1 particle transport

We begin with the particle transport equation:

$$\frac{\partial n_s}{\partial t} + \frac{1}{V'}\frac{\partial}{\partial \psi}\left(V'\Gamma_{\psi,s}\right) = S_n, \tag{1}$$

where n_s is the equilibrium density of species s, ψ is the poloidal flux enclosed by a flux surface, $V' = dV/d\psi$, V is the volume enclosed by the flux surface, S_n is the net particle source, and $\Gamma_{\psi,s}$ is the flux surface-averaged ψ-component of the particle flux. The particle flux consists of both neoclassical and turbulent contributions; the turbulent contribution is given by

$$\Gamma_{\psi,s} = \left\langle \int d^3\mathbf{v}\left(\mathbf{v}_\chi \cdot \nabla\psi\right)\delta f(\mathbf{R})\right\rangle_\psi, \tag{2}$$

where $\mathbf{v}_\chi$ is the drift due to the fluctuating potentials, δf is the lowest order departure of the distribution function from a Maxwellian, $\mathbf{R}$ is the guiding center position, and $\langle . \rangle_\psi$ indicates a flux surface average.

1.1 normalization

Next, we must normalize Eq. (1). We do so by defining the normalized quantities

$$\tilde{\Gamma}_s \equiv \frac{1}{\langle|\nabla\psi|\rangle}\frac{\Gamma_{\psi,s}}{n_r v_{t,r}}\left(\frac{a_r}{\rho_r}\right)^2, \tag{3}$$

$$\tilde{t} \equiv \frac{v_{t,0}}{a_0}\left(\frac{\rho_0}{a_0}\right)^2 t, \tag{4}$$

and $\tilde{n}_s \equiv n_s/n_0$, where: n_r, $v_{t,r}$, and ρ_r are the density, thermal velocity, and gyroradius of the reference species in the flux tube calculation; a_r is the reference length in the flux tube calculation; n_0, $v_{t,0}$, and ρ_0 are the reference density, thermal velocity, and gyroradius defined in the transport calculation; and a_0 is the reference length in the transport calculation. For clarity, we provide definitions of some of these reference quantities here: $v_{t,r} = \sqrt{2T_r/\alpha m_r}$, with m_r and T_r the mass and temperature of the reference species in the flux tube calculation, and $\alpha = 1$ or 2 depending on the choice made in the flux tube code;

$$\begin{aligned}\rho_r &\equiv \frac{v_{t,r}}{\Omega_r} \\ &= \frac{m_r v_{t,r} c}{|e|\,B_a} = \sqrt{\frac{2T_r m_r}{\alpha}}\frac{c}{|e|\,B_a},\end{aligned} \tag{5}$$

with e the electron charge and B_a the reference magnetic field in the flux tube calculation; $n_0 = 10^{20}/m^3$; $v_{t,0} = \sqrt{2T_0/\alpha m_r}$, with $T_0 = 1\ keV$; and ρ_0 is the reference gyroradius in the transport calculation, defined

$$\begin{aligned}\rho_0 &\equiv \frac{v_{t,0}}{\Omega_0} \\ &= \sqrt{\frac{2T_0}{\alpha m_r}}\frac{m_r c}{|e|\,B_0} \\ &= \sqrt{\frac{2T_0 m_r}{\alpha}}\frac{c}{|e|\,B_0},\end{aligned} \tag{6}$$

with $B_0 = 1\ T$.

We now divide Eq. (1) by $n_0(v_{t,0}/a_0)(\rho_0/a_0)^2$ to get

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} + \frac{1}{V'}\frac{\partial}{\partial\psi}\left(V'\langle|\nabla\psi|\rangle a_0 \tilde{n}_r \sqrt{\tilde{T}_r}\left(\frac{\rho_r}{\rho_0}\frac{a_0}{a_r}\right)^2 \tilde{\Gamma}_s\right) = \tilde{S}_n, \tag{7}$$

where $\tilde{T} = T/T_0$ and

$$\begin{aligned}\tilde{S}_n &= (a_0/v_{t,0})(a_0/\rho_0)^2 S_n/n_0 \\ &= 3.23 \times 10^{-26} \, [m]^3 \, [s] \, \tilde{a}_0 \sqrt{\tilde{m}_r} \sqrt{\frac{\alpha}{2}} \rho_*^{-2} S_n\end{aligned} \tag{8}$$

