# Special Relativity: Lecture 5

## 1 Units

Many of the practical applications of relativistic mechanics are in nuclear and particle physics for which SI units are not appropriate. A convenient and standard unit is the MeV ("mega" electron volts) as the unit of energy — it is the energy gained by an electron in falling through a potential of $10^6$ volts:

$$1 \text{ MeV} = 1.6 \times 10^{-13} \text{ J}.$$

Similarly one can use GeV ("giga" $\Rightarrow 10^9$ eV), TeV ("tera" $\Rightarrow 10^{12}$ eV), etc. Since $p = (\mathbf{p},\ iE/c)$ is a 4-vector, its length is a Lorentz invariant and has the value $-m_0^2c^2$. Hence

$$E^2 = \mathbf{p}^2c^2 + m_0^2c^4. \qquad (V.1)$$

This is a very useful relation. It also shows that we can measure momentum and mass in energy units, or more precisely:

| | | |
|---|---|---|
| momentum | $\mathbf{p}$ | in MeV/c |
| mass | $m_0$ | in MeV/c$^2$ |

Quite often $c$ is set equal to 1 — which is fine if you remember at the end to check for factors of $c$, or to measure speeds in units of $c$.

## 2 Techniques

1. Use 4-vectors, particularly 4-momentum $p_\mu = (\mathbf{p},\ iE/c)$. Then under the standard LT $S \to S'$:

$$\begin{pmatrix} p'_x \\ p'_y \\ p'_z \\ iE'/c \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ iE/c \end{pmatrix} \qquad (V.2)$$

2. Use invariants where possible — e.g. lengths of 4-vectors and 4-vector products $(A \cdot B)$.

3. Relativistic parameters $\beta$ and $\gamma$ for a single particle or system:

$$\text{Since} \quad p = m_0\gamma(\mathbf{v},\ ic) = \left(\mathbf{p},\ i\frac{E}{c}\right),$$

$$\gamma = \frac{E}{m_0c^2}, \qquad \text{and} \quad \beta = \sqrt{(\gamma^2-1)/\gamma^2} = pc/E\ . \qquad (V.3)$$

# 3 CMS system:

Consider first a system of two particles with 4-momenta $p_1$ and $p_2$ in some IF $S$. The *centre of mass* system (CMS) for the two particles is defined as the IF in which the total 3-momentum of the two particles is zero, i.e. $\mathbf{p}_1^* + \mathbf{p}_2^* = 0$ (we denote CMS quantities with a star). The 4-vector $p_1 + p_2$ takes the following forms:

$$\begin{aligned} p_1 + p_2 &= (\mathbf{p}_1 + \mathbf{p}_2,\ i(E_1 + E_2)/c) && \text{in IF } S, \\ &= (\mathbf{0},\ i(E_1^* + E_2^*)/c) && \text{in IF CMS.} \end{aligned}$$

Thus the system behaves as though it was a single "particle" of mass $(E_1^* + E_2^*)/c^2$. The velocity of the CMS frame wrt IF $S$ is then:

$$\beta_{\text{CMS}} = \frac{(\mathbf{p}_1 + \mathbf{p}_2)c}{(E_1 + E_2)}\ .$$

The same idea can be extended to a group of particles with 4-momenta $p_k = (\mathbf{p}_k,\ iE_k/c)$ in IF $S$. The CMS frame is defined to be the one for which

$$\Sigma_k \mathbf{p}_k^* = \mathbf{0}.$$

Then

$$\begin{aligned} \beta_{\text{CMS}} &= \Sigma_k \mathbf{p}_k c / \textstyle\sum_k E_k, \\ \gamma_{\text{CMS}} &= \Sigma_k E_k / E_{\text{CMS}}, \end{aligned} \tag{V.4}$$

are the $\beta$ and $\gamma$ for the CMS relative to the IF $S$.

