COMPLEX NUMBERS AND DIFFERENTIAL EQUATIONS

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I. COMPLEX NUMBERS

A. GETTING STARTED

1. Definitions, Cartesian representation

Complex numbers are a natural addition to the number system. Consider the equation

$$x^2 = -1.$$

This is a polynomial in x^2 so it should have 2 roots. To make this work we define i as the square root of -1:

$$i^2 = -1$$

so

$$x^2 = i^2; \qquad x = \pm i.$$

A general complex number is written as

$$z = x + iy.$$

x is the **real part** of the complex number, sometimes written $\mathcal{R}\text{e}(z)$.
y is the **imaginary part** of the complex number, sometimes written $\mathcal{I}\text{m}(z)$.
The **complex conjugate** of z is defined as $z^* = x - iy$.

2. Argand diagram

A pair of numbers (x, y) are needed to specify a complex number z. Therefore z can be represented point in a 2D plane called the **complex plane** or **Argand diagram**. It is sometimes helpful to think of $z = x + iy$ as a vector from the origin to (x,y).

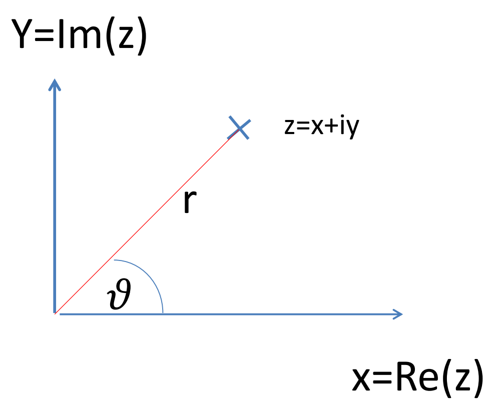

Figure 1: The complex plane

3. Polar form

A complex number z can also be written in terms of polar co-ordinates (r, θ) where

$$x = r\cos\theta \quad , \quad y = r\sin\theta.$$
$$r^2 = x^2 + y^2, \qquad \theta = \tan^{-1}\frac{y}{x}.$$

so

$$z = x + iy = r(\cos\theta + i\sin\theta).$$

r is the **modulus** of z written as $|z|$ or mod(z).
θ is the **argument** of z written as arg(z).

Examples

$$z = 1 + i = \sqrt{2}(\cos\pi/4 + i\sin\pi/4), \qquad z = -1 + \sqrt{3}i = 2(\cos 2\pi/3 + i\sin 2\pi/3)$$

4. Complex exponentials

It is often very useful to write a complex number as an exponential with a complex argument. To justify why we can do this write the polar expression for z and expand the sin and cos using a Taylor expansion:

$$\begin{aligned} z = r(\cos\theta + i\sin\theta) \qquad &= r(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \ldots) + ir(\theta - \frac{\theta^3}{3!} + \ldots) \\ &= r(1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} - \ldots) \\ &= r(1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \ldots) \equiv re^{i\theta}. \end{aligned}$$

We have ended up with the Euler equation

$$re^{i\theta} = z = r(\cos\theta + i\sin\theta). \tag{1}$$

Taking the complex conjugate

$$re^{-i\theta} = z = r(\cos\theta - i\sin\theta). \tag{2}$$

Adding/subtracting Eqns. (1) and (2) gives a pair of very useful identities which you should learn:

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \qquad \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{3}$$

5. Arithmetic manipulation

The next job is to define how to add, subtract, multiply and divide complex numbers. If $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ then

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2), \qquad z_1 - z_2 = x_1 - x_2 + i(y_1 - y_2),$$

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2).$$

To divide two complex numbers note that

$$zz^* = (x + iy)(x - iy) = x^2 + y^2 \equiv |z|^2$$

is real. So multiplying a quotient of complex numbers by the complex conjugate of the denominator gives a tractable expression

$$\frac{z_1}{z_2} = \frac{x_1 + iy_1}{x_2 + iy_2} = \frac{(x_1 + iy_1)(x_2 - iy_2)}{(x_2 + iy_2)(x_2 - iy_2)} = \frac{(x_1x_2 + y_1y_2) + i(y_1x_2 - x_1y_2)}{x_2^2 + y_2^2}.$$

There are simpler formulas for multiplication and division in polars. If $z_1 = r_1e^{i\theta_1}$ and $z_2 = r_2e^{i\theta_2}$

$$z_1z_2 = r_1r_2e^{i(\theta_1+\theta_2)}, \qquad \frac{z_1}{z_2} = \frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}.$$

These expressions immediately imply

$$|z_1z_2| = |z_1|\,|z_2|, \qquad \left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|}.$$

$$\arg(z_1z_2) = \arg(z_1) + \arg(z_2), \qquad \arg\left(\frac{z_1}{z_2}\right) = \arg(z_1) - \arg(z_2).$$

6. Curves in the complex plane

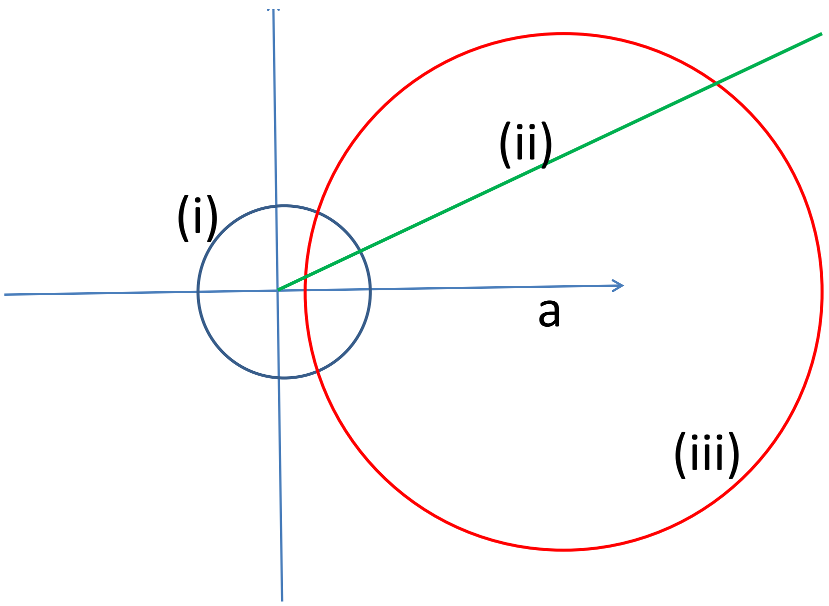

Figure 2: Curves in the complex plane. (i) $|z| = 1$, (ii) $\arg(z) = \pi/6$, (iii) $|z - a| = 3$, a real.

If it is hard to identify a curve by inspection write z as $x + iy$. For the example (iii) in Fig. 2

$$|z - a| = |x - a + iy| = \sqrt{(x-a)^2 + y^2} = 3.$$

So this is the curve $(x - a)^2 + y^2 = 9$ which is a circle, center $(a, 0)$, radius 3.

B. DE MOIVRE'S THEOREM

1. De Moivre's theorem

De Moivre's theorem states

$$z^n = \{r(\cos\theta + i\sin\theta)\}^n = r^n(\cos n\theta + i\sin n\theta).$$

This follows immediately from the properties of complex exponentials:

$$\text{l.h.s.} = (re^{i\theta})^n = r^n e^{in\theta} = \text{r.h.s.}$$

2. Trig. functions of multiple angles → powers of trig. functions

As an example we will use de Moivre's theorem to prove

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta.$$

Consider

$$\cos 3\theta + i\sin 3\theta = e^{3i\theta} = (e^{i\theta})^3 = (\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta.$$

Equating the imaginary parts of the l.h.s. and the r.h.s. of this expression gives the required result. Equating the real parts gives a similar expression for $\cos 3\theta$.

3. Powers of trig. functions → trig. functions of multiple angles

If instead we want to prove

$$\sin^3\theta = -\frac{1}{4}(\sin 3\theta - 3\sin\theta)$$

the easiest way is to use Eq. (3) to write

$$\begin{aligned}\sin^3\theta = \quad & \left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right)^3 = \frac{e^{3i\theta} - 3e^{i\theta} + 3e^{-i\theta} - e^{-3i\theta}}{(2i)^3} \\ = & \frac{1}{(2i)^2}\left\{\frac{(e^{3i\theta} - e^{-3i\theta})}{2i} - 3\frac{(e^{i\theta} - e^{-i\theta})}{2i}\right\} = -\frac{1}{4}(\sin 3\theta - 3\sin\theta).\end{aligned}$$

4. Powers and roots of complex numbers

To find powers and root of complex numbers it is almost always easiest to write them as complex exponentials and it is often important to include a factor $e^{2\pi ir}$ where r is an integer. This is just unity but, as we shall see, it is needed to obtain the correct number of roots. It is good practice to always check that the number of roots is indeed correct.