We next note that the flux surface area, A, satisfies $A \equiv (dV/d\psi) \langle |\nabla\psi| \rangle = (dV/d\rho) \langle |\nabla\rho| \rangle$, where ρ is an arbitrary flux surface label defined so that it is zero at the magnetic axis and one at the LCFS. We then define $\tilde{A} \equiv A/a_0^2$ and $\tilde{\nabla} = a_0 \nabla$. This gives

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} + \frac{\left\langle \left| \tilde{\nabla} \rho \right| \right\rangle}{\tilde{A}} \frac{\partial}{\partial \rho} \left(\tilde{A} \tilde{n}_r \sqrt{\tilde{T}_r} \left(\frac{\rho_r}{\rho_0} \frac{a_0}{a_r} \right)^2 \tilde{\Gamma}_s \right) = \tilde{S}_n, \tag{9}$$

For convenience of future calculation, we explicitly calculate ρ_r/ρ_0:

$$\frac{\rho_r}{\rho_0} = \frac{v_{t,r}}{v_{t,0}} \frac{\Omega_0}{\Omega_r} = \frac{\sqrt{\tilde{T}_r}}{\tilde{B}_a}. \tag{10}$$

Substituting this expression in Eq. (9), we obtain

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} + \frac{\left\langle \left| \tilde{\nabla} \rho \right| \right\rangle}{\tilde{A}} \frac{\partial}{\partial \rho} \left(\frac{\tilde{A}}{\tilde{B}_a^2} \tilde{n}_r \tilde{T}_r^{3/2} \left(\frac{a_0}{a_r} \right)^2 \tilde{\Gamma}_s \right) = \tilde{S}_n. \tag{11}$$

At this point, we assume that a_r is defined so that it does not depend on the flux label, ρ. We are then free to choose $a_0 = a_r$, so that the final form of our normalized particle transport equation bcomes:

$$\frac{\partial \tilde{n}}{\partial \tilde{t}} + \frac{\left\langle \left| \tilde{\nabla} \rho \right| \right\rangle}{\tilde{A}} \frac{\partial}{\partial \rho} \left(\frac{\tilde{A}}{\tilde{B}_a^2} \tilde{n}_r \tilde{T}_r^{3/2} \tilde{\Gamma}_s \right) = \tilde{S}_n. \tag{12}$$

It's useful to also have the normalized quantities in terms of physical units. Here we calculate conversion to physical units for some of our normalized quantities. First, we consider $\rho_* = \rho_0/a_0$:

$$\begin{aligned}\rho_* &= \sqrt{\frac{2T_0 m_r}{\alpha}} \frac{c}{|e| \, B_0 a_0} \\ &= \sqrt{\frac{2}{\alpha}} \frac{\sqrt{\tilde{m}_r}}{\tilde{a}_0} \frac{c\sqrt{m_p}}{|e|} \frac{[keV]^{1/2}}{[m] \, [T]} \\ &= 3.23 \times 10^{-3} \sqrt{\frac{2}{\alpha}} \frac{\sqrt{\tilde{m}_r}}{\tilde{a}_0}\end{aligned} \tag{13}$$

where $\tilde{m} \equiv m/m_p$ and $\tilde{a} = a/\,[m]$. Next, we consider $v_{t,0}/a_0$:

$$\begin{aligned}\frac{v_{t,0}}{a_0} &= \frac{1}{\tilde{a}_0 \sqrt{\tilde{m}_r}} \sqrt{\frac{2}{\alpha}} \frac{1}{\sqrt{m_p}} \frac{[keV]^{1/2}}{[m]} \\ &= \frac{3.094 \times 10^5}{[s]} \frac{1}{\tilde{a}_0 \sqrt{\tilde{m}_r}} \sqrt{\frac{2}{\alpha}}.\end{aligned} \tag{14}$$

Combining the above expressions gives us an expression for $\tilde{t}$:

$$\begin{aligned}\tilde{t} &= \frac{v_{t,0}}{a_0} \rho_*^2 t = 3.23 \left(\frac{2}{\alpha} \right)^{3/2} \frac{\sqrt{\tilde{m}_r}}{\tilde{a}_0^3} \frac{t}{[s]} \\ \Rightarrow \frac{t}{[s]} &= 0.31 \left(\frac{\alpha}{2} \right)^{3/2}_{2} \frac{\tilde{a}_0^3}{\sqrt{\tilde{m}_r}} \tilde{t}.\end{aligned} \tag{15}$$