**An Useful Invariant:**

Consider the 4-vector

$$cP = \Sigma_k c p_k = \Sigma_k (c\mathbf{p}_k,\ iE_k).$$

In the CMS frame $\Sigma_k \mathbf{p}_k^* = \mathbf{0}$ by definition, so $cP = (\mathbf{0},\ i\Sigma_k E_k^*) = (\mathbf{0},\ iE_{\text{CMS}})$ and $(cP)^2 = -E_{\text{CMS}}^2$.

This is an *invariant* quantity (usually denoted by $s$)

$$s = -(cP)^2 = E_{\text{CMS}}^2, \tag{V.5}$$

which is much used in high energy physics.

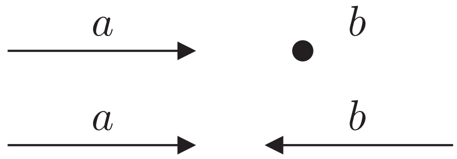
# 4 Example Lab $\leftrightarrow$ CMS

Consider:

$$a + b \to c + d$$

Lab: $E_a^{\mathrm{Lab}}$ — energy of projectile (beam)
$b$ at rest

CMS: $\mathbf{p}_a + \mathbf{p}_b = 0$




Use the invariant $s$ to find the relation between $E_{\mathrm{CMS}}$ and $E_a^{\mathrm{Lab}}$:

$$\begin{array}{rrcl}
\mathrm{Lab}: & s & = & -\{c\mathbf{p}_a,\ i(E_a^{\mathrm{Lab}} + m_b c^2)\}^2 \\
 & & = & -c^2\mathbf{p}_a^2 + (E_a^{\mathrm{Lab}} + m_b c^2)^2 \\
 & & = & (m_a^2 + m_b^2)c^4 + 2m_b E_a^{\mathrm{Lab}} c^2, \quad \text{using } (E_a^{\mathrm{Lab}})^2 - \mathbf{p}_a^2 c^2 = m_a^2 c^4 \\
\mathrm{CMS}: & s & = & E_{\mathrm{CMS}}^2, \qquad \text{by definition,}
\end{array}$$

hence

$$E_{\mathrm{CMS}}^2 = (m_a^2 + m_b^2)c^4 + 2m_b E_a^{\mathrm{Lab}} c^2. \tag{V.6}$$

Note that for energies $\gg$ masses, $E_{\mathrm{CMS}} \sim \sqrt{2m_b c^2 E_a^{\mathrm{Lab}}}$, so that useful energy $E_{\mathrm{CMS}}$ increases only as the square root of $E_a^{\mathrm{Lab}}$.

`(This is why modern particle accelerators are usually colliding-beam rather than fixed-target machines.)`

**Threshold energy for a 2-body reaction:**

The *minimum* energy in the CMS, $E_{\mathrm{CMS}}^{\mathrm{min}}$, is reached when the final particles are produced at rest:

$$E_{\mathrm{CMS}}^{\mathrm{min}} = (m_c + m_d)c^2 \tag{V.7}$$

What is the corresponding lab energy $(E_a^{\mathrm{Lab}})_{\mathrm{min}}$? Invert eq.(V.6) to get:

$$(E_a^{\mathrm{Lab}})_{\mathrm{min}} = \frac{\{(m_c + m_d)^2 - m_a^2 - m_b^2\}c^2}{2m_b}\ , \tag{V.7$'$}$$

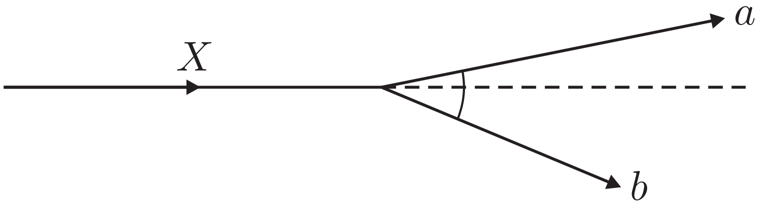
or

$$\begin{array}{lrcl}
\mathrm{KE}: & (T_a^{\mathrm{Lab}})_{\mathrm{min}} & = & (E_a^{\mathrm{Lab}})_{\mathrm{min}} - m_a c^2 \\
 & & = & \dfrac{\{(m_c + m_d)^2 - (m_a + m_b)^2\}c^2}{2m_b}\ .
\end{array} \tag{V.7”}$$

which shows clearly how the difference in mass is related to the minimum KE of the projectile.

