Nonlinear Phase Mixing and Phase-Space Cascade of Entropy in Gyrokinetic Plasma Turbulence

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Electrostatic turbulence in weakly collisional, magnetized plasma can be interpreted as a cascade of entropy in phase space, which is proposed as a universal mechanism for dissipation of energy in magnetized plasma turbulence. When the nonlinear decorrelation time at the scale of the thermal Larmor radius is shorter than the collision time, a broad spectrum of fluctuations at sub-Larmor scales is numerically found in velocity and position space, with theoretically predicted scalings. The results are important because they identify what is probably a universal Kolmogorov-like regime for kinetic turbulence; and because any physical process that produces fluctuations of the gyrophase-independent part of the distribution function may, via the entropy cascade, result in turbulent heating at a rate that increases with the fluctuation amplitude, but is independent of the collision frequency.

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Introduction.—Turbulence is inherently nonlinear and dynamically complicated. In the general case, a broad spectrum of fluctuations is excited, in both wave number and frequency. For turbulent, magnetized plasma, the equations of magnetohydrodynamics provide a pedagogically rich description of the dynamics. However, for those turbulent eddies whose parallel wavelengths (relative to the magnetic field) are comparable to or smaller than the collisional mean free path and whose perpendicular wavelengths are comparable to or smaller than the Larmor radius of one of the constituent species of the plasma, magnetohydrodynamic theory breaks down. In such cases, the gyrokinetic (GK) theory [1,2] represents a rigorous limit of plasma kinetics for anisotropic ($k_\parallel \ll k_\perp$), low-frequency ($\omega \ll \Omega$, the ion cyclotron frequency) fluctuations. In this Letter, we present a GK description of turbulence in a simplified situation, chosen to isolate a novel phenomenon which is a generic component of all GK turbulence: the simultaneous cascade of entropy to smaller scales in both real space and velocity space. This phase-space cascade is the mechanism by which turbulent energy associated with fluctuating fields is brought to small scales in velocity space, where even very infrequent collisions are sufficient to provide irreversibility and thus heating. Below, we present the theory and first-principles simulations of the phase-space cascade in a homogeneous, electrostatic, magnetized plasma.

It is well known that Landau and Barnes damping of electromagnetic plasma fluctuations lead to the generation of small-scale structures in $f(v_\parallel)$, where f is the one-particle distribution function, and $v_\parallel$ is the velocity coordinate along the background magnetic field [3,4]. This is associated with the free-streaming of particles along the field. As t increases, a single Fourier harmonic of the distribution function $f_{k_\parallel} \sim e^{ik_\parallel v_\parallel t}$ gets progressively more oscillatory in $v_\parallel$-space. Eventually, even infrequent collisions are sufficient to smooth these oscillatory features, since the collision operator is roughly a diffusion operator in velocity space. As long as collisions are sufficiently infrequent, the damping rate depends not on the collision rate, but on the nature of the wave and its phase velocity relative to the thermal speeds of the plasma species. Physically, Landau damping is the smearing of spatial perturbations that occurs when there is a spread in the distribution of parallel velocities. We recall for future reference that this generation of velocity-space structure is independent of the fluctuation amplitudes.

Besides this *linear* parallel phase mixing, there exists a *nonlinear* phase mixing process [5] that, in a strongly turbulent plasma and at spatial scales smaller than the Larmor radius, drives the formation of structure in $f(v_\perp)$ much more rapidly than parallel phase mixing drives $f(v_\parallel)$. Physically, this nonlinear phase mixing is the smearing of spatial perturbations due to the *spread* in the distribution of gyroaveraged $\boldsymbol{E} \times \boldsymbol{B}$ velocities (see Fig. 1). Unlike for the parallel phase mixing, the rate of generation of v-space structure by this process is proportional to the fluctuation amplitude. In this Letter, we present a study of this nonlinear process, which we interpret as a turbulent cascade of entropy in phase space [6]. As such, it represents a conceptually novel nonlinear phenomenon, where generation of small scales in the position and velocity space occurs in an intertwined way. This process, which is likely to be a fundamental and ubiquitous feature of magnetized plasma turbulence, has never been numerically diagnosed and analyzed before, although Krommes [3] did point out the general possibility of the coupling between position and velocity space.