We now want to convert the normalized particle flux to physical units:

$$
\begin{aligned}
\frac{\Gamma_s}{\langle|\nabla\psi|\rangle} &= n_r v_{t,r}\left(\frac{\rho_r}{a_r}\right)^2\tilde{\Gamma}_s \\
&= \tilde{n}_r\sqrt{\frac{\tilde{T}_r}{\tilde{m}_r}}\sqrt{\frac{2}{\alpha}}\left(\frac{\rho_r}{\rho_0}\right)^2\rho_*^2\frac{10^{20}}{\sqrt{m_p}}\frac{[keV]^{1/2}}{[m]^3}\left(\frac{a_0}{a_r}\right)^2\tilde{\Gamma}_s \\
&= 1.04\times10^{15}\tilde{n}_r\frac{\sqrt{\tilde{m}_r}\tilde{T}_r^{3/2}}{\tilde{a}_0^2\tilde{B}_a^2}\left(\frac{2}{\alpha}\right)^{3/2}\frac{1}{\sqrt{m_p}}\frac{[keV]^{1/2}}{[m]^3}\left(\frac{a_0}{a_r}\right)^2\tilde{\Gamma}_s \\
&= 3.22\times10^{20}\tilde{n}_r\frac{\sqrt{\tilde{m}_r}\tilde{T}_r^{3/2}}{\tilde{a}_0^2\tilde{B}_a^2}\left(\frac{2}{\alpha}\right)^{3/2}\left(\frac{a_0}{a_r}\right)^2\frac{\tilde{\Gamma}_s}{[m]^2\,[s]}.
\end{aligned}
\tag{16}
$$

Integrating this expression over the flux surface, we get

$$
A\frac{\Gamma_s}{\langle|\nabla\psi|\rangle} = 3.22\times10^{20}\tilde{n}_r\tilde{T}_r^{3/2}\frac{\sqrt{\tilde{m}_r}\tilde{A}}{\tilde{B}_a^2}\left(\frac{2}{\alpha}\right)^{3/2}\left(\frac{a_0}{a_r}\right)^2\frac{\tilde{\Gamma}_s}{[s]}.
\tag{17}
$$

1.2 discretization

We start by defining $F_s \equiv \tilde{A}\tilde{n}_r\tilde{T}_r^{3/2}/\tilde{B}_a^2\tilde{\Gamma}_s$. Then the single-step discretization of Eq. (47) is

$$
\frac{n_j^{m+1}-n_j^m}{\Delta t} + \alpha\left(\frac{\langle|\nabla\psi|\rangle_j}{A_j}\frac{F_+-F_-}{\Delta\rho}\right)^{m+1} + (1-\alpha)\left(\frac{\langle|\nabla\psi|\rangle_j}{A_j}\frac{F_+-F_-}{\Delta\rho}\right) = S_j.
\tag{18}
$$

We can develop a multi-step, backwards difference discretization as follows. First, we use f^m and f^{m-1} about time level $m+1$:

$$
\begin{aligned}
f^m &= f^{m+1} - (\Delta t)_m f'^{m+1} + \frac{(\Delta t)_m^2}{2}f''^{m+1} - \mathcal{O}\left[(\Delta t)_m^3\right] \\
f^{m-1} &= f^{m+1} - \left((\Delta t)_m + (\Delta t)_{m-1}\right)f'^{m+1} + \frac{\left((\Delta t)_m+(\Delta t)_{m-1}\right)^2}{2}f''^{m+1} - \mathcal{O}\left[\left((\Delta t)_m+(\Delta t)_{m-1}\right)^3\right],
\end{aligned}
\tag{19}
$$

where $\Delta t_m = t_{m+1} - t_m$. Combining these expressions, we get

$$
\begin{aligned}
&\frac{f^{m+1}-f^m}{\Delta t_m} - \frac{\Delta t_m}{\Delta t_m+\Delta t_{m-1}}\frac{f^{m+1}-f^{m-1}}{\Delta t_m+\Delta t_{m-1}} = f'^{m+1}\left(\frac{\Delta t_{m-1}}{\Delta t_m+\Delta t_{m-1}}\right) + \mathcal{O}\left[\Delta t^2\right] \\
&\Rightarrow f'^{m+1} \approx f^{m+1}\left[\frac{\Delta t_{m-1}+2\Delta t_m}{\Delta t_m\left(\Delta t_m+\Delta t_{m-1}\right)}\right] - f^m\left[\frac{\Delta t_m+\Delta t_{m-1}}{\Delta t_m\Delta t_{m-1}}\right] + f^{m-1}\left[\frac{\Delta t_m}{\Delta t_{m-1}\left(\Delta t_m+\Delta t_{m-1}\right)}\right].
\end{aligned}
\tag{20}
$$