# 5 Invariant Mass

The invariant mass of two particles with 4-momenta $p_1$ and $p_2$ is defined by $m_{12}^2c^2 = -(p_1+p_2)^2$. Invariant mass is particularly useful for example, in finding the masses of short-lived unstable particles from the momenta of their observed decay products. Consider the decay of a particle $X \rightarrow a + b$:

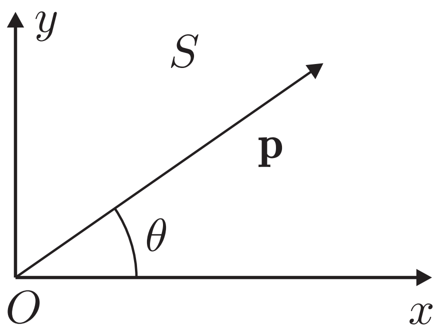


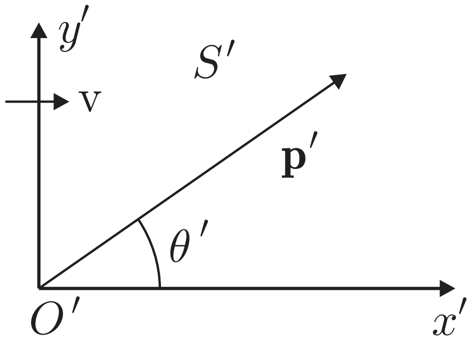

$$\begin{aligned}
\text{since,}\quad p_X &= p_a + p_b \\
-m_X^2c^2 &= (p_a+p_b)^2 \\
&= p_a^2 + p_b^2 + 2p_a.p_b \\
&= -m_a^2c^2 - m_b^2c^2 + 2\mathbf{p}_a.\mathbf{p}_b - 2E_aE_b/c^2 \\
\text{so,}\quad m_X^2c^4 &= (m_a^2+m_b^2)c^4 + 2(E_aE_b - \mathbf{p}_a.\mathbf{p}_bc^2)
\end{aligned}$$

In a high energy experiment, the 3-momenta and masses of the particles $a$ and $b$ must be measured. For charged particles this requires a magnetic field and tracking of the trajectory to measure the bending, as well as some means of particle identification (e.g. through the rate at which they lose energy in passing through matter). One must also identify the vertex and measure the opening angle.

# 6 Relations between Angles

Consider a particle moving in the $xy$ plane and making an angle $\theta$ with the axis $Ox$ in IF $S$. What is the corresponding angle $\theta'$ in IF $S'$, moving with a velocity $v$ along $Ox$?




$$\begin{aligned}
\text{In } S: \qquad p_\mu &= \left(p\cos\theta,\ p\sin\theta,\ 0,\ i\frac{E}{c}\right) \\
\text{In } S': \qquad p'_\mu &= \left(p'\cos\theta',\ p'\sin\theta',\ 0,\ i\frac{E'}{c}\right)
\end{aligned}$$

Apply the Lorentz transformation $S \rightarrow S'$:

$$\begin{aligned} p' \cos\theta' &= \gamma^* \left( p \ \cos\theta - \beta^* \frac{E}{c} \right) \ , \\ p' \sin\theta' &= p \ \sin\theta \ , \end{aligned}$$

so,

$$\begin{aligned} \tan\theta' &= \frac{p \ \sin\theta}{\gamma^*(p \ \cos\theta - \beta^* E/c)} \ , \\ \text{or,} \qquad \tan\theta' &= \frac{\sin\theta}{\gamma^*(\cos\theta - \beta^*/\beta)} \ , \end{aligned} \tag{V.8}$$

where $\beta^* = v/c$ is the velocity of $S'$ wrt $S$ and $\beta = pc/E$ is that of the particle in $S$.