Gyrokinetics in 2D.—Let the distribution function be $f = F_0 + \delta f$, where F_0 is a Maxwellian with density n_0

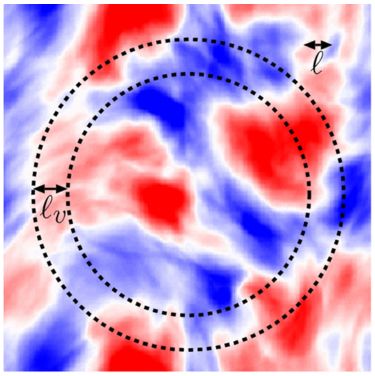

FIG. 1 (color online). Schematic view of the nonlinear phase mixing superimposed on the potential from the Run (iii) (see Table I) at $t/\tau_{\rm init} = 10$ and the largest wavelength mode taken out. When the fluctuation scale $\ell \lesssim \rho$, the gyroaverage of the electric field induces a decorrelation of the distribution function at the velocity-space scale corresponding to the difference in Larmor radii $\ell_v = \delta v/\Omega \sim \ell$ [see (7)].

and temperature T_0, and $\delta f = h - q\varphi F_0/T_0$, where q is the particle charge and φ is the electrostatic potential. To keep the focus on the nonlinear process, we consider electrostatic GK turbulence in slab geometry with $k_\parallel = 0$. Then the non-Boltzmann part h of the perturbed ion distribution function satisfies [1]

$$\frac{\partial h}{\partial t} + \frac{c\hat{z} \times \nabla\langle\varphi\rangle_{\boldsymbol{R}}}{B_0} \cdot \nabla h = \langle C[h]\rangle_{\boldsymbol{R}} + \frac{qF_0}{T_0}\frac{\partial\langle\varphi\rangle_{\boldsymbol{R}}}{\partial t}, \quad (1)$$

where B_0 is the background magnetic field aligned with the z axis and $\langle\cdot\rangle_{\boldsymbol{R}}$ is the gyroaverage holding the guiding center position $\boldsymbol{R}$ constant. The collision operator $C[h]$ used in our simulations contains pitch-angle scattering and energy diffusion with proper conservation properties [7]. The quasineutrality condition yields

$$Q\varphi = q\int\langle h\rangle_{\boldsymbol{r}} d\boldsymbol{v} = q\sum_{\boldsymbol{k}} e^{i\boldsymbol{k}\cdot\boldsymbol{r}}\int J_0\left(\frac{k_\perp v_\perp}{\Omega}\right)h_{\boldsymbol{k}} d\boldsymbol{v}, \quad (2)$$

where $\langle\cdot\rangle_{\boldsymbol{r}}$ denotes the gyroaverage at fixed particle position $\boldsymbol{r}$, J_0 is the Bessel function, $Q = \sum_s q_s^2 n_{0s}/T_{0s}$ for Boltzmann-response (3D) electrons or $Q = q_i^2 n_{0i}/T_{0i}$ for no-response (2D) electrons, and s and i are the species indices. Our results are not affected by the choice of the electron response. For concreteness, we henceforth use no-response electrons since electrons cannot contribute to the potential if $k_\parallel = 0$ exactly. In the absence of collisions, the system has two positive definite conserved integrals [6,8]:

$$\mathfrak{W} = \iint \frac{T_0 \delta f^2}{2F_0} d\boldsymbol{r}\, d\boldsymbol{v} = \int\left(\int\frac{T_0\langle h^2\rangle_{\boldsymbol{r}}}{2F_0}d\boldsymbol{v} - \frac{Q}{2}\varphi^2\right)d\boldsymbol{r}, \quad (3)$$

$$\mathfrak{E} = \frac{Q}{2}\sum_{\boldsymbol{k}}(1-\Gamma_0)|\varphi_{\boldsymbol{k}}|^2, \quad (4)$$

where $\Gamma_0 = I_0(k_\perp^2\rho^2/2)e^{-k_\perp^2\rho^2/2}$, I_0 is the modified Bessel function and ρ is the ion thermal Larmor radius. The invariant $\mathfrak{W}$ is proportional to minus the perturbed part of the entropy of the system, $-\int f \ln f d\boldsymbol{r} d\boldsymbol{v}$ [3,4]. Here we will refer to $\mathfrak{W}$ as "entropy" to emphasize this connection. The second invariant $\mathfrak{E}$ is conserved in the 2D electrostatic case only.

