Model Collision Operators for Numerical Gyrokinetics

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Abstract. We motivate the need to include collisional dissipation in gyrokinetic turbulence simulations, and construct criteria for a physically valid model of such dissipation. A new analytically manageable operator satisfying those criteria is presented and transformed into gyrokinetic variables. The form of conservation laws for collision operators in gyrokinetic variables is explained. The numerical implementation of our new operator in the code `GS2` is outlined and successful tests from this code presented.

INTRODUCTION

Turbulence simulations are of great importance in understanding many problems in modern plasma science, be it the anomalous heat flux in a tokamak or the turbulent spectra in the Solar Wind. The plasmas of interest are typically magnetized, collisionless and support turbulence composed of low frequency fluctuations, forcing us to employ kinetic and in particular gyrokinetic turbulence simulatoins [1, 2]. Due to both the numerical complexity of such simulations and the wide separation between energy containing scales and any collisional scales it might appear impractical to include collisional physics in these simulations.

However similarly to fluid turbulent systems the limit of vanishingly small collision frequency ν is not the same as the exactly dissipationless case. This is because in describing a turbulent system one considers two limits: $t \to \infty$ corresponding to waiting for the turbulence to settle into a statistically steady state and secondly $\nu \to 0$ corresponding to considering a weakly collisional system. These two limits cannot be interchanged [3, 4] if we wish to obtain a correct stationary state; no matter how small the dissipation coefficient is after a sufficiently long time the fluctuations will arrange themselves so as to achieve a finite amount of dissipation [5].

Let us make this statement more precise. We shall work with "δf kinetics," i.e., assume that it is physically reasonable to split the distribution function into a slowly (both spatially and temporally) varying equilibrium part and a rapidly varying fluctuating part: $f = F_0 + \delta f$. We further assume that F_0 is a Maxwellian distribution, $F_0 = (n_0/\pi^{3/2}v_{\mathrm{th}}^3)\exp(-v^2/v_{\mathrm{th}}^2)$, where n_0 is density, $v_{\mathrm{th}} = (2T_0/m)^{1/2}$ the thermal

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speed, T_0 temperature and m the particle mass. In this framework the connection between macroscopic quantities and microscopic dissipation is precisely the statement of energy balance governing the turbulent flutuations, [4, 6, 7, 8, 2, 1, 9, 5]

$$\frac{d}{dt}\left(-\sum_s T_{0s}\delta S_s + U\right) = P + \sum_s \iint \frac{T_{0s}\delta f_s}{F_{0s}} C[\delta f_s] d\boldsymbol{v} d\boldsymbol{r}, \tag{1}$$

where s is the species index, $\delta S = -\iint d\boldsymbol{r} d\boldsymbol{v}\, \delta f^2/2F_0$ is the entropy of the fluctuations, $U = \int d\boldsymbol{r}(E^2 + B^2)/8\pi$ is the energy of the (fluctuating) electromagnetic field, P is the input power (energy source of the turbulence), and $C[\delta f]$ the linearized collision operator. Thus in a stationary state, the input power P is matched precisely by the entropy generated by the collisions. Hence in order for a simulation to achieve a steady state with a finite P (equivalently, with finite macroscopic transport), it must have finite dissipation (which physically can only come from collisions). This means that at small ν very small scale structure is generated in velocity space, thus a collision operator is needed in simulations to smooth these fluctuations and hence avoid generating grid scale structures in velocity space.

The exact effects of small angle Coulomb collisions on an arbitrary distribution function have been computed previously by Landau [10], however using this operator would exceed the numerical resources that can reasonably be expended on modelling collisions. We must thus use a less complex model operator instead. To guide us in choosing a model we set forth the following list of criteria that a physically reasonable collision model must satisfy,

- Conservation of particles, energy and momentum;
- Smoothing of the distribution function, in order to produce regular distribution functions without grid scale structure from our simulations we need smoothing in all velocity variables;
- Boltzmann's H-Theorem,

$$\iint \frac{\delta f}{F_0} C[\delta f] d\boldsymbol{r} d\boldsymbol{v} \leq 0. \tag{2}$$

  This ensures that the dissipatoin in Eq. (1) has the correct sign. If it does not this may lead to unphysical entropy sinks, and hence unphysical cooling effects and transport fluxes (here represented by P);
- Vanishing on a Maxwellian, in order to preserve the correct equilibrium states.

