COMPLEX NUMBERS AND DIFFERENTIAL EQUATIONS

PROBLEM SET 2

Problems and solutions courtesy Julia Yeomans
Comments and corrections to Michael Barnes

1. Explain carefully:
(i) what is meant by the order of a differential equation,
(ii) the difference between independent and dependent variables in a differential equation,
(iii) the difference between an ordinary and a partial differential equation,
(iv) when a differential equation is linear, and why this is important.
(v) Are the following differential equations linear or non-linear?

(i) $\frac{d^2y}{dx^2} + k^2y = f(x)$

(ii) $\frac{d^3y}{dx^3} + 2y\left(\frac{dy}{dx}\right) = \sin x$

(iii) $\frac{dy}{dx} + y^2 = yx$

2. Solve the following differential equations using the method stated:
(a) separable

(i) $\frac{dy}{dx} = \frac{xe^y}{1+x^2}, \quad y = 0 \text{ at } x = 0$

(ii) $\frac{dx}{dt} = \frac{2tx^2+t}{t^2x-x}$

(b) "almost" separable

$\frac{dy}{dx} = 2(2x+y)^2$

(c) homogeneous

$2\frac{dy}{dx} = \frac{xy+y^2}{x^2}$

(d) homogeneous but for constant

$\frac{dy}{dx} = \frac{x+y-1}{x-y-2}$

(e) integrating factor

(i) $\frac{dy}{dx} + \frac{y}{x} = 3, \qquad y = 0 \text{ at } x = 0$

(ii) $\frac{dx}{dt} + x\cos t = \sin 2t$

(f) Bernoulli

$\frac{dy}{dx} + y = xy^{\frac{2}{3}}$

3. Solve the following 1st order differential equations

(i) $\frac{dy}{dx} = \frac{x - y\cos x}{\sin x}$

(ii) $(3x + x^2)\frac{dy}{dx} = 5y - 8$

(iii) $2\frac{dy}{dx} = \frac{y}{x} + \frac{y^3}{x^3}$

(iv) $xy\frac{dy}{dx} - y^2 = (x + y)^2 e^{-y/x}$

(v) $x(x - 1)\frac{dy}{dx} + y = x(x - 1)^2$

(vi) $\frac{dx}{dt} = \cos(x + t), \quad x = \frac{\pi}{2}$ at $t = 0$

(vii) $\frac{dy}{dx} = \frac{x - y}{x - y + 1}$

(viii) $\frac{dx}{dy} = \cos 2y - x\cot y, \quad x = \frac{1}{2}$ at $y = \frac{\pi}{2}$

4. The equation

$$\frac{dy}{dx} + ky = y^n \sin x,$$

where k and n are constants, is linear and homogeneous for $n = 1$. State a property of the solutions to this equation for $n = 1$ that is not true for $n \neq 1$.
Solve the equation for $n \neq 1$ by making the substitution $z = y^{1-n}$.

5. Solve the ordinary differential equation

$$\frac{dy}{dx} = \frac{(3x^2 + 2xy + y^2)\sin x - (6x + 2y)\cos x}{(2x + 2y)\cos x}.$$

(Hint: look for a function $f(x, y)$ whose differential df gives the o.d.e.)

Answers

2. (a) (i) $e^{-y} = 1 - \frac{1}{2}\ln(1+x^2)$

   (ii) $2x^2 + 1 = C(t^2-1)^2$

   (b) $2x + y = \tan(2x + C)$

   (c) $Cx^{\frac{1}{2}}y = y - x$

   (d) $\ln(x - \frac{3}{2}) = \tan^{-1}\left(\frac{y+\frac{1}{2}}{x-\frac{3}{2}}\right) - \frac{1}{2}\ln\left(1 + \frac{(y+\frac{1}{2})^2}{(x-\frac{3}{2})^2}\right) + C$

   (e) (i) $y = \frac{3x}{2}$

   (ii) $x = 2(\sin t - 1) + Ce^{-\sin t}$

   (f) $y^{\frac{1}{3}} = x - 3 + Ce^{-\frac{x}{3}}$

3. (i) $y = \frac{1}{\sin x}(x^2/2 + C)$

   (ii) $\frac{1}{5}\ln(5y-8) = \frac{1}{3}\ln\left(x/(x+3)\right) + C$

   (iii) $x = C(1 - x^2/y^2)$

   (iv) $\ln x = \frac{e^{y/x}}{1+y/x} + C$

   (v) $y = \frac{x}{x-1}(\frac{1}{2}x^2 - 2x + \ln x + C)$

   (vi) $\tan\frac{1}{2}(x+t) = 1 + t$

   (vii) $x - y + 1 = \sqrt{2(x+C)}$

   (viii) $x = \frac{1}{3}\cot y(3 - 2\cos^2 y) + \frac{1}{2\sin y}$

4. $y^{1-n} = Ce^{-ax} + \frac{(1-n)}{(1+a^2)}\{a\sin x - \cos x\}; \quad a = k(1-n)$

5. $(3x^2 + 2xy + y^2)\cos x = C$