Example 1: To find the cube roots of $1+i$ write $1+i$ in polar form, including a factor $e^{2\pi ir}$:

$$1 + i = \sqrt{2}e^{(i\pi/4 + 2\pi ir)}.$$

Take the cube root of the modulus as usual ($\sqrt{2}^{1/3} = 2^{1/6}$, it's easy to forget this step) and the cube root of the exponential by dividing the exponent by 3

$$z = (1+i)^{1/3} = 2^{1/6}e^{2\pi i(1/24 + r/3)}$$

We expect 3 distinct roots, because we are solving a cubic equation $z^3 = 1 + i$, so $r = 0, 1, 2,$ say. (Check that $r = 3$ gives the same value for z as $r = 0$). Without the $e^{2\pi ir}$ term we would have found only one solution.

You might like to try these yourself and then check.

Example 2: Find the fourth root of $-16i$.

$$(-16i)^{1/4} = \{16e^{3\pi i/2+2\pi ir}\}^{1/4} = 2e^{2\pi i(3/16+r/4)}, \quad r = 0, 1, 2, 3$$

Example 3: Find all the values of 1^i.

$$1^i = (e^{2\pi ir})^i = e^{-2\pi r}, \quad \text{r any integer.}$$

5. Polynomials: sums and products of roots

This is a theorem which is useful in complex number problems - and elsewhere. A polynomial equation

$$az^n + bz^{n-1} + cz^{n-2} + \cdots + fz^0 = 0$$

has n roots $z_1, z_2 \ldots z_n$, say. Then the sum and product of all the roots are

$$\sum_{i=1}^{n} z_i = -b/a, \qquad \prod_{i=1}^{n} z_i = (-1)^n f/a.$$

These relations can be derived by considering the following alternate form of the polynomial equation:

$$a\prod_{i=1}^{n}(z - z_i) = 0.$$

Equating the coefficients associated with z^{n-1} and z^0 to their counterparts in the original polynomial equation leads to the desired sum and product rules. Demonstration for $n = 4$:

$$\begin{aligned}(z - z_1)(z - z_2)(z - z_3)(z - z_4) \quad = \quad & z^4 - (z_1 + z_2 + z_3 + z_4)z^3 \\ & +(z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4)z^2 \\ & -(z_1z_2z_3 + z_2z_3z_4 + z_3z_4z_1 + z_4z_1z_2)z + (z_1z_2z_3z_4).\end{aligned}$$

6. Using complex numbers and the roots formulas to prove trig. identities

Example: By finding the roots of the equation

$$z^{2n+1} + 1 = 0 \tag{4}$$

show that

$$\sum_{r=-n}^{n} \cos\frac{(2r+1)\pi}{(2n+1)} = 0. \tag{5}$$

We shall do this by using the sum of roots of a polynomial formula. First find the roots of Eq. (4):

$$z = (-1)^{1/(2n+1)} = e^{(i\pi+2\pi ir)/(2n+1)} = e^{(2r+1)i\pi/(2n+1)}, \qquad r = -n, -n+1, -n+2 \ldots n-2, n-1, n$$

where I have chosen the $2n+1$ values of r needed to specify distinct roots to match the limits on the sum in the expression (5). Note that the roots are equispaced around the unit circle in the complex plane. Eq. (4) has no term in z^{2n} so the sum of the roots is zero:

$$\sum_{r=-n}^{n} e^{(2r+1)i\pi/(2n+1)} = 0.$$

Taking the real part of both sides immediately gives Eq. (5).

C. OTHER APPLICATIONS OF COMPLEX NUMBERS

Here are two more examples of the use of complex numbers.

1. Summing trigonometric series

For example, to find

$$S_I = \sum_{r=1}^{n} \sin r\theta.$$

consider

$$S = \sum_{r=1}^{n} (\cos r\theta + i \sin r\theta) = \sum_{r=1}^{n} e^{ir\theta}.$$

This is a geometric progression with n terms. The first term is $a = e^{i\theta}$ and the common ratio is $r = e^{i\theta}$. So, summing,

$$S = \frac{a(1-r^n)}{1-r} = \frac{e^{i\theta}(1-e^{in\theta})}{1-e^{i\theta}}.$$

We want $S_I = \text{Im}(S)$. The easiest way to find this is the 'half-angle trick'. Factorise the brackets in the numerator and denominator to give sin functions:

$$\begin{aligned} S = \frac{e^{i\theta}\; e^{in\theta/2}(e^{-in\theta/2} - e^{in\theta/2})}{e^{i\theta/2}(e^{-i\theta/2} - e^{i\theta/2})} \quad &= \quad e^{i(n+1)\theta/2}\left(\frac{-2i\sin(n\theta/2)}{-2i\sin(\theta/2)}\right) \\ &= \quad \left\{\cos\frac{(n+1)\theta}{2} + i\sin\frac{(n+1)\theta}{2}\right\}\frac{\sin(n\theta/2)}{\sin(\theta/2)}. \end{aligned}$$

from which we can immediately read off

$$S_I = \left\{\sin\frac{(n+1)\theta}{2}\right\}\frac{\sin(n\theta/2)}{\sin(\theta/2)}.$$

2. Integration

To integrate

$$I_R = \int e^{ax}\cos bx\; dx$$

consider

$$I = \int e^{ax}e^{ibx}\; dx = \frac{e^{(a+ib)x}}{a+ib} + C = \frac{e^{a}(\cos bx + i\sin bx)(a-ib)}{a^2+b^2} + C.$$

Taking the real part of I gives

$$I_R = \frac{e^{a}(a\cos bx + b\sin bx)}{a^2+b^2} + C.$$

Another way of finding I is to write

$$I_R = \int e^{ax} \frac{(e^{ibx} + e^{-ibx})}{2}\, dx.$$

Check that this gives the same answer. The small advantage of this approach is that an imaginary answer signals an error in arithmetic.
The short option *Complex Variables* covers much more interesting ways to use complex analysis in integration.

D. FUNCTIONS OF A COMPLEX VARIABLE

1. Exponential

$$e^z = e^{x+iy} = e^x e^{iy} = e^x(\cos y + i \sin y)$$

2. Logarithms

$$\ln z = \ln(re^{i\theta}) = \ln r + \ln(e^{i\theta}) = \ln r + i\theta, \qquad n \text{ an integer}$$

3. Trig. and hyperbolic

Recall

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} \qquad \cosh z = \frac{e^z + e^{-z}}{2}$$
$$\sin z = \frac{e^{iz} - e^{-iz}}{2i} \qquad \sinh z = \frac{e^z - e^{-z}}{2}$$

so

$$\sin iz = i \sinh z \qquad \sinh iz = i \sin z$$
$$\cos iz = \cosh z \qquad \cosh iz = \cos z$$

4. Inverse trig. and hyperbolic

To find $\sin^{-1} z$:
Let $w = \sin^{-1} z$, then

$$z = \sin w = \tfrac{e^{iw} - e^{-iw}}{2i}$$
$$e^{iw} - 2iz - e^{-iw} = 0$$
$$e^{2iw} - 2ize^{iw} - 1 = 0$$

This is a quadratic in e^{iw} so

$$e^{iw} = \tfrac{2iz \pm (-4z^2+4)^{1/2}}{2}$$
$$= iz \pm (1 - z^2)^{1/2}$$
$$\Rightarrow \quad iw = \ln\{iz \pm (1 - z^2)^{1/2}\}$$
$$\sin^{-1} z = w = \quad -i \ln\{(iz \pm (1 - z^2)^{1/2}\}$$

II. FIRST ORDER DIFFERENTIAL EQUATIONS

0. Terminology

A differential equation is one involving derivatives, eg $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$.

In an *ordinary* differential equation y depends on just one variable, $y(x)$. An example is simple harmonic motion $\frac{d^2y}{dx^2} = -k^2 y$ In a *partial* differential equation y depends on more than one variable $y(x,t)$. An example is the wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$.

The *order* of a differential equation is the order of the highest derivative. For example

$$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 6y = f(x)$$

is a second order, ordinary differential equation. The solution is $y(x)$: y is called the *dependent* variable, x is the *independent* variable.

An equation is *linear* if it is linear in the dependent variable. ie terms like y, $\frac{d^2y}{dx^2}$ can occur in a linear differential equation, terms like y^2, $y\frac{dy}{dx}$, $\sin y$ cannot.