The inverse relation is:

$$\tan\theta = \frac{\sin\theta'}{\gamma^*(\cos\theta' + \beta^*/\beta')} \ , \tag{V.8'}$$

where $\beta' = p'c/E'$ is the velocity of the particle in $S'$.

# 7 Two body decay

Consider the decay process $M \rightarrow a + b$.

The frame in which M is at rest is the CMS frame for $a$ and $b$, in which $a$ and $b$ have equal and opposite 3-momentum $\mathbf{q}$. Then conservation of energy gives:

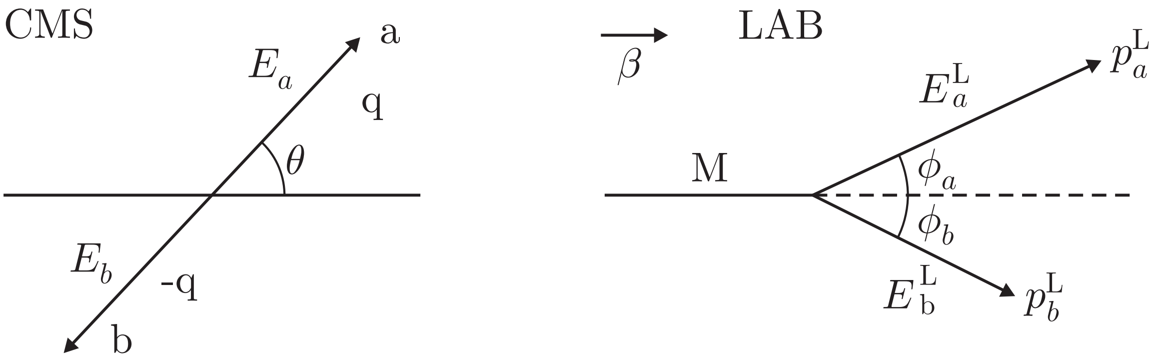
$$\begin{aligned} Mc^2 &= E_a + E_b \\ \text{but,} \qquad E_a^2 &= m_a^2 c^4 + \mathbf{q}^2 c^2 \\ E_b^2 &= m_b^2 c^4 + \mathbf{q}^2 c^2 \ . \end{aligned}$$

Taken together these three relations imply:

$$\begin{aligned} E_a &= \frac{(M^2 + m_a^2 - m_b^2)c^2}{2M} \\ E_b &= \frac{(M^2 + m_b^2 - m_a^2)c^2}{2M} \ . \end{aligned} \tag{V.9}$$

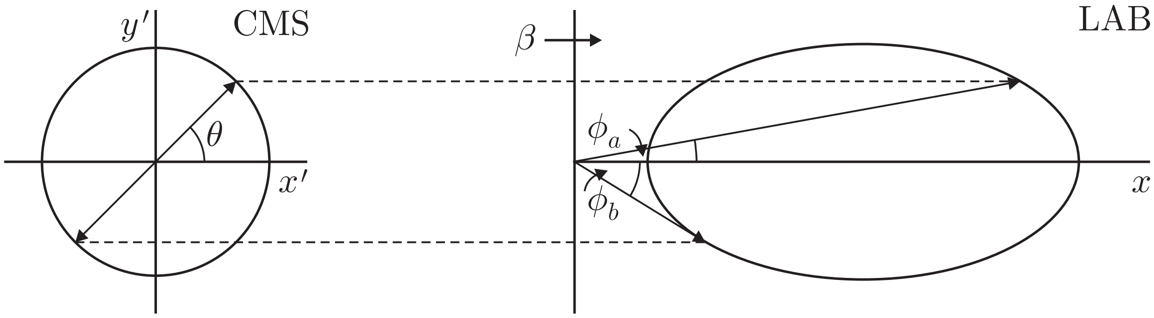
Note that the energies of the decay products are completely determined by the particle masses involved; it is irrelevant what the K.E. of the decaying particle is!