Scalings.—A scaling theory of the entropy cascade in the sub-Larmor scale range can be developed in a way reminiscent of the Kolmogorov-style turbulence theories [6]. Assume that at (perpendicular) scales $\ell \ll \rho$, the transfer of entropy is local in scale. On dimensional grounds, the entropy flux is

$$\frac{v_{\rm th}^2}{\tau_\ell}\left(\frac{h v_{\rm th}^3}{n_0}\right)^2 = \text{const} \quad (5)$$

until it reaches the collisional dissipation scale, where $v_{\rm th}$ is the thermal speed and τ_ℓ is the nonlinear decorrelation time at scale ℓ. The neglect of the φ^2 term in $\mathfrak{W}$ [see (3)] is justified *post hoc* due to its smallness in the $\ell \ll \rho$ regime [see (9) and Fig. 3(a)]. There is a self-consistent electrostatic potential at the scale ℓ: from (2),

$$\frac{q\varphi}{T_0} \sim \left(\frac{\ell}{\rho}\right)^{1/2}\left(\frac{\delta v_\perp}{v_{\rm th}}\right)^{1/2} \sim \frac{h v_{\rm th}^3}{n_0}\frac{\ell}{\rho}. \quad (6)$$

Here we have assumed that the nonlinear phase mixing produces velocity-space structures correlated with the spatial scale via (see Fig. 1 and Refs. [6,8])

$$\frac{\delta v_\perp}{v_{\rm th}} \sim \frac{\ell}{\rho}. \quad (7)$$

This has allowed us to estimate the velocity integral in (2) as a random-walk-like accumulation of the integrand represented by the product of $h_{\boldsymbol{k}}$, which is a random function of $v_\perp$ whose "step size" is given by (7) with $\ell \sim k_\perp^{-1}$, and of the Bessel function, which introduces a reduction factor of $(\ell/\rho)^{1/2}$.

The decorrelation time τ_ℓ may be estimated by balancing the ∂_t term with the nonlinear term in (1), leading to

$$\tau_\ell \sim \frac{\ell^2}{c\langle\varphi\rangle_{\boldsymbol{R}}/B_0} \sim \left(\frac{\rho}{\ell}\right)^{1/2}\frac{\ell^2}{c\varphi/B_0}. \quad (8)$$

Substituting (6) into (8) and (8) into (5) yields $h \sim \ell^{1/6}$ and $\varphi \sim \ell^{7/6}$. Therefore, the spectra of h and φ are

$$E_h(k_\perp) \sim k_\perp^{-4/3}, \qquad E_\varphi(k_\perp) \sim k_\perp^{-10/3}, \quad (9)$$

where $E_h(k_\perp) = \sum_{|\boldsymbol{k}_\perp|=k_\perp}\int T_0|h_{\boldsymbol{k}}|^2/2F_0 d\boldsymbol{v}$ and $E_\varphi(k_\perp) = \sum_{|\boldsymbol{k}_\perp|=k_\perp} q^2 n_0|\varphi_{\boldsymbol{k}}|^2/2T_0$. Note that the total entropy (3) can be expressed as $\mathfrak{W} = \int[E_h(k_\perp) - E_\varphi(k_\perp)]dk_\perp$.