In Abel et al. [11] we have developed a model collision operator that satisfies these criteria:

$$\begin{aligned} C[\delta f] = &\frac{\nu_D}{2}\left[\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial\delta f}{\partial\xi} + \frac{1}{1-\xi^2}\frac{\partial^2\delta f}{\partial\vartheta^2}\right] + \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{1}{2}v^4\nu_\parallel F_0\frac{\partial}{\partial v}\frac{\delta f}{F_0}\right) \\ &+ \nu_s\frac{2\boldsymbol{v}\cdot\boldsymbol{U}[\delta f]}{v_{\text{th}}^2}F_0 + \nu_E\frac{v^2}{v_{\text{th}}^2}Q[\delta f]F_0. \end{aligned} \tag{3}$$

Where the velocity dependant collision frequencies ν_D, ν_E, ν_s and $\nu_\parallel$ are the standard ones found in Helander and Sigmar [12] and in Abel et al. [11]. The operator (3) consists

of the exact Landau test-particle operator, to correctly capture the diffusive effect of collisions, and prescriptions for the field-particle operator ensuring particle, energy and momentum conservation (c.f. [13, 14]) :

$$\boldsymbol{U}[\delta f] = \frac{3}{2}\int \nu_s \boldsymbol{v}\delta f\, d\boldsymbol{v} \Big/ \int (v/v_{\rm th})^2 \nu_s F_0\, d\boldsymbol{v}, \tag{4}$$

$$Q[\delta f] = \int v^2 \nu_E \delta f\, d\boldsymbol{v} \Big/ \int v^2 (v/v_{\rm th})^2 \nu_E F_0\, d\boldsymbol{v}. \tag{5}$$

COLLISIONS IN GYROKINETICS

Repeating the standard derivation of nonlinear gyrokinetics found in Frieman and Chen [15] whilst explicitly including weak collisions leads to a nonlinear evolution equation for the non-adiabatic part of δf which is denoted by h [2]. It is important to note that in this formulation, even for very weak collisions, F_0 is constrained to be a Maxwellian. The resulting gyrokinetic equation for h includes a contribution due to collisions, which is just the usual collision operator with the coordinate change from $(\boldsymbol{r},\boldsymbol{v})$ to $(\boldsymbol{R},v)$, $\boldsymbol{R} = \boldsymbol{r} - \boldsymbol{v}\times\boldsymbol{b}/\Omega$ the guiding centre position, and averaged over gyroangle. If we consider a flux tube where we can Fourier transform perpendicular to the field lines then this transformation can be done explicitly [16, 11, 1] to obtain for each Fourier mode $h_{\boldsymbol{k}}$,

$$\begin{aligned} C_{\rm GK}[h_{\boldsymbol{k}}] &= \frac{\nu_D}{2}\frac{\partial}{\partial\xi}(1-\xi^2)\frac{\partial h_{\boldsymbol{k}}}{\partial\xi} + \frac{1}{v^2}\frac{\partial}{\partial v}\left(\frac{1}{2}v^4\nu_\parallel F_0 \frac{\partial}{\partial v}\frac{h_{\boldsymbol{k}}}{F_0}\right) \\ &\quad -\frac{1}{4}\left[\nu_D(1+\xi^2)+\nu_\parallel(1-\xi^2)\right]\frac{v^2}{v_{\rm th}^2}k_\perp^2\rho^2 h_{\boldsymbol{k}} \\ &\quad +2\nu_s\frac{v_\perp J_1(a)U_\perp[h_{\boldsymbol{k}}]+v_\parallel J_0(a)U_\parallel[h_{\boldsymbol{k}}]}{v_{\rm th}^2}F_0 + \nu_E\frac{v^2}{v_{\rm th}^2}J_0(a)Q[h_{\boldsymbol{k}}]F_0, \end{aligned} \tag{6}$$

where $\rho = v_{\rm th}/\Omega$ is the thermal Larmor radius, $a = k_\perp v_\perp/\Omega$, J_0 and J_1 are Bessel functions and

$$U_\perp[h_{\boldsymbol{k}}] = \frac{3}{2}\int \nu_s v_\perp J_1(a) h_{\boldsymbol{k}}\, d\boldsymbol{v} \Big/ \int (v/v_{\rm th})^2 \nu_s F_0\, d\boldsymbol{v}, \tag{7}$$

$$U_\parallel[h_{\boldsymbol{k}}] = \frac{3}{2}\int \nu_s v_\parallel J_0(a) h_{\boldsymbol{k}}\, d\boldsymbol{v} \Big/ \int (v/v_{\rm th})^2 \nu_s F_0\, d\boldsymbol{v}, \tag{8}$$

$$Q[h_{\boldsymbol{k}}] = \int v^2 \nu_E J_0(a) h_{\boldsymbol{k}}\, d\boldsymbol{v} \Big/ \int v^2 (v/v_{\rm th})^2 \nu_E F_0\, d\boldsymbol{v}. \tag{9}$$

