Special Relativity: Lecture 4

1 Velocity

How does velocity behave under a Lorentz transformation? Newtonian velocity $\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}}$ cannot be part of a 4-vector without modification as t is not a Lorentz scalar, and $\frac{\mathrm{dx}_\mu}{\mathrm{dt}} = \left(\frac{\mathrm{d}\mathbf{x}}{\mathrm{dt}}, ic\right)$ is not a 4-vector. However using proper time τ we can define a 4-vector "velocity" $U_\mu = \frac{\mathrm{dx}_\mu}{\mathrm{d}\tau}$ which transforms as,

$$U'_\mu = L_{\mu\nu} U_\nu . \qquad (IV.1)$$

How is U_μ related to the usual 3-velocity? Suppose a particle is moving with a velocity $\mathbf{u}$ directed along Ox in S. Then using the chain rule and the relation $\Delta\tau = \Delta t\sqrt{1 - u^2/c^2}$ we have,

$$\begin{aligned} \frac{\mathrm{dx}_\mu}{\mathrm{d}\tau} &= \frac{\mathrm{dx}_\mu}{\mathrm{dt}}\frac{\mathrm{dt}}{\mathrm{d}\tau} , \\ \text{or} \quad \frac{\mathrm{dx}_\mu}{\mathrm{d}\tau} &= (1 - u^2/c^2)^{-1/2}\frac{\mathrm{dx}_\mu}{\mathrm{dt}} , \\ \text{so} \quad U_\mu &= \gamma(u)(\mathbf{u}, ic) , \end{aligned} \qquad (IV.2)$$

with an obvious notation for $\gamma(u)$.

2 Transformation of Velocity

Suppose in S; $U_\mu = \gamma(u)(\mathbf{u}, ic)$ and in S'; $U'_\mu = \gamma(u')(\mathbf{u}', ic)$.
Then applying eq.(IV.1) gives,

$$\gamma(u')\begin{pmatrix} u'_x \\ u'_y \\ u'_z \\ ic \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & i\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\beta\gamma & 0 & 0 & \gamma \end{pmatrix}\begin{pmatrix} u_x \\ u_y \\ u_z \\ ic \end{pmatrix}\gamma(u) \qquad (IV.3)$$

or equivalently the four equations,

$$\begin{aligned} \gamma(u')u'_x &= \gamma(u)\gamma(v)(u_x - v) , \\ \gamma(u')u'_y &= \gamma(u)u_y , \\ \gamma(u')u'_z &= \gamma(u)u_z , \\ \gamma(u') &= \gamma(u)\gamma(v)\left(1 - u_x v/c^2\right) . \end{aligned}$$

Using the last equation to eliminate $\gamma(u')$ we get,

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - u_x v/c^2} \, , \\ u'_y &= \frac{u_y}{\gamma(v)\,(1 - u_x v/c^2)} \, , \\ u'_z &= \frac{u_z}{\gamma(v)\,(1 - u_x v/c^2)} \, . \end{aligned} \tag{IV.4}$$

Notice that u_x, the component of $\mathbf{u}$ parallel to the boost, transforms differently to the transverse components but *all are changed.* From the inverse of the first of eqs.(IV.4) we get the relativistic velocity addition law,

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2}. \tag{IV.5}$$

These results for the transformation of velocity can also be deduced by differentiating the standard LT (II.6) for position and time, e.g. $\frac{dx'}{\mathrm{dt}'} = \frac{dx'}{\mathrm{dt}}\frac{\mathrm{dt}}{\mathrm{dt}'}$ gives the relation between u'_x and u_x.

Note: While we can often make plausible guesses at how to extend a Newtonian 3-vector to a 4-vector there is usually no *unique* generalisation. Ultimately we have to appeal to experiment for corroboration.

3 Limiting Velocity

The velocity addition formula (IV.5) shows almost immediately that c is a limiting velocity. Suppose a body has velocity $c - \epsilon$ approaching that of light from below in S', what velocity will it have in S? (All velocities are parallel to Ox.) Using eq.(IV.5) with $c > v >> \epsilon$, we have

$$u_x = \frac{c - \epsilon + v}{1 + (c - \epsilon)v/c^2} \, .$$

We see that

$$u_x \to c\left(1 - \frac{\epsilon}{c+v}\left[1 - \frac{v}{c}\right] + \ldots\right)$$

so the velocity in S also approaches c from below.