Dissipation cutoff.—From the balance between the nonlinear decorrelation time (8) and the collision time ν^{-1}, one obtains an estimate of the dissipation cutoff scales in both the velocity space and the real space [see (7)]. Using $C[h] \sim \nu v_{\rm th}^2 h/\delta v_\perp^2$, we find the cutoffs

$$\frac{\delta v_{\perp c}}{v_{\rm th}} \sim \frac{1}{k_{\perp c}\rho} \sim D^{-3/5}, \qquad D = \frac{1}{\nu\tau_\rho}, \quad (10)$$

where τ_ρ is the nonlinear decorrelation time measured at $\ell = \rho$. We have introduced a new dimensionless number D to characterize the scale separation in gyrokinetic turbulence: analogous to the Reynolds number in fluid turbulence, large D corresponds to a broader scaling range over which the entropy cascade extends, and to dissipation at smaller scales. Here, however, the smallest spatial scale observed is determined by the v-space scale for which diffusion in velocities becomes important, through the correlation between real and velocity space given by (7). The fact that D increases with the amplitude of the fluctuations at the Larmor scale clearly distinguishes this process from linear Landau damping. We note that for 3D gyrokinetic turbulence, the nonlinear phase mixing is a much faster process than the linear one if the fluctuation amplitude is sufficiently large.

Numerical simulations.—We now report the first-of-a-kind numerical investigation of the entropy cascade in phase space, carried out with the GK code AstroGK. The code uses a Fourier pseudospectral scheme for the real-space dimensions perpendicular to the background magnetic field and a Legendre collocation scheme for the velocity-space integrations. The velocity space is discretized in energy $\varepsilon = v^2$ and $\lambda = v_\perp^2/\varepsilon$. In the absence of collisions, AstroGK conserves the invariants (3) and (4) with a high precision.

The results reported below were obtained in three runs at decreasing collision frequency ν and correspondingly increasing spatial and velocity resolution. They are indexed in Table I, where $N_x \times N_y$ is number of collocation points in the real space and $N_\varepsilon \times 2N_\lambda$ is the number of grid points in velocity space—the factor of 2 corresponds to the sign of $v_\parallel = \pm\sqrt{\varepsilon(1-\lambda)}$. Our highest-resolved run required 36 wallclock hours on 8192 processors.

The code evolves $g = h - qF_0\langle\varphi\rangle_{\boldsymbol{R}}/T_0$ and φ via Eqs. (1) and (2). We take the box size $L_x = L_y = 2\pi\rho$ and start from the initial condition $g_{\rm init} = g_0[\cos(2x/\rho) + \cos(2y/\rho) + \chi(x, y)]F_0$, where g_0 is a constant and $\chi(x, y)$ is a small-amplitude white noise superimposed on all Fourier modes. From (2), we can calculate $\varphi_{\rm init}$.

Time evolution.—The initial $|k_x\rho|, |k_y\rho| = 2$ configuration is unstable: the amplitudes of φ corresponding to $|k_x\rho|, |k_y\rho| = 1$ grow and then saturate around $t/\tau_{\rm init} \simeq 9$, where $\tau_{\rm init} = 2\pi B_0/(ck_\perp^2 \| \langle\varphi_{\rm init}\rangle_{\boldsymbol{R}} \|)$ is the turnover time associated with the initial condition and $\| \langle\varphi\rangle_{\boldsymbol{R}} \| = [(1/n_0)\iint |\langle\varphi\rangle_{\boldsymbol{R}}|^2 F_0 d\boldsymbol{v} d\boldsymbol{R}]^{1/2}$. The nonlinear interactions between modes produce smaller scales down to a cutoff determined by D [see (10)]. The turbulent spectra fill up by $t/\tau_{\rm init} \simeq 10$, then decay with time.

The time evolution of the collisionless conserved quantities $\mathfrak{W}$ and $\mathfrak{E}$ [see (3) and (4)] is shown in Fig. 2. During the initial growth of the instability of the $|k_x\rho|, |k_y\rho| = 1$, $\mathfrak{W}$ decays very slowly at a rate $\sim\nu$, consistent with a collisional decay rate associated with the large-scale phase-space variation of $g_{\rm init}$. Once turbulence develops, $\mathfrak{W}$ decays more rapidly as the entropy cascade transfers it nonlinearly to smaller scales in phase space, until the fluctuations of the distribution function are thermalized (dissipated) at the collisional cutoff.