We can Taylor expand $F_\pm^{m+1}$ about its value at the m time level and keep terms through linear order:

$$
F_\pm^{m+1} \approx F_\pm + (\mathbf{y}-\mathbf{y}_0)\left.\frac{\partial F_\pm}{\partial\mathbf{y}}\right|_{\mathbf{y}_0}.
\tag{21}
$$

The derivative term can be written explicitly as

$$
\left.\frac{\partial F_\pm}{\partial\mathbf{y}}\right|_{\mathbf{y}_0} = \sum_k\left(n_k^{m+1}-n_k\right)\frac{\partial F_\pm}{\partial n_k} + ...
\tag{22}
$$

In the local approximation, this becomes

$$
\left.\frac{\partial F_\pm}{\partial\mathbf{y}}\right|_{\mathbf{y}_0} = \sum_k\left(n_k^{m+1}-n_k\right)\left(\frac{\partial F_\pm}{\partial n_\pm}\frac{dn_\pm}{dn_k} + \frac{\partial F_\pm}{\partial(a_0/L_n)_\pm}\frac{d(a_0/L_n)_\pm}{dn_k}\right) + ...
\tag{23}
$$

For now, we assume that the dependence of $F_\pm$ on $n_\pm$ and $p_\pm$ is weak so that

$$\left.\frac{\partial F_\pm}{\partial \mathbf{y}}\right|_{\mathbf{y}_0} \approx \sum_k \left(n_k^{m+1} - n_k\right) \frac{\partial F_\pm}{\partial (a_0/L_n)_\pm} \frac{d(a_0/L_n)_\pm}{dn_k} + ... \tag{24}$$

We express grad scale lengths discretely as

$$\left(\frac{a_0}{L_n}\right)_\pm = \mp \frac{1}{n_\pm} \frac{n_{j\pm1} - n_j}{\Delta\rho}. \tag{25}$$

We then have

$$\begin{aligned} \frac{d(a_0/L_n)_\pm}{dn_k} &= \mp \frac{1}{n_\pm \Delta\rho} \left(-\frac{1}{n_\pm} \frac{dn_\pm}{dn_k} \left(n_{j\pm1} - n_j\right) + \delta_{k,j\pm1} - \delta_{k,j} \right) \\ &= \mp \frac{1}{n_\pm \Delta\rho} \left(-\frac{1}{2n_\pm} \left(\delta_{k,j\pm1} + \delta_{k,j}\right) \left(n_{j\pm1} - n_j\right) + \delta_{k,j\pm1} - \delta_{k,j} \right) \\ &= \mp \frac{1}{n_\pm \Delta\rho} \left(\delta_{k,j\pm1} \left(\frac{n_j - n_{j\pm1}}{2n_\pm} + 1 \right) + \delta_{k,j} \left(\frac{n_j - n_{j\pm1}}{2n_\pm} - 1 \right) \right), \end{aligned} \tag{26}$$

which gives

$$\begin{aligned} \sum_k \left(n_k^{m+1} - n_k\right) \frac{d(a_0/L_n)_\pm}{dn_k} &= \mp \frac{1}{n_\pm \Delta\rho} \left(\left(n_{j\pm1}^{m+1} - n_{j\pm1}\right) \left(\frac{n_j - n_{j\pm1}}{2n_\pm} + 1 \right) + \left(n_j^{m+1} - n_j\right) \left(\frac{n_j - n_{j\pm1}}{2n_\pm} - 1 \right) \right) \\ &= \mp \frac{1}{\Delta\rho} \left(\frac{n_{j\pm1}^{m+1}}{n_\pm} \frac{n_j}{n_\pm} - \frac{n_j^{m+1}}{n_\pm} \frac{n_{j\pm1}}{n_\pm} \right). \end{aligned} \tag{27}$$