1. **Separable**

A first order differential equation is *separable* if it can be written

$$\frac{dy}{dx} = \frac{f(x)}{g(y)}.$$

Then

$$\int g(y)dy = \int f(x)dx$$

which, with luck, can be integrated directly.

Example

$$\frac{dy}{dx} = -y^2 e^x \quad \Rightarrow \quad -\int \frac{dy}{y^2} = \int e^x dx \quad \Rightarrow \quad \frac{1}{y} = e^x + C$$

1′. Almost separable

If

$$\frac{dy}{dx} = f(ax+by), \qquad a, b \text{ constants}$$

a simple change of variable leads to a separable equation. Let

$$z = ax + by \quad \Rightarrow \quad \frac{dz}{dx} = a + b\frac{dy}{dx} = a + bf(ax+by) = a + bf(z)$$

which can be integrated by separation of variables

$$\int \frac{dz}{a + bf(z)} = \int dx$$

Example

$$\frac{dy}{dx} = (x+2y+3)^2 - \frac{1}{2}$$

Let $z = x + 2y + 3$. Then

$$\frac{dz}{dx} = 1 + 2\frac{dy}{dx} = 1 + 2(z^2 - \frac{1}{2}) = 2z^2.$$

Integrating

$$\int \frac{dz}{z^2} = \int 2\,dx \quad \Rightarrow \quad -\frac{1}{z} = 2x + C \quad \Rightarrow \quad \frac{-1}{x+2y+3} = 2x + C$$

2. Homogeneous

A first order differential equation of the form

$$\frac{dy}{dx} = \frac{g(x,y)}{h(x,y)}$$

is of the homogeneous type if $g(\alpha x, \alpha y)/h(\alpha x, \alpha y) = g(x,y)/h(x,y)$.

By choosing $\alpha = 1/x$, we find

$$\frac{dy}{dx} = \frac{g(x,y)}{h(x,y)} = \frac{g(1, y/x)}{h(1, y/x)} = f\left(\frac{y}{x}\right).$$

For such an equation, the substitution $v = \frac{y}{x}$ leads to a separable equation. Let

$$y = vx \quad \Rightarrow \quad \frac{dy}{dx} = x\frac{dv}{dx} + v = f(v) \quad \Rightarrow \quad \frac{dv}{dx} = \frac{f(v)-v}{x} \quad \Rightarrow \quad \int \frac{dv}{f(v)-v} = \frac{dx}{x}.$$

Example

$$\frac{dy}{dx} = \frac{y^2 + xy}{x^2} = \frac{y^2}{x^2} + \frac{y}{x}$$

$$\text{Let} \quad v = \tfrac{y}{x} \quad \Rightarrow \quad y = vx \quad \Rightarrow \quad \tfrac{dy}{dx} = x\tfrac{dv}{dx} + v = v^2 + v \quad \Rightarrow$$
$$\int \tfrac{dv}{v^2} = \int \tfrac{dx}{x} \quad \Rightarrow \quad -\tfrac{1}{v} = \ln x + \ln C^{-1} = \ln \tfrac{x}{C} \quad \Rightarrow \quad x = Ce^{-\frac{1}{v}} = Ce^{-\frac{x}{y}}.$$

2′. Homogeneous but for constant

Consider an equation like

$$\frac{dy}{dx} = \frac{y + x - 5}{y - 3x - 1}. \tag{6}$$

The numerator and denominator are linear and the equation is homogeneous apart from the constants -5 and -1. These can be eliminated by changing variables to $y' = y + a$, $x' = x + b$, where a and b are constants, with a sensible choice of a, b. Substituting the change of variable in Eq. (6) gives

$$\frac{dy}{dx} = \frac{dy'}{dx'} = \frac{y' - a + x' - b - 5}{y' - a - 3x' + 3b - 1}.$$

The new equation, in the primed variables, will have no constant terms if $a + b = -5$ and $a - 3b = -1$. So as long as there is a solution for a, b (see next section for what happens if not) the differential equation becomes

$$\frac{dy'}{dx'} = \frac{y' + x'}{y' - 3x'}$$

which is homogeneous and can be solved using the substitution $v = y'/x'$.

2″. Looks like 'homogeneous but for constant' but is 'almost separable'

A special case – what happens if there is no solutions for a, b?

Consider the differential equation

$$\frac{dy}{dx} = \frac{y - 3x - 2}{2y - 6x - 5}. \tag{7}$$

Substituting $y' = y + a$, $x' = x + b$ gives

$$\frac{dy'}{dx'} = \frac{y' - a - 3x' - 3b - 2}{2y' - 2a - 6x' - 6b - 5}.$$

To get rid of the constant terms choose $a + 3b = -2$, $2a + 6b = -5$. However these are parallel lines in the (a, b) plane and there is therefore no solution. Oh dear! But looking carefully at Eq. (7) it is apparent that the rhs is a function of just one variable, $z = y - 3x$. This means that it is 'almost separable' and method $1'$ works. Partial fractions are needed to give a solution

$$\frac{1}{25}(2u - \ln u) = x + C \quad \text{where} \quad u = -5(y - 3x) + 13.$$

3. Integrating factor

The most general linear, first-order equation is

$$\frac{dy}{dx} + P(x)\, y = Q(x)$$

because we can always divide through by the coefficient of $\frac{dy}{dx}$ to make it unity. This equation can be solved by multiplying through by the integrating factor

$$I \equiv e^{\int P(x)\, dx}$$

to give

$$\frac{dy}{dx}\, e^{\int P(x)\, dx} + P(x)\, y\, e^{\int P(x)\, dx} = Q(x)\, e^{\int P(x)\, dx}.$$

Integrating

$$y\, e^{\int P(x)\, dx} = \int Q(x)\, e^{\int P(x)\, dx} dx.$$

(Differentiate back to check.) Hence

$$y = e^{-\int P(x)\, dx} \int Q(x)\, e^{\int P(x)\, dx} dx.$$

Example

$$x^2\frac{dy}{dx} + 3xy = 1$$

Divide through by the coefficient of the derivative to get the equation into standard form

$$\frac{dy}{dx} + \frac{3}{x}y = \frac{1}{x^2}.$$

Find the integrating factor

$$I = e^{\int \frac{3}{x}dx} = e^{3\ln x} = e^{\ln x^3} = x^3.$$

Multiply both sides by I and integrate, don't forget the constant

$$yx^3 = \int \frac{x^3}{x^2}dx = \frac{x^2}{2} + C$$

giving

$$y = \frac{1}{2x} + \frac{C}{x^3}.$$

Remember

- The coefficient of $\frac{dy}{dx}$ must be unity before calculating I.
- Remember to multiply both sides of the equation by I.
- Remember to add the constant of integration immediately after integrating.

4. The Bernoulli equation

The non-linear equation

$$\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n \tag{8}$$

can be solved by the change of variable

$$z = y^{1-n}, \qquad \frac{dz}{dx} = (1-n)y^{-n}\frac{dy}{dx}.$$

Multiplying Eq. (8) by $(1-n)y^{-n}$ gives

$$(1-n)y^{-n}\frac{dy}{dx} + (1-n)P(x)y^{1-n} = Q(x)(1-n).$$

Changing variables from y to z

$$\frac{dz}{dx} + (1-n)zP(x) = (1-n)Q(x)$$

which is a linear equation in z which can be solved by using an integrating factor.

5. Exact equations

The differential equation

$$\frac{dy}{dx} = -\frac{p(x,y)}{q(x,y)} \tag{9}$$

can be integrated by inspection if $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}$. This is because this equality of partial derivatives is the condition for $p(x,y)dx + q(x,y)dy$ to be an exact derivative and so we may write

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = p(x,y)dx + q(x,y)dy = 0$$

with a solution f=C.

Example

$$\frac{dy}{dx} = -\frac{6x + y + y^2}{x + 2xy}$$

Comparing to Eq. (9), $p(x,y) = 6x+y+y^2$, $q(x,y) = x+2xy$, $\frac{\partial q}{\partial x} = \frac{\partial p}{\partial y} = 1+2y$ so the equation is exact. Writing

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (6x + y + y^2)dx + (x + 2xy)dy = 0$$

we can identify

$$\frac{\partial f}{\partial x} = 6x + y + y^2, \qquad \frac{\partial f}{\partial y} = x + 2xy.$$

So, by inspection, the differential equation can be integrated to give

$$f = xy + xy^2 + 3x^2 = C.$$

6. Oddments

1. 'One-off' changes of the dependent or independent variable sometimes work.
2. Don't forget that equations like $\frac{dy}{dx} = f(x)$ can be integrated directly.
3. If all else fails use a computer.