TABLE I. Index of the runs.

Run	$N_x \times N_y$	$N_\varepsilon \times 2N_\lambda$	$\nu\tau_{\rm init}$	D	$k_{\perp c}\rho$
(i)	64^2	32^2	5.6×10^{-3}	48	20
(ii)	128^2	64^2	1.9×10^{-3}	118	35
(iii)	256^2	128^2	7.4×10^{-4}	440	77

The decrease of $\mathfrak{W}$ from its initial value corresponds to the amount of entropy (heat) production due to the irreversible collisional smearing of the distribution function. The turbulence that follows the initial instability enhances the heating, suggesting that small-scale velocity-space structure is generated (this is confirmed below). As expected, the rate of dissipation is not strongly affected by the collision frequency, i.e., there is a finite amount of dissipation even as the collision frequency tends to zero. The dissipation rate is determined instead by the nonlinear cascade rate.

While $\mathfrak{W}$ decays, $\mathfrak{E}$ stays almost constant. If we increase the size of the simulation box, the $|k_x\rho|, |k_y\rho| = 1$ modes themselves become unstable to even longer-wavelength modes. We attribute both this instability and the failure of $\mathfrak{E}$ to decay to the intrinsic tendency of $\mathfrak{E}$ to have an inverse cascade [8,9], which we do not discuss here.

Spectra and scalings.—The wave-number spectra of the decaying developed turbulence are given in Fig. 3(a). They are angle integrated over wave-number shells $|\boldsymbol{k}_\perp| = k_\perp$, normalized by $\mathfrak{W}(t)$ at each time and then averaged over time for $10 \le t/\tau_{\rm init} \le 15$. As resolution is increased, the spectra appear to converge to the theoretically predicted scalings (9), which supports the validity of our dimensional and physical considerations of the entropy cascade.

To characterize the entropy cascade in the velocity space, Plunk *et al.* [8] introduced velocity-space spectra

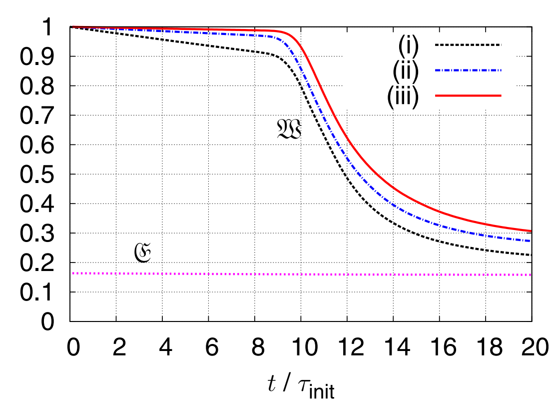

FIG. 2 (color online). Time evolution of $\mathfrak{W}$ and $\mathfrak{E}$ [Eqs. (3) and (4)] normalized to initial $\mathfrak{W}$. The runs (i)–(iii) are indexed in Table I. Evolution of $\mathfrak{E}$ does not differ among runs significantly, and is given for run (iii).

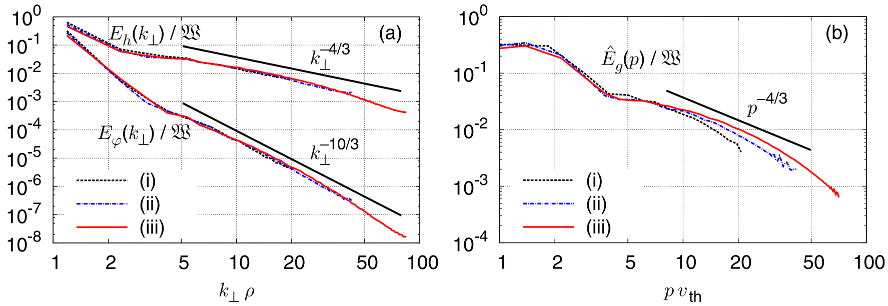

FIG. 3 (color online). Time-averaged normalized (a) wave-number (Fourier) spectra $E_h(k_\perp)/\mathfrak{W}$ and $E_\varphi(k_\perp)/\mathfrak{W}$ [cf. (9)], and (b) velocity-space (Hankel) spectrum $\hat{E}_g(p)/\mathfrak{W}$ for the runs indexed in Table I. Theoretically predicted slopes are given for comparison.