2 momentum transport

We begin with the equation for the transport of toroidal angular momentum in the high flow ($\mathbf{u}_0 \sim v_{t,i}$) limit:

$$\frac{\partial L}{\partial t} + \frac{1}{V'} \frac{\partial}{\partial \psi} \left[V' \sum_s \left(\langle \pi_{\psi\phi,s} \rangle + m_s \omega(\psi) \left\langle R^2 \Gamma_{\psi,s} \right\rangle \right) \right] = S_L, \tag{28}$$

where $L = \sum_s m_s \left\langle n_s R^2 \right\rangle \omega(\psi)$, $\mathbf{u}_0 = R\omega(\psi)\hat{\mathbf{e}}_\phi$, $\langle . \rangle$ is the flux surface average, S_L is the flux-surface averaged external momentum source, $\Gamma_{\psi,s}$ is defined by Eq. (2), and π is a turbulent momentum flux given by

$$\pi_{\psi\phi,s} = \int d^3\mathbf{v}\ m_s R^2 \left(\mathbf{v} \cdot \nabla\phi\right) \left(\mathbf{v}_\chi \cdot \nabla\psi\right) \delta f(\mathbf{R}). \tag{29}$$

We identify the total toroidal angular momentum flux to be

$$\Pi \equiv \sum_s \left(\langle \pi_{\psi\phi,s} \rangle + m_s \omega(\psi) \left\langle R^2 \Gamma_{\psi,s} \right\rangle \right), \tag{30}$$

so that the momentum transport equation takes the simple form

$$\frac{\partial L}{\partial t} + \frac{1}{V'} \frac{\partial}{\partial \psi} \left(V'\Pi\right) = S_L. \tag{31}$$

2.1 normalization

Next, we must normalize Eq. (31). We do so by defining the normalizing quantities

$$\tilde{\Pi} \equiv \frac{1}{\langle |\nabla\psi| \rangle} \frac{\Pi}{m_r a_r n_r v_{t,r}^2} \left(\frac{a_r}{\rho_r}\right)^2, \tag{32}$$

$$\tilde{L} \equiv \frac{L}{m_p a_0 n_0 v_{t,0}}, \tag{33}$$

$$\tilde{t} \equiv \frac{v_{t,0}}{a_0} \left(\frac{\rho_0}{a_0}\right)^2 t, \tag{34}$$

where: m_p is the proton mass; a_0 is the half-diameter of the LCFS at the elevation of the magnetic axis; $n_0 = 10^{20}/m^3$; $T_0 = 1\ keV$; $v_{t,0}$ is the reference thermal velocity for the transport calculation, defined

$$v_{t,0} \equiv \sqrt{\frac{2T_0}{\alpha m_p}}; \tag{35}$$

$\alpha = 1$ (or 2) if $v_{t,r}$ is defined with (or without) the $\sqrt{2}$ factor; ρ_0 is the reference gyroradius in the transport calculation, defined

$$\begin{aligned}\rho_0 &\equiv \frac{v_{t,0}}{\Omega_0} \\ &= \sqrt{\frac{2T_0}{m_p}}\frac{m_p c}{|e|\,B_0} \\ &= \sqrt{2T_0 m_p}\frac{c}{|e|\,B_0};\end{aligned} \tag{36}$$

$B_0 = 1\ T$; m_r, n_r, and $v_{t,r}$ are the mass, density, and thermal velocity of the reference species in the flux tube calculation; a_r is the reference length in the flux tube calculation; and ρ_r is the gyroradius of the reference species in the flux tube calculation, defined

$$\begin{aligned}\rho_r &\equiv \frac{v_{t,r}}{\Omega_r} \\ &= \frac{m_r v_{t,r} c}{Z_r\,|e|\,B_a},\end{aligned} \tag{37}$$

with B_a equal to the toroidal magnetic field at R_a, which is the average of the minimum and maximum major radius of the flux surface.