III. SECOND ORDER DIFFERENTIAL EQUATIONS

0. More terminology and the principle of superposition

A general, linear, second order differential equation takes the form

$$a(x)\frac{d^2y}{dx^2} + b(x)\frac{dy}{dx} + c(x)y = f(x).$$

If $f(x) = 0$ the equation is called homogeneous.
If $f(x) \neq 0$ the equation is called inhomogeneous.

Principle of superposition:
For linear, homogenous equations, if $y_1(x)$ and $y_2(x)$ are solutions then any linear combination

$c_1y_1(x) + c_2y_2(x)$ is also a solution.
This can be demonstrated very easily. We know

$$a(x)\frac{d^2y_1}{dx^2} + b(x)\frac{dy_1}{dx} + c(x)y_1 = 0, \tag{10}$$

$$a(x)\frac{d^2y_2}{dx^2} + b(x)\frac{dy_2}{dx} + c(x)y_2 = 0. \tag{11}$$

Adding $c_1\times$ Eq. (15) and $c_2\times$ Eq. (16) shows that $c_1y_1(x) + c_2y_2(x)$ is also a solution. (Check that this no longer works if $f(x) \neq 0$.)

We shall focus on the case where the coefficients $a(x)$, $b(x)$, $c(x)$ are constants, considering first homogeneous equations, then inhomogeneous equations.

1. Second order, linear, homogeneous differential equations with constant coefficients

Consider the equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0, \qquad a, b, c \text{ constants.} \tag{12}$$

Try a solution

$$y = Ae^{mx}.$$

Differentiating and substituting into Eq. (12)

$$am^2Ae^{mx} + bmAe^{mx} + cAe^{mx} = 0.$$

This gives the auxiliary equation

$$am^2 + bm + c = 0,$$

$$\Rightarrow \quad m = \frac{-b \pm (b^2 - 4ac)^{1/2}}{2a}. \tag{13}$$

case 1: auxiliary equation has real roots $b^2 > 4ac$

$$y = Ae^{\left\{\frac{-b+(b^2-4ac)^{1/2}}{2a}\right\}x} + Be^{\left\{\frac{-b-(b^2-4ac)^{1/2}}{2a}\right\}x}$$

$$\equiv e^{-\frac{bx}{2a}}(Ae^{\alpha x} + Be^{-\alpha x}), \qquad \text{where} \quad \alpha = \frac{(b^2-4ac)^{1/2}}{2a}.$$

Notes:
1. The two solutions can be added because of the principle of superposition.
2. Q: How do we know this is the most general solution?
   A: It contains two arbitrary constants A, B which we expect for a second order differential equation because we have integrated twice to reach the solution.
3. Q: How do we find A, B?
   A: From the boundary conditions e.g. the values of y and $\frac{dy}{dx}$ at $x = 0$.

case 2: auxiliary equation has complex roots $b^2 < 4ac$

Taking a factor -1 out of the bracket in Eq. (13) **and noting that the term in the bracket is then positive**

$$m = \frac{-b \pm i(4ac - b^2)^{1/2}}{2a}. \tag{14}$$

So the general solution of the differential equation is

$$\begin{aligned} y &= Ae^{\left\{\frac{-b+i(4ac-b^2)^{1/2}}{2a}\right\}x} + Be^{\left\{\frac{-b-i(4ac-b^2)^{1/2}}{2a}\right\}x} \\ &\equiv e^{-\frac{bx}{2a}}(Ae^{i\beta x} + Be^{-i\beta x}), \qquad \text{where} \quad \beta = \frac{(4ac-b^2)^{1/2}}{2a}. \end{aligned}$$

There are other ways of writing this. Expanding the complex exponentials in terms of sin and cos:

$$\begin{aligned} y &= e^{-\frac{bx}{2a}}(A\cos\beta x + iA\sin\beta x + B\cos\beta x - iB\sin\beta x) \\ &= e^{-\frac{bx}{2a}}((A+B)\cos\beta x + i(A-B)\sin\beta x) \\ &\equiv e^{-\frac{bx}{2a}}(C\cos\beta x + D\sin\beta x) \end{aligned} \tag{15}$$

where C and D are constants. Yet another way of writing the solution, where the two arbitrary constants are now E and φ, is

$$y = e^{-\frac{bx}{2a}} E\cos(\beta x - \varphi). \tag{16}$$

Eqs. (15) and (16) are equivalent if $E\cos\varphi = C$, $E\sin\varphi = D$.
Choose the form of solution that is most convenient to match the boundary conditions. It is fine to write down the expressions (15) and (16) directly from Eq. (14) without going through all the intermediate steps.

case 3: auxiliary equation has repeated roots $b^2 = 4ac$

In this case, we appear to have the solution $y = y_1 = A\exp(-bx/2a)$. However, this involves only one undetermined constant, and we know there should be two since we must integrate a second order ODE twice to obtain y. To obtain the other part of the general solution, rewrite $y = (y/y_1)y_1 = g(x)y_1$, where we define $g(x) \doteq (y/y_1)$. Plugging this form of y into the homogeneous ODE gives

$$g\left(a\frac{d^2y_1}{dx^2} + b\frac{dy_1}{dx} + cy_1\right) + a\left(\frac{d^2g}{dx^2}y_1 + 2\frac{dg}{dx}\frac{dy_1}{dx}\right) + b\frac{dg}{dx}y_1 = 0.$$

Since y_1 is a solution to the homogeneous ODE, the first set of terms in parentheses is zero. Noting that g no longer appears outside a differential, we make the substitution $h \doteq dg/dx$ to reduce the equation to a first order ODE. Finally, making use of $dy_1/dx = my_1$, we arrive at the following equation for h:

$$a\frac{dh}{dx} + (2am + b)\,h = 0. \tag{17}$$

For the case of repeated roots, $m = -b/2a$, eliminating the term proportional to h so that the solution for h is a constant, say α. Integrate h to get

$$g = \alpha x + \beta$$

and

$$y = (Ax + B)\exp\left(-\frac{bx}{2a}\right).$$

2. The damped oscillator

To a good approximation, for small amplitude, many simple mechanical oscillators, e.g. a mass on a spring or a pendulum, execute simple harmonic motion

$$\frac{d^2y}{dt^2} + \omega_0^2 y = 0$$

where y is displacement, t is time, and ω_0 is the natural frequency. Trying a solution $y = Ae^{mt}$ gives an auxiliary equation

$$m^2 + \omega_0^2 = 0.$$

So $m = \pm i\omega_0$ and the solution is

$$y = C\cos\omega_0 t + D\sin\omega_0 t.$$

(You should be able to recognise the differential equation for simple harmonic motion and write down the solution immediately.) The oscillations continue forever. However in any real system there is damping. This is often well modeled as a term proportional to the velocity giving a differential equation

$$\frac{d^2y}{dt^2} + \gamma\frac{dy}{dt} + \omega_0^2 y = 0.$$

The auxiliary equation is

$$m^2 + \gamma m + \omega_0^2 = 0 \quad \Rightarrow \quad m = \frac{-\gamma \pm (\gamma^2 - 4\omega_0^2)^{1/2}}{2} = -\frac{\gamma}{2} \pm \left(\frac{\gamma^2}{4} - \omega_0^2\right)^{1/2}.$$

The three possible behaviours of the solutions are:

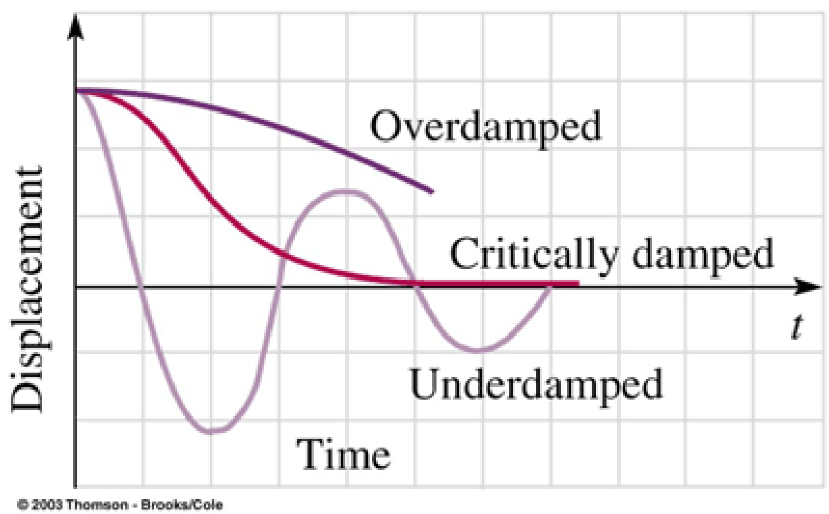

Figure 3: Displacement with time of a damped oscillator. Note that the exact shape of the curves also depends on the initial conditions, here $\dot{y} = 0$ and $y > 0$.

