos\theta\right), \\ \text{or} \quad \nu_0 &= \gamma\nu\left(1+\beta\cos\theta\right). \end{aligned} \qquad (VI.5')$$

This equation gives ν_0 in terms of quantities measured by the observer. It is sometimes written as $\lambda = \lambda_0\gamma(1+\beta\cos\theta)$.

3 Angles

From the inverse LT (VI.5) we get

$$\tan\theta = \frac{\sin\theta_0}{\gamma(\beta - \cos\theta_0)}\ ,$$

or, using the identity $1 + \tan^2\theta = 1/\cos^2\theta$:

$$\cos\theta = \frac{\beta - \cos\theta_0}{\beta\cos\theta_0 - 1}\ . \qquad (VI.6)$$

Special cases:

(refer to eq.(VI.5))

1. $\theta_0 = 0$ gives $\nu = \nu_0\left(\frac{1-\beta}{1+\beta}\right)^{1/2} \quad \to \nu_0(1-\beta)$ in the NR limit.
   This corresponds to the source moving away from observer (Note $\theta = 0$).

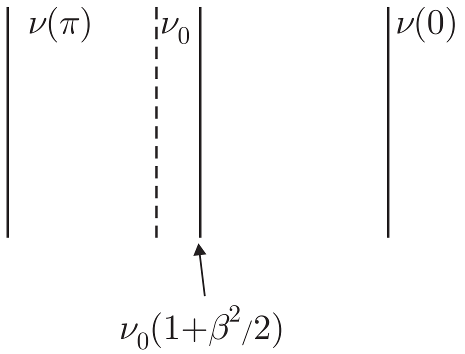
2. $\theta_0 = \pi$ gives $\nu = \nu_0\left(\frac{1+\beta}{1-\beta}\right)^{1/2}$.
   Here the source is moving towards the observer (Note $\theta = \pi$).

3. $\theta_0 = \pi/2$ gives $\nu = \nu_0\gamma$. $\qquad (VI.7)$
   This last is the transverse Doppler effect — a second order relativistic effect.
   It can be thought of as arising from the dilation of time in the moving frame.

Key Experiment Ives & Stilwell (1938)

It is too difficult to test eq.(VI.7) directly, so they used (1) and (2) above to look for evidence of the second order effect in motion directly towards and away from an observer.
Balmer lines from H atoms (486nm) ionised in a discharge tube were employed.
The velocity was determined by the accelerating voltage. Very accurate spectra were produced using a diffraction grating and photographed — $\delta\lambda \sim 2\times 10^{-4}$nm.

towards: $\nu(0) = \nu_0\gamma(1+\beta)$
away: $\nu(\pi) = \nu_0\gamma(1-\beta)$
average: $\dfrac{\nu(0)+\nu(\pi)}{2} = \nu_0\gamma \approx \nu_0(1+\beta^2/2+\cdots)$



The predicted shift was confirmed.

4 Angular Transformation

Suppose the light is emitted isotropically in a star's rest frame S', i.e. $\mathrm{d}N/\mathrm{d}\Omega_0 = k$ (constant), where dN is the number of photons in solid angle $\mathrm{d}\Omega_0$. What is the angular distribution in IF S?
Now

$$\frac{\mathrm{d}N}{\mathrm{d}\Omega} = \frac{\mathrm{d}N}{\mathrm{d}\Omega_0}\frac{\mathrm{d}\Omega_0}{\mathrm{d}\Omega}, \qquad \text{so we need} \qquad \frac{\mathrm{d}\cos\theta_0}{\mathrm{d}\cos\theta} \; .$$

Invert eq.(VI.6) above to get

$$\cos\theta_0 = \frac{\beta + \cos\theta}{\beta\cos\theta + 1} \; ,$$

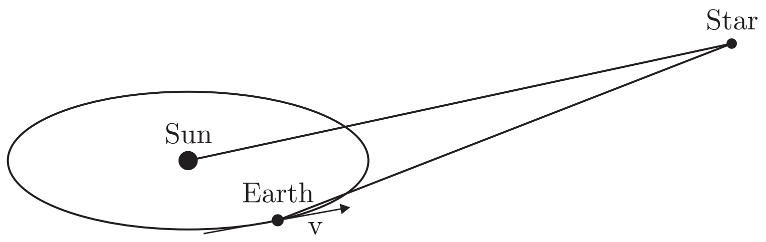
and so

$$\frac{\mathrm{d}N}{\mathrm{d}\Omega} = \frac{k(1-\beta^2)}{(1+\beta\cos\theta)^2} \; . \qquad (VI.8)$$

NB: This is for a source moving away from O. To get the result for motion towards O, just replace $\beta \to -\beta$.