Dividing Eq. 31 by $(\rho_0/a_0)^2 m_p n_0 v_{t0}^2$, we obtain

$$\frac{\partial\tilde{L}}{\partial\tilde{t}} + \frac{1}{V'}\frac{\partial}{\partial\psi}\left(V'\left\langle|\nabla\psi|\right\rangle a_r \frac{m_r}{m_p}\frac{n_r}{n_0}\left(\frac{v_{t,r}}{v_{t,0}}\frac{\rho_r}{\rho_0}\frac{a_0}{a_r}\right)^2\tilde{\Pi}\right) = \tilde{S}_L., \tag{38}$$

where $\tilde{S}_L \equiv (a_0/\rho_0)^2 S_L/(m_p n_0 v_{t0}^2)$. We next note that the flux surface area, A, satisfies $A \equiv (dV/d\psi)\left\langle|\nabla\psi|\right\rangle = (dV/d\rho)\left\langle|\nabla\rho|\right\rangle$, where ρ is an arbitrary flux surface label defined so that it is zero at the magnetic axis and one at the LCFS. We then define $\tilde{A} \equiv A/a_0^2$ and $\tilde{\nabla} = a\nabla$. This gives

$$\frac{\partial\tilde{L}}{\partial\tilde{t}} + \frac{\left\langle\left|\tilde{\nabla}\rho\right|\right\rangle}{\tilde{A}}\frac{\partial}{\partial\rho}\left(\tilde{A}\frac{a_0}{a_r}\frac{m_r}{m_p}\frac{n_r}{n_0}\left(\frac{v_{t,r}}{v_{t,0}}\frac{\rho_r}{\rho_0}\right)^2\tilde{\Pi}\right) = \tilde{S}_L. \tag{39}$$

For convenience of future calculations, we explicitly calculate $v_{t,r}/v_{t,0}$ and ρ_r/ρ_0:

$$\frac{v_{t,r}}{v_{t,0}} = \sqrt{\frac{\tilde{T}_r}{\tilde{m}_r}}, \tag{40}$$

$$\begin{aligned}\frac{\rho_r}{\rho_0} &= \frac{v_{t,r}}{v_{t,0}}\frac{\Omega_0}{\Omega_r} \\ &= \frac{\sqrt{\tilde{m}_r\tilde{T}_r}}{Z_r\tilde{B}_a}\end{aligned} \tag{41}$$

where we have defined $\tilde{T} \equiv T/T_0$, $\tilde{m} \equiv m/m_p$, and $\tilde{B}_a = B_a/B_0$. Additionally defining $\tilde{a} = a_r/a_0$ and $\tilde{n} = n/n_0$, we obtain the final form of our normalized equation:

$$\frac{\partial\tilde{L}}{\partial\tilde{t}} + \frac{\left\langle\left|\tilde{\nabla}\rho\right|\right\rangle}{\tilde{A}}\frac{\partial}{\partial\rho}\left(\frac{\tilde{A}}{\tilde{a}Z_r^2\tilde{B}_a^2}\tilde{m}_r\tilde{n}_r\tilde{T}_r^2\tilde{\Pi}\right) = \tilde{S}_L. \tag{42}$$

2.2 discretization

3 energy transport

We begin with the energy transport equation:

$$\frac{3}{2}\frac{\partial p_s}{\partial t} + \frac{1}{V'}\frac{\partial}{\partial \psi}\left(V' Q_{\psi,s}\right) + \frac{3}{2} n_s \sum_u \nu^{\varepsilon}_{su}\left(T_s - T_u\right) = S_p, \tag{43}$$

where p_s is the equilibrium pressure, S_p is the total external energy input, ν^{ε}_{su} is the collisional temperature equilibration frequency, and $Q_{\psi,s}$ is flux surface averaged ψ-component of the heat flux, given by

$$Q_{\psi,s} \equiv \left\langle \int d^3\mathbf{v}\ \frac{m_s v^2}{2}\left(\mathbf{v}_{\chi}\cdot\nabla\psi\right)\delta f(\mathbf{R})\right\rangle_{\psi}. \tag{44}$$

3.1 normalization

$$\tilde{Q}_s \equiv \frac{1}{\langle|\nabla\psi|\rangle}\frac{Q_{\psi,s}}{n_r T_r v_{t,r}}\left(\frac{a_r}{\rho_r}\right)^2. \tag{45}$$