(i) $\underline{\gamma > 2\omega_0}$ overdamped

$$y = e^{-\frac{\gamma}{2}t}(Ae^{(\frac{\gamma^2}{4}-\omega_0^2)^{1/2}t} + Be^{-(\frac{\gamma^2}{4}-\omega_0^2)^{1/2}t})$$

(ii) $\underline{\gamma = 2\omega_0}$ critically damped

$$y = e^{-\frac{\gamma}{2}t}(At + B)$$

(iii) $\underline{\gamma < 2\omega_0}$ underdamped

$$y = e^{-\frac{\gamma}{2}t}\left\{A\cos\left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2} t + B\sin\left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2} t\right\} \tag{18}$$

Check that this simplifies to the simple harmonic motion solution for $\gamma = 0$.

Notes:
1. Be careful with the sign in the argument of the sin and cos in the underdamped case.
2. The maths of the LCR series circuit is identical. The differential equation is

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{I}{C} = 0.$$

3. Second order, linear, inhomogeneous differential equations with constant coefficients:

We aim to solve

$$a\frac{d^2y(x)}{dx^2} + b\frac{dy(x)}{dx} + cy(x) = f(x), \qquad a, b, c \text{ constants.} \tag{19}$$

Let the solution of Eq. (19) with $f(x) = 0$ be $y_{CF}(x)$. $y_{CF}(x)$ is called the **complementary function**. We know how to find it from Sec. III.1.
Let any solution of Eq. (19) be $y_{PI}(x)$. $y_{PI}(x)$ is called the **particular integral**. We shall find it by inspection/informed guesswork. It cannot be the general solution as it has no arbitrary constants.
Then the sum $y(x) = y_{CF}(x) + y_{PI}(x)$ is also a solution as can easily be checked by substituting it into Eq. (19). This function does have 2 arbitrary constants, which appear in y_{CF}, so it is the general solution.

Summary: the solution of Eq. (19) is

$$\mathbf{y(x) = y_{CF}(x) + y_{PI}(x)}. \tag{20}$$

Finding the particular integral

The strategy is to guess the particular integral based on the form of the inhomogeneous term $f(x)$ and then to find the constants by substituting the trial solution into the differential equation. It is easiest to see how this works by trying a few examples.

Example 1:

$f(x)$	guess for particular integral
polynomial of degree n	polynomial of degree n
sum of $\sin ax$, $\cos ax$	$A\sin ax + B\cos ax$
e^{ax}	Ae^{ax}

Solve

$$2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 2x. \tag{21}$$

First find the CF, the solution to Eq. (21) with the right hand side zero. Try a solution Ae^{mx} giving the auxiliary equation $2m^2 - 7m + 3 = 0$ with roots $m = 3$ and $m = 1/2$. So the CF is

$$y_{CF} = Ae^{3x} + Be^{x/2}.$$

Next find the PI. The right hand side of the equation is a polynomial of degree 1 in x so try $y_{PI} = \alpha x + \beta$. Differentiating and substituting into Eq. (21) gives

$$-7\alpha + 3\alpha x + 3\beta = 2x.$$

This must be true for all x so matching the coefficients of x and the constants

$$-7\alpha + 3\beta = 0, \quad 3\alpha = 2 \qquad \Rightarrow \qquad \alpha = \frac{2}{3}, \quad \beta = \frac{14}{9}$$

giving

$$y_{PI} = \frac{2}{3}x + \frac{14}{9}$$

and a general solution to Eq. (21)

$$y = Ae^{3x} + Be^{x/2} + \frac{2}{3}x + \frac{14}{9}.$$

Example 2:

Solve

$$2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = \sin 4x. \tag{22}$$

As for Example 1 the CF is

$$y_{CF} = Ae^{3x} + Be^{x/2}.$$

Next find the PI. The right hand side is a sin function so try $y_{PI} = \alpha\sin 4x + \beta\cos 4x$. Differentiating and substituting into Eq. (22) gives

$$-32\alpha\sin 4x - 32\beta\cos 4x - 28\alpha\cos 4x + 28\beta\sin 4x + 3\alpha\sin 4x + 3\beta\cos 4x = \sin 4x$$

This must be true for all x so matching the coefficients of the sin terms and of the cos terms

$$\begin{aligned} \sin 4x: &\quad -32\alpha + 28\beta + 3\alpha = 1 \\ \cos 4x: &\quad -32\beta - 28\alpha + 3\beta = 0 \\ &\Rightarrow \qquad \alpha = -\frac{29}{1625}, \quad \beta = \frac{28}{1625} \end{aligned}$$

giving

$$y_{PI} = -\frac{29}{1625}\sin 4x + \frac{28}{1625}\cos 4x$$

and a general solution to Eq. (22)

$$y = Ae^{3x} + Be^{x/2} - \frac{29}{1625}\sin 4x + \frac{28}{1625}\cos 4x.$$

Special cases:

If $f(x)$ contains a term that appears in the CF add an extra factor x in the trial PI. Here's why:

If the trial PI contains a term that appears in the CF, then that term will vanish when substituted into the homogeneous ODE. To overcome this problem, we use the same trick we used when treating homogeneous ODEs with repeated roots: rewrite $y = (y/y_1)y_1 = g(x)y_1$, where $f(x) = \alpha y_1$ and y_1 is a solution to the homogeneous ODE. Repeating the analysis from the homogeneous ODE with repeated roots then leads to an equation similar to Eq. (17):

$$a\frac{dh}{dx} + (2am + b)\,h = \frac{f}{y_1} = \alpha.$$

Distinct roots:
If there are two distinct roots to the auxiliary equation, i.e., $2am + b \neq 0$, the solution is $h = \alpha/(2am + b)$ and so

$$g = \frac{\alpha x}{2am + b} + \beta.$$

We drop the constant β as this can be absorbed into the complementary function, giving

$$y_{PI} = \frac{f(x)x}{2am + b}.$$

Repeated root:
If there is a repeated root, then $2am + b = 0$ and the solution for h is $h = \alpha x/a + \beta$. Then

$$g = \frac{\alpha x^2}{2a} + \beta x + \gamma.$$

We drop the terms in β and γ as these can both be absorbed into the complementary function, giving

$$y_{PI} = \frac{x^2 f(x)}{2a}.$$

Example 3:

Solve

$$2\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 3y = 4e^{3x}. \tag{23}$$

As before

$$y_{CF} = Ae^{3x} + Be^{x/2}.$$

Next find the PI. The right hand side of the equation is an exponential so the first thought is to try $y_{PI} = \alpha e^{3x}$. However this will not work because it appears in the CF and therefore when substituted into the left hand side of Eq. (23) it gives zero. Let's try multiplying $y_1 = e^{3x}$ by a function $g(x)$ and see what we get. When we plug $y = y_1(x)g(x)$ into Eq. (23) and collect terms, we find

$$g\left(2\frac{d^2y_1}{dx^2} - 7\frac{dy_1}{dx} + 3y_1\right) + \left(5\frac{dg}{dx} + 2\frac{d^2g}{dx^2}\right)y_1 = 4y_1.$$

The first set of terms in the parentheses evaluates to zero because we chose y_1 to be a solution to the homogeneous equation, leaving

$$5\frac{dg}{dx} + 2\frac{d^2g}{dx^2} = 4,$$

with solution $dg/dx = 4/5$, or $g = 4x/5 + C$. We can drop the C since it can be absorbed in y_{CF}, so

$$y_{PI} = \frac{4}{5}xe^{3x}$$

and

$$y = Ae^{3x} + Be^{x/2} + \frac{4}{5}xe^{3x}.$$

In general, when the inhomogeneous term $f(x)$ is part of the complementary function, try $y_{PI} = \alpha x f(x)$ (unless $xf(x)$ is also part of the complementary function, as we address below).