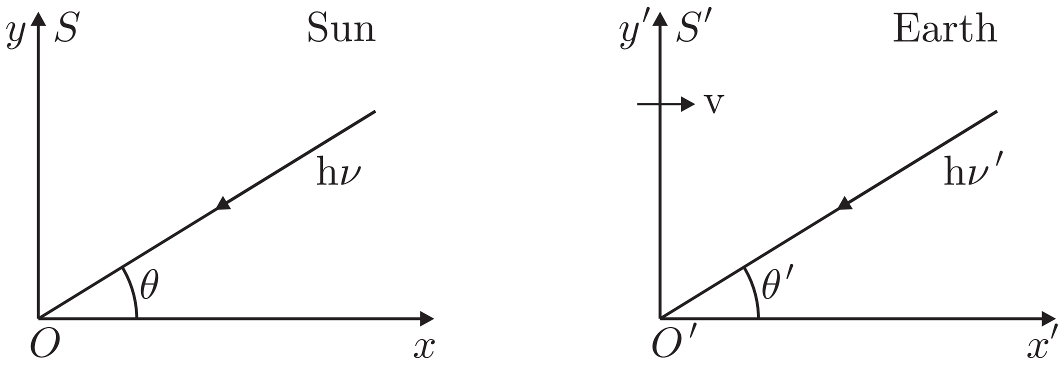
Note that an approaching source looks bright and blue while a receeding source looks dim and red. `(In Problem 8, Set 2 we consider an application of this to observations of the 2.73 K cosmic relic background radiation.)`

5 Stellar Aberration



This is the apparent motion of a star due to rotation of the Earth about the Sun.

Consider an incoming photon from star with 4-momentum p.
S is Sun's frame and S' is Earth frame moving with velocity v.



$$p = \left(-\frac{h\nu}{c}\cos\theta,\ -\frac{h\nu}{c}\sin\theta,\ 0,\ \frac{ih\nu}{c}\right)$$
$$p' = \left(-\frac{h\nu'}{c}\cos\theta',\ -\frac{h\nu'\sin\theta'}{c},\ 0,\ \frac{ih\nu'}{c}\right)$$

Do the direct LT from $S \to S'$ to get:

$$\begin{aligned} \nu'\cos\theta' &= \nu\gamma(\cos\theta+\beta) && \text{(i)} \\ \nu'\sin\theta' &= \nu\sin\theta && \text{(ii)} \\ \nu' &= \nu\gamma(1+\beta\cos\theta) && \text{(iii)} \end{aligned} \qquad (VI.9)$$

Now use (iii) to eliminate ν and ν' giving:

$$\cos\theta' = \left(\frac{\beta+\cos\theta}{1+\beta\cos\theta}\right),$$
$$\text{and,} \qquad \sin\theta' = \frac{\sin\theta}{\gamma(1+\beta\cos\theta)}\,.$$

For small β, this gives $\cos\theta' \approx \cos\theta + \beta\sin^2\theta$.
Define the angle of aberration α by $\theta' = \theta - \alpha$, then

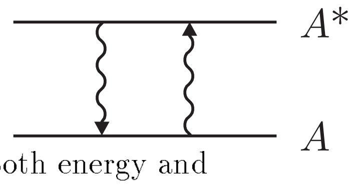
$$\cos(\theta-\alpha) = \cos\theta + \alpha\sin\theta, \qquad \text{since } \alpha \text{ is small.}$$

$$\text{therefore}: \quad \alpha \approx \beta\sin\theta.$$

This is in agreement with the data of Bradley (1729) — a result which caused problems for the aether theory of EM waves (see French's book for details).

6 Recoil Shifts in Emission and Absorption

Consider photon emission or absorption by nucleus A of rest mass M. Suppose the energy level diagram is as shown with $E_0 = (M^* - M)c^2$ where



$M^*c^2 = E_0 + Mc^2$ is the energy of the excited state A^*. Both energy and momentum must be conserved.