# 8 Angular Distribution of Fragments in the Lab




Suppose the particle M is moving with velocity $\beta = |\mathbf{p}c|/E$ in the Lab system. Use the inverse LT.

$$
\begin{aligned}
p_a^{\mathrm{Lab}} \ \cos\phi_a &= \gamma\ (q\ \cos\theta + \beta E_a/c)\ , \\
p_b^{\mathrm{Lab}} \ \cos\phi_b &= \gamma\ (-q\ \cos\theta + \beta E_b/c)\ .
\end{aligned}
$$




**Example:** $\pi^0 \ \to \ 2\gamma$

What is the opening angle between the two $\gamma$s for $\pi^0$ decay in flight? In the CMS $E_a = E_b = qc$. Use equation (V.8') twice to get:

$$
\tan\phi_1 = \frac{\sin\theta}{\gamma(\cos\theta + \beta)}\ , \qquad \tan\phi_2 = \frac{\sin\theta}{\gamma(-\cos\theta + \beta)}\ ,
$$

so

$$
\begin{aligned}
\tan(\phi_1 + \phi_2) &= \frac{\tan\phi_1 + \tan\phi_2}{1 - \tan\phi_1 \tan\phi_2}\ , \\
&= \frac{2\beta\gamma\sin\theta}{(\gamma^2 - 1)\sin^2\theta - 1}\ .
\end{aligned}
\tag{V.10}
$$

Here $\beta$ and $\gamma$ refer to the parent $\pi^0$.

# 9 Two body scattering

Consider again the reaction $a + b \rightarrow c + d$. In the Lab system $b$ is at rest (fixed target) and a is the projectile with energy $E_a^{\text{Lab}}$ travelling in direction $Ox$.


The velocity of the CMS system in the Lab is

$$\beta_{\text{CMS}} = \frac{\mathbf{p}_a^{\text{Lab}} c}{E_a^{\text{Lab}} + m_b c^2} \ .$$

The CMS energy $E_{\text{CMS}}$ is given by eq.(V.6). Suppose that the final state particles have 4-momenta (in the CMS):

$$\begin{aligned} p_c &= (q\cos\theta,\ q\sin\theta,\ 0,\ i\frac{E_c}{c})\ , \\ p_d &= (-q\cos\theta,\ -q\sin\theta,\ 0,\ i\frac{E_d}{c})\ . \\ \text{Now :} \qquad E_c + E_d &= E_{\text{CMS}}\ , \\ \text{and} \qquad E_c^2 &= m_c^2 c^4 + q^2 c^2\ , \\ E_d^2 &= m_d^2 c^4 + q^2 c^2. \end{aligned}$$

A calculation similar to eq.(V.9) above gives:

$$E_c = \frac{E_{\text{CMS}}^2 + (m_c^2 - m_d^2)c^4}{2E_{\text{CMS}}}\ , \qquad \text{etc}$$

In the case of two-body scattering the energies of the particles depend *only* on the masses and $E_a^{\text{Lab}}$. Once the CMS quantities have been evaluated, the Lab 4-momenta can be found using the inverse LT with velocity $\beta_{\text{CMS}}$.

The angular distribution of final state particles $c$ and $d$ in the Lab can be found in a manner similar to the decay example above. Again the reaction products will be boosted forward along the direction $\mathbf{p}_a^{\text{Lab}}$.

In a colliding beam accelerator, such as LEP (CERN, Geneva) or the Tevatron (Fermilab, nr Chicago), particles of equal mass and opposite charge travel in opposite directions on the same machine orbit. Hence at collision;

$$p_a = (p,\ 0,\ 0,\ iE/c), \qquad p_b = (-p,\ 0,\ 0,\ iE/c)$$

and the Lab frame is also the CMS frame with $E_{\text{CMS}} = 2E$.