4 4-momentum

Momentum plays a key role in relativisitic mechanics. In Newtonian mechanics the conservation of mass is assumed and the conservation of momentum and energy follow from the equations of motion. We will continue to assume the conservation of momentum and energy `(which follow from fundamental symmetry principles - invariance under space and time translations respectively).`

The conservation of 3-momentum for a closed system can be summarized by the equation $\sum_i \mathbf{p}^i = 0$. Suppose we have identified a 4-vector momentum p with space components $\mathbf{p}$, we would expect that in a closed system a similar result will hold, i.e. $\sum_i p^i = 0$. What is the fourth component of p and what does its conservation mean?

Remembering the form of the 4-velocity, an obvious candidate for 4-momentum is

$$p_\mu = m_0 U_\mu \tag{IV.6}$$

where m_0 is a constant.
Consider now the non-relativistic (NR) limit of the space components of p_μ. From eq.(IV.2), we have $p_\mu = m_0\gamma(v)(\mathbf{v}, ic)$ so $\mathbf{p} \to m_0\mathbf{v}$ as $v/c \to 0$ so m_0 can be identified as the Newtonian inertial mass or *rest mass*.
Now consider the fourth component

$$p_4 = i m_0 c \gamma(v), \tag{IV.7}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$. For small v/c we can expand:

$$\begin{aligned} p_4 &= i m_0 c\,(1 + \frac{v^2}{2c^2} + \cdots) \\ \text{or} \qquad p_4 c &= i\,(m_0 c^2 + \frac{m_0 v^2}{2} + \cdots) \end{aligned}$$

The second term in the bracket can be recognised as the expression for the NR kinetic energy of a particle.

Using this clue, write $p_4 c = iE$ where E is to be identified with energy; then

$$\begin{aligned} E &= m_0 c^2 + \frac{m_0 v^2}{2} + \cdots \qquad \text{(non-relativistic)} \\ \text{or} \qquad E &= m_0 c^2 \gamma \qquad \text{(for the relativistic case)} \end{aligned}$$

Later we shall consider other reasons for believing this expression to be correct. The Newtonian kinetic energy for a relativistic particle becomes,

$$T = E - m_0 c^2. \tag{IV.8}$$

The major difference in the relativistic expression for energy is to associate an energy with the rest mass $E_0 = m_0 c^2$. Notice that for conservation of energy to be correct relativistically the rest mass term must be included and the possibility of transformations between mass and energy arises. The theory of relativity gives no clue as to *how* such transformations could take place. As we know they do occur and with dramatic consequences.

The relativistic 3-momentum is defined by,

$$\mathbf{p} = m_0 \gamma(v)\mathbf{v}. \tag{IV.9}$$

$m_0\gamma(v)$ can be identified as the relativistic inertial mass and is a function of velocity.

5 Transformation of 4-momentum

We write the 4-vector p_μ as $(\mathbf{p}, i\frac{E}{c})$, and its transformation from $S \to S'$ is $p'_\mu = L_{\mu\nu} p_\nu$ or:

$$\begin{aligned} p'_x &= \gamma\left(p_x - \beta\frac{E}{c}\right) \\ p'_y &= p_y \\ p'_z &= p_z \\ \frac{E'}{c} &= \gamma\left(\frac{E}{c} - \beta p_x\right) \end{aligned} \tag{$IV.10$}$$

The inverse transformation is:

$$\begin{aligned} p_x &= \gamma\left(p'_x + \beta\frac{E'}{c}\right) \\ p_y &= p'_y \\ p_z &= p'_z \\ \frac{E}{c} &= \gamma\left(\frac{E'}{c} + \beta p'_x\right) \end{aligned} \tag{$IV.10'$}$$

These transformation equations for momentum are much simpler than those for velocity.