$\hat{E}_g(p) = \sum_{\boldsymbol{k}} p|\hat{g}_{\boldsymbol{k}}(p)|^2$, where $\hat{g}_{\boldsymbol{k}}(p) = \int J_0(pv_\perp) g_{\boldsymbol{k}}(\boldsymbol{v}) d\boldsymbol{v}$ is a Hankel transform. The theoretical expectation is that $\hat{E}_g(p) \sim p^{-4/3}$ because the real- and velocity-space scales should be related according to (7), which, in terms of the dual variable p, becomes $k_\perp \rho \sim p v_{\rm th}$. The time-averaged Hankel spectrum $\hat{E}_g(p)$ obtained in our simulations is shown in Fig. 3(b). This again shows approximate consistency with the theoretical prediction and confirms that small-scale structure is formed in the velocity space.

Dissipation cutoff.—In Table I, we show for each of our runs the dimensionless number $D = (\nu \tau_\rho)^{-1}$, where $\tau_\rho = 2\pi B_0/(c k_\perp^2 \, \| \langle \varphi' \rangle_{\boldsymbol{R}} \|)$ measured at $t/\tau_{\rm init} = 10$ and φ' is φ with the $|k_x \rho|$, $|k_y \rho| = 1$ modes taken out (see also Fig. 1). Also shown is the theoretical estimate (10) for the wave-number cutoff $k_{\perp c}\rho = \alpha D^{3/5}$, where $\alpha = 2$ is an empirical value that corresponds to our particular set up. Comparing with the wave-number and velocity-space spectra in Fig. 3, we see that (10) describes the resolution requirements quite well. With fewer velocity grid points, we find shallower wave-number spectra than the resolved ones, while with more, we resolve below the velocity cutoff without any change in the wave-number spectra. Thus D is a good indicator of necessary and sufficient resolution in full 4D phase space.

Conclusions.—We have presented electrostatic, decaying turbulence simulations for weakly collisional, magnetized plasmas using the gyrokinetic model in 4D phase space (two real-space and two velocity-space dimensions). Landau damping was removed from the system by ignoring variation along the background magnetic field. Nonlinear interactions introduce an amplitude-dependent perpendicular phase mixing of the gyrophase-independent part of the perturbed distribution function and create structure in $v_\perp$ which is finer for higher $k_\perp$. We have found that the wave-number (Fourier) and velocity-space (Hankel) spectra of the perturbed distribution function and the resulting electrostatic fluctuations at sub-Larmor scales agree well with theoretical predictions based on the interpretation of the nonlinear phase mixing as a cascade of entropy in phase space [6,8]. We have introduced a dimensionless number D (analogous to Reynolds number) that characterizes the scale separation between the thermal Larmor scale and the collisional cutoff in phase space [see (10)], and showed that this number correctly predicts the resolution requirements for our simulations.

We note that there are, in general, entropy cascades for each plasma species. Equations for the gyrokinetic turbulence at and below the electron Larmor scale are mathematically similar to the model simulated here and identical arguments apply [6,8]. Similar considerations are also possible for ion-scale electromagnetic turbulence [6] and for minority species.

The small-scale phase-space structure that we have discovered is likely to be a universal feature of strong, magnetized plasma turbulence. Understanding it theoretically and diagnosing it numerically is akin to the inertial-range studies for Kolmogorov turbulence, extended to the kinetic phase space. One should expect rich and interesting physics to emerge and it is likely that predicting large-scale dynamics will require effective models for the small-scale cascade. An immediate key physical implication of the existence of the entropy cascade is a turbulent heating rate independent of collisionality in weakly collisional plasmas.

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