We now divide Eq. (43) by $n_0 T_0 (v_{t,0}/a_0)(\rho_0/a_0)^2$ and switch to the generalized flux label ρ. The algebra is almost identical to that from the particle transport section. The final equation is

$$\frac{3}{2}\frac{\partial \tilde{p}_s}{\partial \tilde{t}} + \frac{\langle|\nabla\rho|\rangle}{\tilde{A}}\frac{\partial}{\partial\rho}\left(\frac{\tilde{A}}{\tilde{B}_a^2}\tilde{n}_r\tilde{T}_r^{5/2}\tilde{Q}_s\right) + \frac{3}{2}\tilde{n}_s\sum_u \tilde{\nu}^{\varepsilon}_{su}\left(\tilde{T}_s - \tilde{T}_u\right) = \tilde{S}_p, \tag{46}$$

where

$$\begin{aligned}\tilde{S}_p &\equiv \frac{S_p}{n_0 T_0}\frac{a_0}{v_{t,0}}\rho_*^{-2} \\ &= 3.1\times 10^{-21}\frac{[m]^3\,[s]}{[keV]}\frac{\tilde{a}_0^3}{\sqrt{\tilde{m}_r}}\left(\frac{\alpha}{2}\right)^{3/2} S_p \\ &= 1.9\times 10^{-5}\frac{[m]^3}{[W]}\frac{\tilde{a}_0^3}{\sqrt{\tilde{m}_r}}\left(\frac{\alpha}{2}\right)^{3/2} S_p\end{aligned} \tag{47}$$

and

$$\begin{aligned}\tilde{\nu}^{\varepsilon}_{su} &\equiv 6.86\frac{\sqrt{m_s m_u}\,q_s^2 q_u^2 n_u \lambda_{su}}{\left(m_s T_u + m_u T_s\right)^{3/2}}\frac{a_0}{v_{t,0}}\rho_*^{-2} \\ &= 1.54\tilde{\nu}_{su}\sqrt{\frac{m_u}{m_s}}\left(\frac{T_u}{T_s} + \frac{m_u}{m_s}\right)^{-3/2}\frac{v_{t,r}}{v_{t,0}}\rho_*^{-2},\end{aligned} \tag{48}$$

where

$$\begin{aligned}\tilde{\nu}_{su} &\equiv \frac{4\pi q_s^2 q_u^2 n_u \lambda_{su}}{\sqrt{m_s}\left(2T_s\right)^{3/2}}\frac{a_0}{v_{t,r}} \\ &= \frac{4\pi}{2^{3/2}}\frac{Z_s^2 Z_u^2 \tilde{n}_u \lambda_{su}}{\sqrt{\tilde{m}_s \tilde{T}_r}\tilde{T}_s^{3/2}}\frac{a_0}{v_{t,0}}\frac{e^4}{\sqrt{m_p}}\frac{10^{20}}{[m]^3\,[keV]^{3/2}} \\ &= 9.22\times 10^{-4}\sqrt{\frac{\alpha}{2}}\frac{\tilde{a}_0\sqrt{\tilde{m}_r}Z_s^2 Z_u^2 \tilde{n}_u \lambda_{su}}{\sqrt{\tilde{m}_s\tilde{T}_r}\tilde{T}_s^{3/2}}.\end{aligned} \tag{49}$$

Let's get the heat flux in physical units:

$$\begin{aligned}\frac{Q_{\psi,s}}{\langle|\nabla\psi|\rangle} &= n_r T_r v_{t,r}\left(\frac{\rho_r}{a_r}\right)^2 \tilde{Q}_s \\ &= 3.22\times10^{20}\tilde{n}_r\frac{\sqrt{\tilde{m}_r}\tilde{T}_r^{5/2}}{\tilde{a}_0^2\tilde{B}_a^2}\left(\frac{2}{\alpha}\right)^{3/2}\frac{[keV]}{[m]^2\,[s]}\tilde{Q}_s \\ &= 5.16\times10^{4}\tilde{n}_r\frac{\sqrt{\tilde{m}_r}\tilde{T}_r^{5/2}}{\tilde{a}_0^2\tilde{B}_a^2}\left(\frac{2}{\alpha}\right)^{3/2}\frac{[W]}{[m]^2}\tilde{Q}_s.\end{aligned} \tag{50}$$

If we want to consider power balance, we need to integrate this expression over the flux surface. Since none of the quantities vary over the flux surface, this simply involves multiplying by the flux surface area:

$$A\frac{Q_{\psi,s}}{\langle|\nabla\psi|\rangle} = 5.16\times10^{4}\tilde{n}_r\tilde{T}_r^{5/2}\frac{\sqrt{\tilde{m}_r}\tilde{A}}{\tilde{B}_a^2}\left(\frac{2}{\alpha}\right)^{3/2}\tilde{Q}_s\,[W]\,. \tag{51}$$