We note that due to the mixing of velocity space and real space through the use of the guiding centre position $\boldsymbol{R}$ the collision operator has gained spatial dependence in the form of the $k_\perp^2\rho^2$ terms, which represent diffusion of guiding centres through space due to collisions. These terms were omitted from some early collision models and later found to be important e.g. in microtearing turbulence [17, 18].

NUMERICAL IMPLEMENTATION

Now that we have the complete model for including collisions in the theoretical description of gyrokinetics we turn to the question of how to efficiently implement this in numerical simulations. The full details of the implementation of this operator in the flux tube Eulerian code `GS2` can be found in Barnes et al. [19]. We present an outline of the algorithm here.

As we are using spectral methods perpendicular to the field line to obtain Eq. (6), we are left with the task of choosing an appropriate timestepping and a discretization scheme for velocity space. Considering first the timestepping we can use Godunov dimensional splitting [20] to separate the collisional and collisionless dynamics. As this restricts us to first order accuracy, we will also use a simple first-order backward differencing scheme in time. Eq. (6) can be split into pitch-angle derivatives plus an integral operator and velocity derivatives plus an integral operator. We use Godunov splitting again to separate these. If we time advace the collisional dynamics explicitly small scales in velocity space would drastically limit the overall timestep, thus we choose to advance the system fully implicitly. Thus we have to solve the following system,

$$\frac{h_{\boldsymbol{k}}^{n+1} - h_{\boldsymbol{k}}^{n}}{\Delta t} = D h_{\boldsymbol{k}}^{n+1} + I h_{\boldsymbol{k}}^{n+1}, \tag{10}$$

with superscripts denoting the timestep that $h_{\boldsymbol{k}}$ is evaluated at, and D and I the appropriate differential and integral operators. The differential operator D contains either ξ derivatives only, or v derivatives only. We discretize ξ and v so that we can invert the right hand side of Eq. (10).

The innovation employed here (detailed in Barnes et al. [19]) is that our integration scheme in ξ or v, written as a weighted sum over the gridpoints of the integrand evaluated at the gridpoints, can be thought of as the dot product of the vector containing $h_{\boldsymbol{k}}$ evaluated at each gridpoint in turn with some constant (in time) vector containing the weights of the gridpoints and the rest of the integrand. We discretize ξ and v into a finite number of gridpoints chosen to achieve increased accuracy in velocity space integrals [**?**]. We can thus consider $h_{\boldsymbol{k}}^{n}$ as a vector with i-th component equal to $h_{\boldsymbol{k}}^{n}$ evaluated at the i-th (ξ, v) pair (denoted $h_{\boldsymbol{k}i}^{n}$). Our integration scheme can be written as a weighted sum over gridpoints, thus an integral of some function $g_{\boldsymbol{k}}$ multiplied by $h_{\boldsymbol{k}}^{n}$ over velocity space becomes

$$\int d\xi dv v^2 g_{\boldsymbol{k}}(\xi, v) h_{\boldsymbol{k}}^{n} \approx \sum_i w_i g_{\boldsymbol{k}i} h_{\boldsymbol{k}i}^{n} = \tilde{\boldsymbol{g}} \cdot h_{\boldsymbol{k}}^{n}, \tag{11}$$

where we have absorbed the weights w_i into the vector $\tilde{\boldsymbol{g}}$. This allows us to write $I h_{\boldsymbol{k}}^{n+1}$ as $\boldsymbol{e}\boldsymbol{f} \cdot h_{\boldsymbol{k}}^{n+1}$ where $\boldsymbol{e}$ and $\boldsymbol{f}$ are constant vectors. In this form we can apply the Sherman-Morrison formula [21] to invert the matrix $1 - \Delta t(D + I)$ if we can cheaply invert D. This is possible as in our scheme D will be tridiagonal.