Example 4:

The equation

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 4e^{-x/2} \tag{24}$$

has an auxiliary equation with coincident roots. The CF is

$$y_{CF} = e^{-x/2}(Ax + B).$$

For the PI neither $\alpha e^{-x/2}$ nor $\alpha xe^{-x/2}$ will work as they both appear in the CF. Therefore try $\alpha x^2e^{-x/2}$. Check by substituting in to Eq. (24) that $\alpha = 1/2$ giving the general solution

$$y = e^{-x/2}(Ax + B) + \frac{1}{2}x^2e^{-x/2}.$$

Example 5:

This is one that can be confusing. The equation

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = \sin 2x \tag{25}$$

has an auxiliary equation with imaginary roots. The CF is

$$y_{CF} = e^x(A\cos 2x + B\sin 2x).$$

For the PI try $y_{PI} = \alpha\sin 2x + \beta\cos 2x$. This is fine as $\sin 2x$ and $\cos 2x$ are not solutions of the homogeneous equation. (The solutions are $e^x\cos 2x$ and $e^x\sin 2x$.)

Comments

1. If $f(x)$ is a sum of terms, the PI is the sum of the PIs for each term. eg if

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = e^{2x} + x^3 \tag{26}$$

try $y_{PI} = \alpha e^{2x} + \beta x^3 + \gamma x^2 + \delta x + \epsilon$.

2. If $f(x)$ is a product of terms, the PI is the product of the PIs for each term. eg consider

$$4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + y = 4x^2e^{-x/2}.$$

Normally the trial PI would be $y_{PI} = (\alpha x^2 + \beta x + \gamma)e^{-x/2}$. However the CF is $y_{CF} = e^{-x/2}(Ax + B)$ so the terms in β and γ cannot be used. Instead the trial PI must be $y_{PI} = (\alpha x^4 + \beta x^3 + \gamma x^2)e^{-x/2}$. Find the coefficients to show that Eq. (26) has the solution

$$y = e^{-x/2}(Ax + B) + \frac{1}{12}x^4 e^{-x/2}.$$

When in doubt add more terms to the trial PI. It is hard to go wrong (except in the arithmetic). Too few terms will lead to equations that do not have a solution for $\alpha, \beta \ldots$. Too many terms, and some of the $\alpha, \beta \ldots$ will be zero.

4. Oddments

a. Euler's equation

Euler's equation

$$ax^2\frac{d^2y}{dx^2} + bx\frac{dy}{dx} + cy = f(x), \qquad a, b, c \text{ constants} \tag{27}$$

can be solved using the substitution $x = e^t$. Noting $\frac{dx}{dt} = e^t = x$, $\frac{dt}{dx} = \frac{1}{x}$,

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{1}{x}\frac{dy}{dt}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x}\frac{d^2y}{dt^2}\frac{dt}{dx} = -\frac{1}{x^2}\frac{dy}{dt} + \frac{1}{x^2}\frac{d^2y}{dt^2}.$$

Substituting the derivatives into Eq. (27) gives

$$a\frac{d^2y}{dt^2} + (b-a)\frac{dy}{dt} + cy = f(t),$$

a second order, linear differential equation with constant coefficients that we know how to solve.

b. A useful formula that allows integration with respect to the dependent variable

$$y'' = y'\frac{dy'}{dy} \qquad \text{where} \quad y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2} \tag{28}$$

Proof:

$$\frac{d^2y}{dx^2} = \frac{dy'}{dx} = \frac{dy'}{dy}\frac{dy}{dx} = y'\frac{dy'}{dy}$$

Example - simple harmonic motion

(Conventionally $\dot{y}$ denotes a time derivative, y' a derivative with respect to any other variable.)

The equation of simple harmonic motion is

$$\ddot{y} = -\omega^2 y \quad \Rightarrow \quad \dot{y}\frac{d\dot{y}}{dy} = -\omega^2 y \quad \Rightarrow \quad \int \dot{y}\, d\dot{y} = -\omega^2 \int y\, dy \quad \Rightarrow \quad \frac{\dot{y}^2}{2} = -\frac{\omega^2 y^2}{2} + C.$$

Assume that at $t = 0$ $y = 0$ and $\dot{y} = v_0$. Then $C = \frac{v_0^2}{2}$ and

$$\dot{y} = (v_0^2 - \omega^2 y^2)^{1/2}.$$

Integrating with respect to t

$$\int \frac{dy}{(v_0^2 - \omega^2 y^2)^{1/2}} = \int dt \quad \Rightarrow \quad \frac{1}{\omega} \sin^{-1} \frac{\omega y}{v_0} = t + C$$

The boundary conditions give $C = 0$ and we are left with the familiar

$$y = \frac{v_0}{\omega} \sin \omega t.$$

c. Dependent variable 'missing'

If there is no term in y, let $\frac{dy}{dx} = p$ and transform to a first order equation.

Example:

$$\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^2 = 0 \quad \Rightarrow \quad \frac{dp}{dx} + 2p^2 = 0 \quad \Rightarrow \quad -\int \frac{dp}{p^2} = 2\int dx \quad \Rightarrow \quad \frac{1}{p} = 2x + C.$$

Integrating again

$$p = \frac{dy}{dx} = \frac{1}{2x + C} \quad \Rightarrow \quad y = \frac{1}{2} \ln(2x + C).$$

IV. FORCED OSCILLATORS AND RESONANCE

1. The forced oscillator

The equation we will be considering in this section describes the physics of a forced, damped, harmonic oscillator:

$$m\frac{d^2x}{dt^2} + m\gamma\frac{dx}{dt} + m\omega_0^2 x = F \cos \omega t. \tag{29}$$

Notation: m mass, x displacement, t time, γ damping coefficient, ω_0 natural frequency of the oscillator, F amplitude of the driving force, ω frequency of the driving force.

The first term is mass $\times$ acceleration.
The second term is the damping term which we assume is proportional to the velocity.
The third term is the restoring force, proportional to the displacement.
The final term is the harmonic driving force.

Note:
If $F = 0$ we recover the unforced damped, harmonic oscillator of Sec. 3.1.
If $F = 0$ and $\gamma = 0$ we are back to simple harmonic motion with natural frequency ω_0.

2. Transient solution

The solution to Eq. (29) is the sum of the complementary function and the particular integral. The complementary function is just the solution found in Sec. III.2, Eq. (18), where we assume light damping or else it would be a pretty useless oscillator:

$$x_{CF} = e^{-\frac{\gamma}{2}t}(A \cos \beta t + B \sin \beta t), \qquad \beta = \left(\omega_0^2 - \frac{\gamma^2}{4}\right)^{1/2}. \tag{30}$$

This is a transient solution. It depends on the initial conditions (which determine A and B) and it decays to zero. In the steady state it will have died away and it can be ignored.

3. Steady state solution

On the basis of what we have covered so far, to find the steady state solution (ie the PI) we would try $x = D\cos\omega t + E\sin\omega t$ or, equivalently, $x = C\cos(\omega t - \varphi)$. However it is MUCH easier to use complex numbers as follows:
Consider

$$m\ddot{x}_R + m\gamma\dot{x}_R + m\omega_0^2 x_R = F\cos\omega t, \tag{31}$$

$$m\ddot{x}_I + m\gamma\dot{x}_I + m\omega_0^2 x_I = F\sin\omega t. \tag{32}$$

Adding Eq. (31) $+i\times$ Eq. (32)

$$m\ddot{x} + m\gamma\dot{x} + m\omega_0^2 x = Fe^{i\omega t} \tag{33}$$

where $x = x_R + ix_I$. We shall solve Eq. (33) for x and take the real part of the answer to get the solution of Eq. (31).

To find the PI of Eq. (33) try $x = Ce^{i\omega t}$. Substituting in gives

$$-\omega^2 mCe^{i\omega t} + m\gamma i\omega Ce^{i\omega t} + m\omega_0^2 Ce^{i\omega t} = Fe^{i\omega t} \qquad \Rightarrow \qquad C = \frac{F}{m\{(\omega_0^2 - \omega^2) + i\gamma\omega\}}$$

so

$$x = \frac{Fe^{i\omega t}}{m\{(\omega_0^2 - \omega^2) + i\gamma\omega\}}. \tag{34}$$

To obtain the real part of x, and get the solution in the most useful form to look at the physics, we next write x in polar form. Note that

$$m\{(\omega_0^2 - \omega^2) + i\gamma\omega\} = re^{i\varphi} \quad \text{where} \quad r = m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}, \quad \tan\varphi = \frac{\gamma\omega}{(\omega_0^2 - \omega^2)}. \tag{35}$$

So the displacement, Eq. (34) becomes

$$x = \frac{Fe^{i\omega t}}{\left(m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}\right)e^{i\varphi}} = \frac{Fe^{(i\omega t - \varphi)}}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}.$$

Taking the real part

$$x_R = \frac{F\cos(\omega t - \varphi)}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \equiv A\cos(\omega t - \varphi). \tag{36}$$

The steady state solution is harmonic. It has a different amplitude A, which differs from that of the driving force, and it lags the driving force by a phase φ.

