COMPLEX NUMBERS AND DIFFERENTIAL EQUATIONS

PROBLEM SET 4

more challenging problems for eg the vacation or revision

Problems and solutions courtesy Julia Yeomans
Comments and corrections to Michael Barnes

Complex Numbers

1. (i) Obtain and sketch the locus in the complex plane defined by Re $z^{-1} = 1$. On the same picture sketch the locus defined by Im $z^{-1} = 1$. At what angle do these loci intersect one another? Show that the unit circle touches both loci but crosses neither of them.

(ii) Make a sketch of the complex plane showing a typical pair of complex numbers z_1 and z_2 which satisfy the equations

$$z_2 - z_1 = (z_1 - a)e^{2\pi i/3},$$
$$a - z_2 = (z_2 - z_1)e^{2\pi i/3}$$

where a is a real positive constant. Describe the geometrical figure whose vertices are z_1, z_2 and a.

2. The polynomial $f(z)$ is defined by

$$f(z) = z^5 - 6z^4 + 15z^3 - 34z^2 + 36z - 48.$$

Show that the equation $f(z) = 0$ has two purely imaginary roots. Hence, or otherwise, factorize $f(z)$, and find all of its roots. Check that the sum and product of the roots take the expected values.

3. Show that the equation $(z+1)^n - e^{2in\theta}(z-1)^n = 0$ has roots $z = -i\cot(\theta + r\pi/n)$, $\quad r = 0, n-1$. Hence show that

$$\Pi_{r=1}^{n} \cot\left(\theta + \frac{r\pi}{n}\right) = \begin{cases} (-1)^{n/2}, & \text{for n even,} \\ (-1)^{(n-1)/2}\cot n\theta, & \text{for n odd.} \end{cases}$$

4. Find all the roots, real and complex, of the equation $z^3 - 1 = 0$. If ω is one of the complex roots prove that $1 + \omega + \omega^2 = 0$. Find the sums of the following series:

$$S_1 = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \ldots; \qquad S_2 = x + \frac{x^4}{4!} + \frac{x^7}{7!} + \ldots; \qquad S_3 = \frac{x^2}{2!} + \frac{x^5}{5!} + \frac{x^8}{8!} + \ldots.$$

Differential Equations

5. When a varying couple $I\cos nt$ is applied to a torsional pendulum with natural period $2\pi/m$ and moment of inertia I, the angle of the pendulum satisfies the equation of motion $\frac{d^2\theta}{dt^2} + m^2\theta = \cos nt$. The couple is first applied at time $t = 0$ when the pendulum is at rest in equilibrium. Show that in the subsequent motion the root mean square angular displacement is $1/\left|m^2 - n^2\right|$ when the average is taken over a time large compared to $1/\left|m - n\right|$. Discuss the motion as $\left|m - n\right| \mapsto 0$.

6. Prove that

$$\frac{d^2\theta}{dt^2} = \frac{1}{2}\frac{du}{d\theta}$$

where $u = \left(\frac{d\theta}{dt}\right)^2$.

A simple pendulum with damping proportional to the square of its velocity is described by the equation

$$2\frac{d^2\theta}{dt^2} + k\left(\frac{d\theta}{dt}\right)^2 = -\lambda \sin\theta$$

where θ is the angular displacement from the downwards vertical and k and λ are constants. By writing this equation in terms of the variable u, or otherwise, obtain an expression for the square of the angular velocity of the pendulum as a function of θ.

The pendulum is given an initial angular velocity ω_0 at its equilibrium position $\theta = 0$. Show that it will just reach the horizontal if

$$\omega_0^2 = \frac{\lambda}{1+k^2}\left(ke^{\frac{k\pi}{2}} + 1\right).$$

7. The position vector of a particle of charge q and mass m that moves in a region of space where there is a magnetic field $\mathbf{B}$ and an electric field $\mathbf{E}$ obeys the equation of motion

$$m\frac{d^2\mathbf{r}}{dt^2} = q\left(\mathbf{E} + \frac{\mathbf{dr}}{\mathbf{dt}} \times \mathbf{B}\right).$$

If $\mathbf{B}$ is along the x-axis and $\mathbf{E}$ is along the z-axis, write down the equations of motion for each component of $\mathbf{r}$, and solve the resulting coupled differential equations.

At time t = 0 the charge is at the origin with velocity
(a) $v = (0; E = B; 0)$,
(b) $v = (0; E = 2B; 0)$,
(c) $v = (0; E = B; E = B)$.
Find, and sketch, its trajectory for each initial condition.

8. A mass m is constrained to move in a straight line and is attached to a spring of strength $\lambda^2 m$ and a dashpot which produces a retarding force $\alpha m v$ where v is the velocity of the mass. Find the displacement of the mass when an amplitude-modulated periodic force $Am \cos pt \sin \omega t$ with $p \ll \omega$ and $\alpha \ll \omega$ is applied to it.

Show that for $\omega = \lambda$ the displacement is the amplitude-modulated wave

$$x = -2\frac{\cos\omega t \sin(pt + \phi)}{\sqrt{4\omega^2 p^2 + \alpha^2\omega^2}} \qquad \text{where} \qquad \tan\phi = \alpha/2p.$$

9. $y(x) = 1/x$ is one of the solutions of the differential equation

$$F(x,y) = x(x+1)\frac{d^2y}{dx^2} + (2 - x^2)\frac{dy}{dx} - (2+x)y = 0.$$

Find a second linearly independent solution by setting $y_2(x) = y_1(x)u(x)$ (or by noting the sum of the coefficients in the equation).

Hence, using the variation of parameters method, find the general solution of

$$F(x,y) = (x+1)^2.$$