3.2 discretization

We start by defining $F_s \equiv (2/3)\tilde{A}\tilde{n}_r\tilde{T}_r^{5/2}/\tilde{B}_a^2\tilde{Q}_s$. Then the single-step discretization of Eq. (47) is

$$\frac{p_j^{m+1}-p_j^m}{\Delta t}+\alpha\left(\frac{\langle|\nabla\psi|\rangle_j}{A_j}\frac{F_+-F_-}{\Delta\rho}\right)^{m+1}+(1-\alpha)\left(\frac{\langle|\nabla\psi|\rangle_j}{A_j}\frac{F_+-F_-}{\Delta\rho}\right)=\frac{2}{3}S_j. \tag{52}$$

We can Taylor expand $F_\pm^{m+1}$ about its value at the m time level and keep terms through linear order:

$$F_\pm^{m+1}\approx F_\pm+(\mathbf{y}-\mathbf{y}_0)\left.\frac{\partial F_\pm}{\partial\mathbf{y}}\right|_{\mathbf{y}_0}. \tag{53}$$

The derivative term can be written explicitly as

$$\left.\frac{\partial F_\pm}{\partial\mathbf{y}}\right|_{\mathbf{y}_0}=\sum_k\left(n_k^{m+1}-n_k\right)\frac{\partial F_\pm}{\partial n_k}+... \tag{54}$$

In the local approximation, this becomes

$$\left.\frac{\partial F_\pm}{\partial\mathbf{y}}\right|_{\mathbf{y}_0}=\sum_k\left(n_k^{m+1}-n_k\right)\left(\frac{\partial F_\pm}{\partial n_\pm}\frac{dn_\pm}{dn_k}+\frac{\partial F_\pm}{\partial(R/L_n)_\pm}\frac{d(R/Ln)_\pm}{dn_k}\right)+... \tag{55}$$

For now, we assume that the dependence of $F_\pm$ on $n_\pm$ and $p_\pm$ is weak so that

$$\left.\frac{\partial F_\pm}{\partial\mathbf{y}}\right|_{\mathbf{y}_0}\approx\sum_k\left(n_k^{m+1}-n_k\right)\frac{\partial F_\pm}{\partial(R/L_n)_\pm}\frac{d(R/Ln)_\pm}{dn_k}+... \tag{56}$$

We express grad scale lengths discretely as

$$\left(\frac{R}{L_n}\right)_\pm=\mp\frac{R}{n_\pm}\frac{n_{j\pm1}-n_j}{\Delta\rho}. \tag{57}$$

We then have

$$\begin{aligned}\frac{d(R/L_n)_\pm}{dn_k}&=\mp\frac{R}{n_\pm\Delta\rho}\left(-\frac{1}{n_\pm}\frac{dn_\pm}{dn_k}\left(n_{j\pm1}-n_j\right)+\delta_{k,j\pm1}-\delta_{k,j}\right)\\&=\mp\frac{R}{n_\pm\Delta\rho}\left(-\frac{1}{2n_\pm}\left(\delta_{k,j\pm1}+\delta_{k,j}\right)\left(n_{j\pm1}-n_j\right)+\delta_{k,j\pm1}-\delta_{k,j}\right)\\&=\mp\frac{R}{n_\pm\Delta\rho}\left(\delta_{k,j\pm1}\left(\frac{n_j-n_{j\pm1}}{2n_\pm}+1\right)+\delta_{k,j}\left(\frac{n_j-n_{j\pm1}}{2n_\pm}-1\right)\right),\end{aligned} \tag{58}$$

which gives

$$\sum_k \left(n_k^{m+1} - n_k\right) \frac{d(R/Ln)_\pm}{dn_k} = \mp \frac{R}{n_\pm \Delta\rho} \left(\left(n_{j\pm1}^{m+1} - n_{j\pm1}\right) \left(\frac{n_j - n_{j\pm1}}{2n_\pm} + 1 \right) + \left(n_j^{m+1} - n_j\right) \left(\frac{n_j - n_{j\pm1}}{2n_\pm} - 1 \right) \right). \tag{59}$$