To find such a D we note that if x is either ξ or v then the operator D takes the form

$$D h_{\boldsymbol{k}}^{n+1} = \frac{\partial}{\partial x} G(x) \frac{\partial}{\partial x} h_{\boldsymbol{k}}^{n+1} \tag{12}$$

for some function G. It is known for other collision operators that certain discretisations of the derivatives allow exact discrete conservation laws to be proven [22], i.e. hard-wired into the overall numerical scheme. It is advantageous to try and find such a discretisation scheme considering that satisfaction of the conservation laws was one of the basic criteria used in deriving our model operator [Eq. (3)] originally. For our operator if the grids are uniform in ξ and v and uniform weights chosen in the integration scheme then a three point centered difference for D is a suitable scheme. However for the nonuniform grids in use in modern gyrokinetic codes a more complex scheme must be found. This is done in Barnes et al. [19], leading to a numerical implementation of the gyrokinetic collision operator that conserves particles, energy and momentum to machine precision.

NUMERICAL TESTS

In order to verify the accuracy of the numerical implementation of the collision operator, a suite of numerical tests can been performed. We present a simple and striking such test here. We try to reproduce the well known linear damping of the slow magnetosonic wave in a high β_i plasma. This is a particularly useful test as the numerical code can be compared to theory in three disticnt collisionality regimes [1]:

- When $k_\parallel \lambda_{mfp} \ll 1$ the fluctuations are damped viscously with frequency $\omega = \pm k_\parallel v_A \sqrt{1 - \left(\frac{k_\parallel \nu_\parallel}{2 v_A}\right)^2} - i\frac{k_\parallel^2 \nu_\parallel}{2}$, where $\nu_\parallel$ is the ion parallel viscosity;
- If $k_\parallel \lambda_{mfp} \gg 1$ the slow mode is collisionlessly damped by Barnes damping $\omega = -i\frac{k_\parallel v_A}{\sqrt{\pi \beta_i}}$.

The test was performed at $k_\perp \rho_i = 10^{-5}$ and $\beta_i = 50$ with $T_i = T_e$. In this regime we can set the electron response to be $\delta n_e = 0$. The results are plotted in Fig. (1), the solid line the numerical results from `GS2` and the dashed lines the analytic results.

Nonlinear simulations have also been carried out [19] and it has been found that our collision model allows nonlinear simulations to be performed for parameters that previously required hyperdiffusion [19, 23] in order to achieve a steady state.

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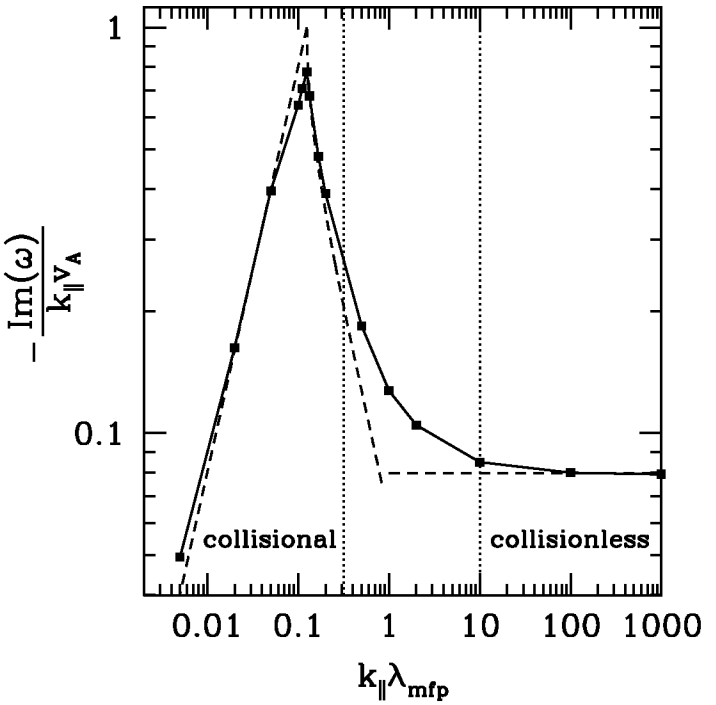

FIGURE 1. High β_i damping of $\delta B_\parallel$ as a function of $k_\parallel \lambda_{\mathrm{mfp}}$. See text.

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