The derivation in this section is tricky at first but easy once you have practised it a few times. You will have to do it often so it is worth getting it straight.

4. The amplitude response

We have just shown that, for a forcing term with amplitude F, the displacement x (we shall

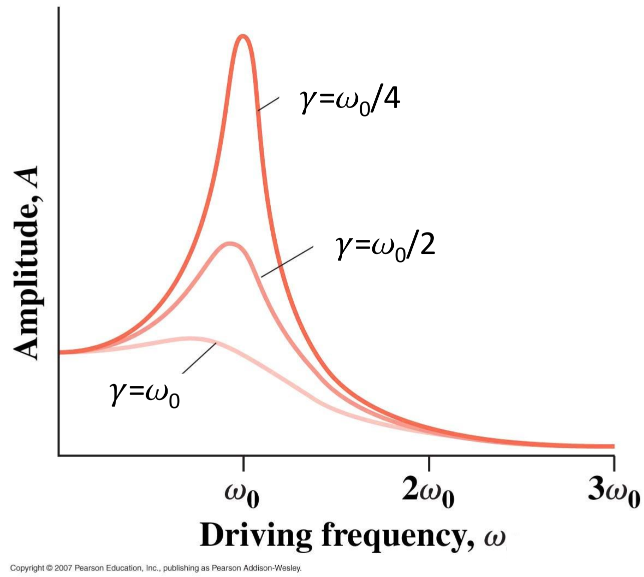

Figure 4: Amplitude of a damped, harmonic oscillator for different damping strengths.

lose the subscript R from now on) has amplitude

$$A = \frac{F}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}. \tag{37}$$

The variation of A with ω is plotted in Fig. 4. Note:

- As $\omega \to \infty$, $A \to 0$. This is becaue the oscillator cannot respond if the driving is too fast.
- For $\omega = 0$, $A = \frac{f}{m\omega_0^2}$. The static force is causing a Hookean displacement.
- The curve has a maximum at $\omega := \omega_R = \left(\omega_0^2 - \frac{\gamma^2}{2}\right)^{1/2}$.

Let's confirm the last statement. A is a maximum when the denominator in Eq. (37) is a minimum.

$$\frac{d}{d\omega}\{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\}\,|_{\omega_R} = 0 \quad \Rightarrow \quad -4\omega(\omega_0^2 - \omega_R^2) + 2\gamma^2\omega = 0 \quad \Rightarrow \quad \omega_R^2 = \omega_0^2 - \frac{\gamma^2}{2}. \tag{38}$$

Note:
Differentiating Eq. (36) gives the velocity response

$$\dot{x} = \frac{-\omega F \sin(\omega t - \varphi)}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} \equiv -A_V \sin(\omega t - \varphi) \tag{39}$$

with amplitude

$$A_V = \frac{\omega F}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}} = \frac{F}{m\sqrt{(\frac{\omega_0^2}{\omega} - \omega)^2 + \gamma^2}} \tag{40}$$

ω's only appear in the first bracket in the denominator in the last expression so we can read off that the velocity amplitude has a maximum at $\omega = \omega_0$.

5. Width of the resonance and the Q-factor

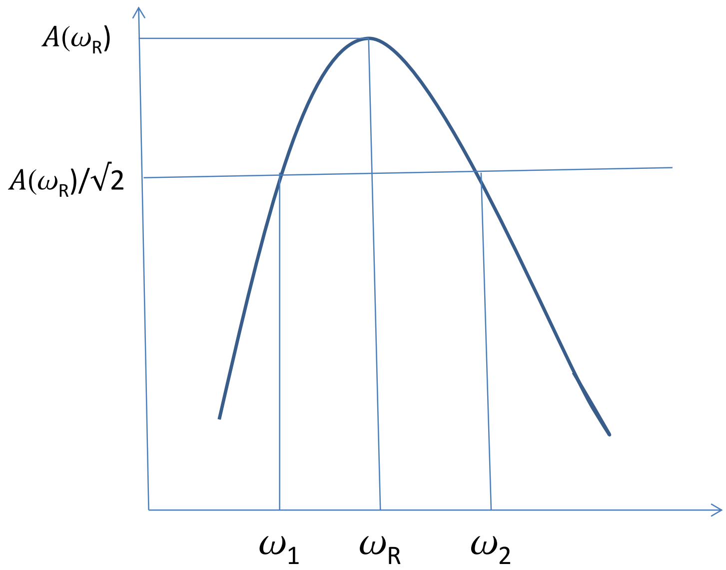

Figure 5: Defining the width of the resonance.

For small damping the amplitude response $A(\omega)$ can be very sharply peaked. This is called resonance, and ω_R is the resonant frequency. It is useful to have a measure of the width of the resonance. A sensible definition is (see Fig. 5)

$$\Delta\omega = \omega_2 - \omega_1 \quad \text{where} \quad A(\omega_1) = A(\omega_2) = \frac{1}{\sqrt{2}} A(\omega_R) \quad (\text{with} \quad \omega_2 > \omega_R > \omega_1).$$

(The choice of measuring the width of the curve at $\frac{1}{\sqrt{2}} A(\omega_R)$ corresponds to the stored energy $\sim A^2$ being 1/2 of its maximum value.)

To find $\Delta\omega$:

From Eq. (37)

$$A(\omega_1) = \frac{F}{m\sqrt{(\omega_0^2 - \omega_1^2)^2 + \gamma^2\omega_1^2}} = \frac{1}{\sqrt{2}} A(\omega_R) = \frac{F}{\sqrt{2}m\sqrt{(\omega_0^2 - \omega_2^2)^2 + \gamma^2\omega_2^2}}$$

$$\Rightarrow \qquad (\omega_0^2 - \omega_1^2)^2 + \gamma^2\omega_1^2 = 2\{(\omega_0^2 - \omega_R^2)^2 + \gamma^2\omega_R^2\}.$$

Recall (Eq. (38)) that $\omega_R^2 = \omega_0^2 - \frac{\gamma^2}{2}$ so $\omega_0^2 - \omega_R^2 = \frac{\gamma^2}{2}$ and substituting for ω_R

$$(\omega_0^2 - \omega_1^2)^2 + \gamma^2\omega_1^2 = 2\left\{\frac{\gamma^4}{4} + \gamma^2\omega_0^2 - \frac{\gamma^4}{2}\right\} = 2\gamma^2\omega_0^2 - \frac{\gamma^4}{2}. \qquad (41)$$

This can be solved as a quadratic in ω_1^2 but the answer is messy. It is much neater to identify a small parameter and expand the solution in terms of it (a useful approach for many problems).

The relevant small parameter is γ because we expect, on physical grounds, that the damping is small for a sharp resonance. γ has the dimensions of frequency so we may write

$$\omega_1 = \omega_0 + a\gamma + O(\gamma^2) \tag{42}$$

and see if we get a consistent solution. Noting that $O(\gamma^2)$

$$(\omega_0^2 - \omega_1^2)^2 = (\omega_0 - \omega_1)^2(\omega_0 + \omega_1)^2 \approx a^2\gamma^2 4\omega_0^2$$

and substituting Eq. (42) into Eq. (41)

$$a^2\gamma^2 4\omega_0^2 + \gamma^2\omega_0^2 = 2\gamma^2\omega_0^2 \qquad \Rightarrow \qquad 4a^2 = 1 \qquad \Rightarrow \qquad a = \pm\frac{1}{2}.$$

So

$$\omega_1 = \omega_0 - \frac{\gamma}{2}, \qquad \omega_2 = \omega_0 + \frac{\gamma}{2}$$

and, for small damping, the full width of the amplitude response is

$$\Delta\omega = \omega_2 - \omega_1 = \gamma.$$

Comments:

1. $\Delta\omega$ is related to the decay of free (unforced) oscillations, $e^{-\frac{\gamma}{2}t}$, see Eq. (30). Small damping $\Rightarrow$ slow decay of oscillations $\Rightarrow$ sharp resonance.

2. The quality factor Q is a dimensionless measure of the width of the resonance.

$$Q = 2\pi\frac{\text{stored energy}}{\text{energy lost per cycle}} = \frac{\omega_0}{\gamma}.$$

So large Q $\Leftrightarrow$ sharp resonance. We will prove this formula in the next section.

3. Note that the resonance frequency is the same as the natural frequency, ω_0, for the velocity amplitude but differs by a term $O(\gamma^2)$ for the displacement amplitude (Eq. (38)). Check that the expression for $\Delta\omega$ is the same to leading order in γ.