6 Particles with Zero Mass

Since p_μ is a 4-vector, its length is an invariant so from eqs (IV.6) and (IV.2) we have that,

$$p^2 = p_\mu p_\mu = m_0^2 U^2 = m_0^2 \gamma(u)^2 (u^2 - c^2) = -m_0^2 c^2 \tag{$IV.11$}$$

which gives $\mathbf{p}^2 - \dfrac{E^2}{c^2} = -m_0^2 c^2$. For particles with zero rest mass, such as the photon: $E = |\mathbf{p}c|$. This result can also be deduced from classical EM (use the Poynting vector). Photons have momentum as well as energy and can exert a pressure. All particles with zero mass travel in vacuum at the speed of light.

`(In Nature` **`406`** `(2000) 277, Wang` *et al* `claim to have observed faster-than-`c `propagation of light pulses in atomic caesium gas! However this is` *not* `in violation of Special Relativity - just a manifestation of anomalous dispersion.  For a pictorial explanation see http://www.astro.ucla.edu/~wright/anomalous-dispersion.html).`

7 Force

Let us try

$$F_\mu = m_0 \frac{\mathrm{d}^2 x_\mu}{\mathrm{d}\tau^2} = m_0 \frac{\mathrm{d}U_\mu}{\mathrm{d}\tau}$$

or

$$F_\mu = \frac{\mathrm{d}p_\mu}{\mathrm{d}\tau}.$$

These expressions appear to be identical if m_0 is independent of τ, but to be consistent we will use the second form.

Since $\dfrac{\mathrm{d}t}{\mathrm{d}\tau} = \gamma, \quad F_\mu = \dfrac{\mathrm{d}p_\mu}{\mathrm{d}t}\dfrac{\mathrm{d}t}{\mathrm{d}\tau}$, we write

$$F_\mu = \gamma \frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{p}, i\frac{E}{c}\right). \qquad (IV.12)$$

Define $\mathbf{f} = \dfrac{\mathrm{d}\mathbf{p}}{\mathrm{d}t}$ to be the relativistic 3-force,

$$\text{then} \qquad F_\mu = \gamma(u)\left(\mathbf{f}, \frac{i}{c}\frac{\mathrm{d}E}{\mathrm{d}t}\right). \qquad (IV.12')$$

So far we have been working in a frame in which the particle has velocity u.

8 Rate of Working

Consider the 4-scalar product $U \cdot F$ (velocity·force)

$$\begin{aligned} U \cdot F &= (\gamma\mathbf{u}, i\gamma c) \cdot (\gamma\mathbf{f}, \tfrac{i\gamma}{c}\tfrac{\mathrm{d}E}{\mathrm{d}t}) \\ &= \gamma^2(\mathbf{u}.\mathbf{f} - \tfrac{\mathrm{d}E}{\mathrm{d}t}), \end{aligned}$$

but this is an invariant and can therefore be evaluated in the particle rest frame in which $\mathbf{u} = 0, \gamma(u) = 1$ and $E = m_0 c^2$:

$$U \cdot F|_{\text{rest frame}} = -c^2 \frac{\mathrm{d}m_0}{\mathrm{d}t}.$$

Suppose m_0 is a constant (no change in rest-mass of the particle) then $U \cdot F = 0$ and we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \mathbf{u} \cdot \mathbf{f} \qquad (IV.13)$$

In the NR limit the RHS of this equation is the rate of working of the force $\mathbf{f}$. The LHS is the time rate of change of the (relativistic) energy. We thus have a consistent definition of force and a second line of argument for identifying the fourth component of the 4-momentum with energy.

We will leave force at this point to develop relativistic kinematics but in a later lecture we will return to the question of dynamics and relativistic equations of motion.

9 Summary

The 4-momentum $p_\mu = (\mathbf{p}, iE/c)$ is the natural variable to use in relativistic mechanics. It is much easier to use than velocity. To summarise the steps we have taken to establish the 4-momentum:

1. 4-vector formalism and the LT in the form $x'_\mu = L_{\mu\nu} x_\nu$.
2. Construction of $p_\mu = m_0 \dfrac{\mathrm{d}x_\mu}{\mathrm{d}\tau}$ and the identification of m_0 as the inertial mass.
3. Identification of the fourth component p_4 with energy (including the rest mass term $m_0 c^2$)
4. Conservation of 4-momentum and its transformation equations.