• <u>Emission</u>: $A^* \to A + \gamma$ with A^* at rest. Nucleus A must recoil to conserve momentum, therefore energy of the photon is less than E_0

$$E_\gamma = E_0 - \frac{E_0^2}{2M^*c^2} \qquad (VI.10)$$

• <u>Absorption</u>: $\gamma + A \to A^*$ with A at rest. A^* will also recoil therefore for absorption $E_\gamma > E_0$

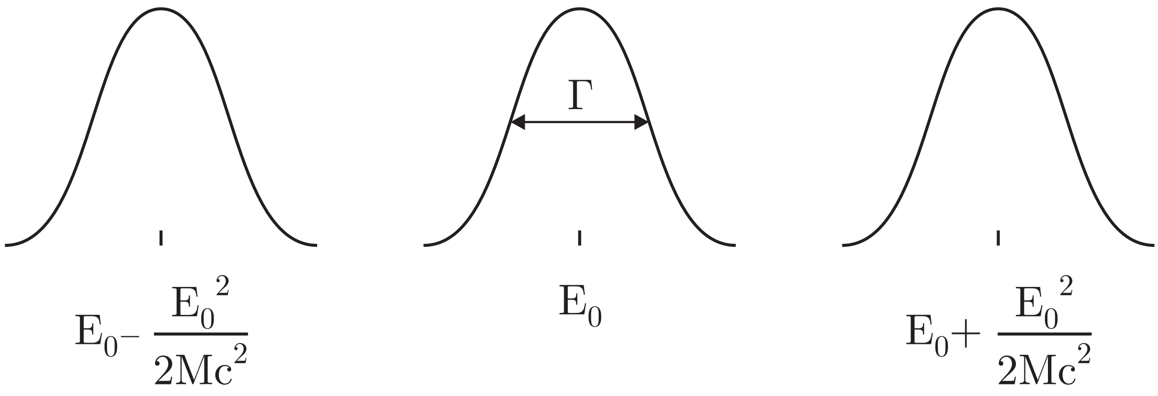
$$E'_\gamma = E_0 + \frac{E_0^2}{2Mc^2} \qquad (VI.10')$$

For most nuclei $E_0 \ll Mc^2$, so the difference between M and M^* can be ignored.

- <u>Magnitude of effect</u>: e.g. for ^{57}Fe

$$\frac{E_0^2}{Mc^2} = \frac{(14.4\ \text{keV})^2}{57 \times 940\ \text{MeV}} = 3.8 \times 10^{-3}\text{eV}$$

How does this compare to the line width?
Quantum mechanics tells us that the mean life-time τ of A^* is related to the line-width Γ by $\Gamma = \hbar/\tau$. For the 14.4 keV line in ^{57}Fe, $\Gamma = 4.7 \times 10^{-9}$ eV, so the recoil shift between emission and absorption is 8×10^5 times the line-width!



Thus under normal circumstances a ^{57}Fe absorber will *not* exhibit resonance absorption of photons from a ^{57}Fe* source.

7 Mössbauer Spectroscopy

The Mössbauer effect is when atoms become locked into a crystal lattice in such a way that the crystal (or part of it) *recoils as a whole*, then in eqs. (VI.10) above, $M \to N_A M$ where N_A is Avogadro's number, and the recoil shift becomes effectively zero. Under these circumstances one can use the sensitivity of the *natural* line width — for ^{57}Fe (14.4 keV)

$$\frac{\Delta\nu}{\nu} = \frac{4.7 \times 10^{-9}}{14.4 \times 10^3} = 3.3 \times 10^{-13}$$

Consider Doppler shifts for small β:

$$\frac{\nu'}{\nu} = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} \approx 1 - \beta, \quad \text{therefore } \frac{\Delta\nu}{\nu} = \beta$$

and one could be sensitive to velocities of the order of 10^{-4} m/s.

Mössbauer tests of SR

(a) Isaak *et al*: "Aether drift" (1963,1970) — discussed in Lecture 1

(b) Hay *et al*: (1960)

Rotating ^{57}Fe* source — experiment can be viewed as test of transverse Doppler effect.