6. Power and Energy

Using Eq. (39) for the velocity of the oscillator we can write down the power supplied

$$P = \text{driving force} \times \text{velocity} = F\cos\omega t \times \frac{-\omega F\sin(\omega t - \varphi)}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}.$$

Averaging the time-dependent terms over a cycle, the mean power is

$$\bar{P} = \frac{-\omega F^2}{m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}\overline{\cos\omega t\sin(\omega t - \varphi)}.$$

where

$$\overline{\cos\omega t\sin(\omega t - \varphi)} = \overline{\cos\omega t\sin\omega t\cos\varphi} - \overline{\cos^2\omega t\sin\varphi} = -\frac{1}{2}\sin\varphi$$

because $\overline{\cos\omega t\sin\omega t} = 0$ and $\overline{\cos^2\omega t} = \frac{1}{2}$. So

$$\bar{P} = \frac{\omega F^2\sin\varphi}{2m\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

but, from Eq. (35),

$$\tan\varphi = \frac{\gamma\omega}{(\omega_0^2 - \omega^2)} \qquad \Rightarrow \qquad \sin\varphi = \frac{\gamma\omega}{\sqrt{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2}}$$

so

$$\Rightarrow \quad \bar{P} = \frac{\gamma\omega^2 F^2}{2m\{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\}}.$$

At $\omega = \omega_0$

$$\bar{P} = \frac{F^2}{2m\gamma}.$$

You should check that this is the same as the energy lost per cycle of a lightly damped, unforced oscillator (see problem set 3).

The energy stored by the oscillator is

$$\begin{aligned} E &= \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega_0^2 x^2 \\ &= \frac{m}{2}\frac{\omega^2 F^2 \sin^2(\omega t - \varphi)}{m^2\{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\}} + \frac{m\omega_0^2}{2}\frac{F^2\cos^2(\omega t - \varphi)}{m^2\{(\omega_0^2 - \omega^2)^2 + \gamma^2\omega^2\}} \\ &= \frac{F^2}{2\gamma^2 m} \end{aligned}$$

at $\omega = \omega_0$.
Hence the quality factor is

$$Q = 2\pi\frac{\text{stored energy}}{\text{energy lost per cycle}} = 2\pi\frac{F^2}{2\gamma^2 m}\frac{2m\gamma}{F^2}\frac{\omega_0}{2\pi} = \frac{\omega_0}{\gamma}.$$

7. Phase

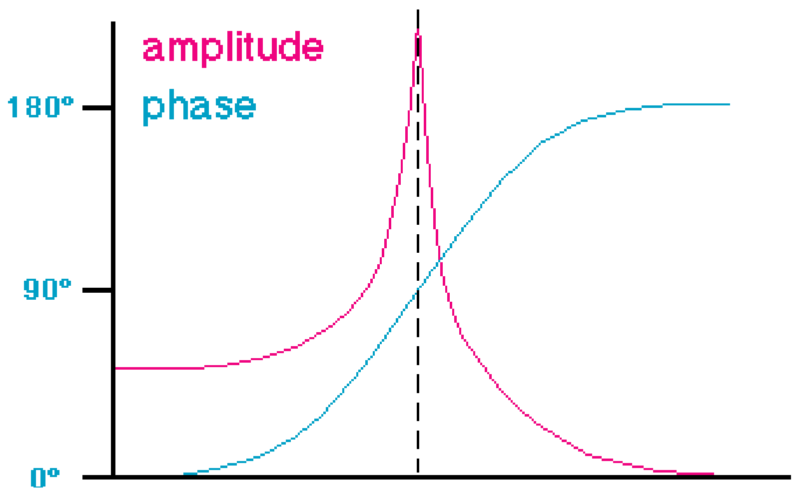

Figure 6: Phase of a damped, harmonic oscillator.

We found in Sec. IV.3 that for a force $F\cos\omega t$ the displacement of a damped harmonic oscillator is $x = A\cos(\omega t - \varphi)$. The displacement lags the force by a phase φ, Eq. (35),

$$\tan\varphi = \frac{\gamma\omega}{(\omega_0^2 - \omega^2)}.$$

The velocity of the oscillator is

$$\dot{x} = -\omega A \sin(\omega t - \varphi) = \omega A \sin(\varphi - \omega t) = \omega A \cos(\frac{\pi}{2} - \varphi + \omega t)$$
$$= \omega A \cos(\omega t - (\varphi - \frac{\pi}{2}) := \omega A \cos(\omega t - \varphi_V)$$

so the velocity lags the force by $\varphi_V = \varphi - \frac{\pi}{2}$. Note that the velocity and the driving force are in phase at resonance.

8. The LCR circuit

An example of a damped oscillator that can be described by identical mathematics is the series LCR circuit, which you will cover in Circuit Theory lectures. For an inductance L, resistance R and capacitance C, in series with an oscillating voltage $V_0 \cos \omega t$, Kirchoff's law gives

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = V_0 \cos \omega t$$

or, because $I = \frac{dQ}{dt}$,

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = V_0 \cos \omega t.$$

This is identical to Eq. (29) given the replacements $L \Leftrightarrow m$, $R \Leftrightarrow m\gamma$, $\frac{1}{C} \Leftrightarrow m\omega_0^2$, $\omega_0^2 \Leftrightarrow \frac{1}{LC}$, $\gamma \Leftrightarrow \frac{R}{L}$. So the resonance condition for the displacement $\Leftrightarrow$ charge is

$$\omega_R = \left(\omega_0^2 - \frac{\gamma^2}{2}\right)^{1/2} \quad \Leftrightarrow \quad \omega_R = \left(\frac{1}{LC} - \frac{R^2}{2L^2}\right)^{1/2}$$

and for the velocity $\Leftrightarrow$ current

$$\omega_V = \omega_0 \quad \Leftrightarrow \quad \omega_V = \frac{1}{\sqrt{LC}}.$$

V. COUPLED DIFFERENTIAL EQUATIONS

There are many ways to solve coupled differential equations. If you can see an easy way, use it. Here are two approaches:

Method 1: Complementary function and particular integral

$$\dot{x} - x + 2y = e^{2t}, \tag{43}$$
$$\dot{y} - y + 2x = 0. \tag{44}$$

To find the complementary function try $x = Ae^{mt}$, $y = Be^{mt}$ $\Rightarrow$

$$(m-1)A + 2B = 0,$$
$$2A + (m-1)B = 0.$$

These equations have a solution if the determinant of the coefficients

$$\begin{vmatrix} m-1 & 2 \\ 2 & m-1 \end{vmatrix}$$

is zero (see Vectors and Matrices lectures) giving $m = -1, 3$. If $m = -1$ $B = A$ and if $m = 3$ $B = -A$ so we may write down the complementary functions:

$$
\begin{aligned}
x &= \alpha e^{-t} + \beta e^{3t}, \\
y &= \alpha e^{-t} - \beta e^{3t}.
\end{aligned}
$$

There are two arbitrary constants as expected for two first order differential equations.

For the particular integral try $x = De^{2t}$, $y = Ee^{2t}$. Substituting in to Eqs. (44) gives $D = -\frac{1}{3}$, $E = \frac{2}{3}$ so the full solution is

$$
\begin{aligned}
x &= \alpha e^{-t} + \beta e^{3t} - \frac{1}{3}e^{2t}, \\
y &= \alpha e^{-t} - \beta e^{3t} + \frac{2}{3}e^{2t}.
\end{aligned}
$$

Method 2: Differentiate one equation and substitute into the other

Using the same example

$$
\begin{aligned}
\dot{x} - x + 2y &= e^{2t}, \\
\dot{y} - y + 2x &= 0.
\end{aligned}
$$

From the second equation

$$
x = \frac{y}{2} - \frac{\dot{y}}{2} \qquad \Rightarrow \qquad \dot{x} = \frac{\dot{y}}{2} - \frac{\ddot{y}}{2}. \tag{45}
$$

Substituting into the first equation

$$
\frac{\dot{y}}{2} - \frac{\ddot{y}}{2} - \frac{y}{2} + \frac{\dot{y}}{2} + 2y = e^{2t} \qquad \Rightarrow \qquad \ddot{y} - 2\dot{y} - 3y = -2e^{2t}.
$$

Solving for the complementary function and particular integral in the usual way

$$
y = \alpha e^{-t} - \beta e^{3t} + \frac{2}{3}e^{2t}.
$$

Substituting for y and $\dot{y}$ in Eq. (45) immediately gives

$$
x = \alpha e^{-t} + \beta e^{3t} - \frac{1}{3}e^{2t}
$$

as before.