Transport scalings for critically-balanced ITG turbulence in tokamaks

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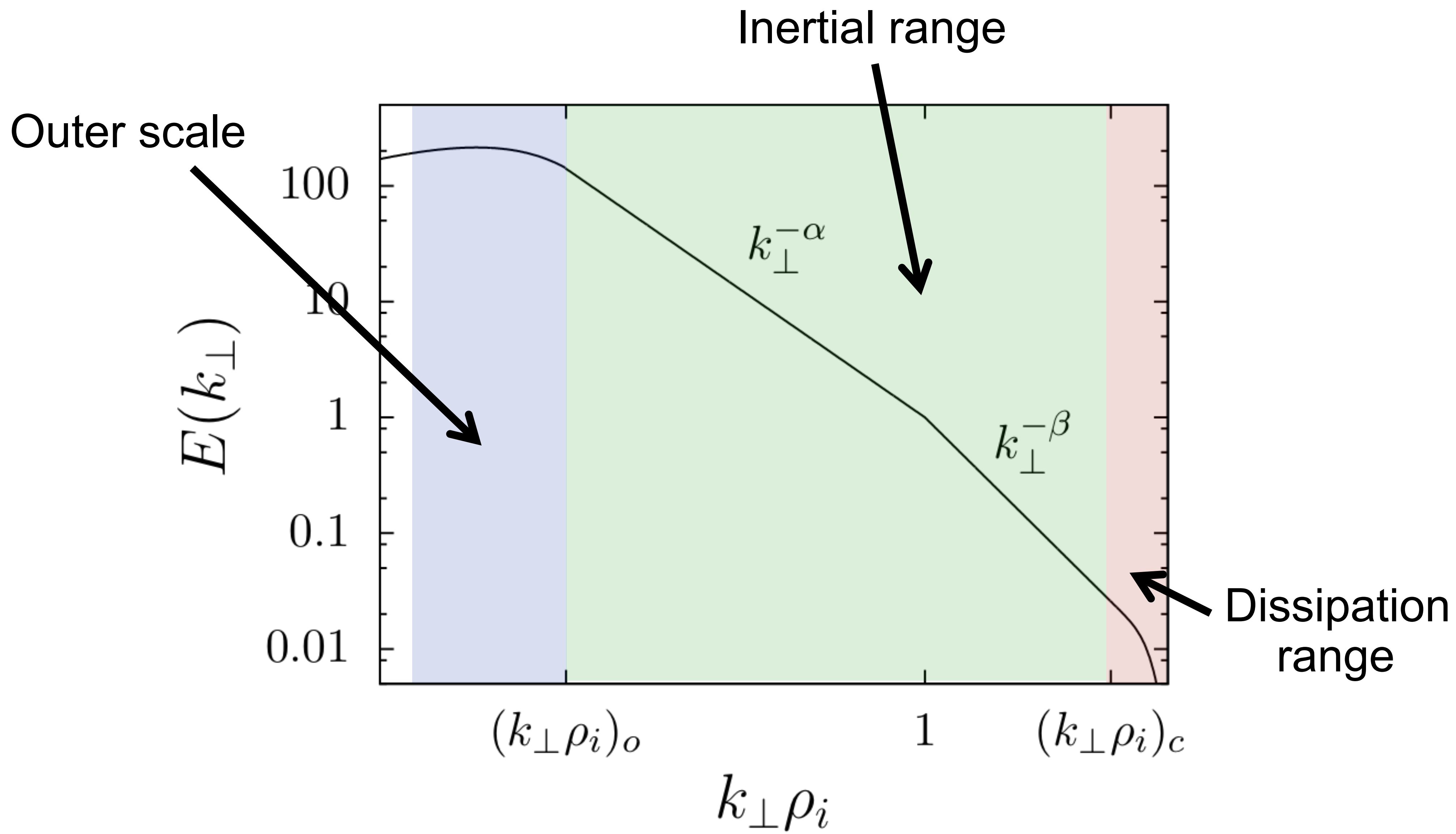
Why do we care?

- Plasma confinement properties depend strongly on quantities such as mean plasma current and ion temperature gradient
- Analytical results for turbulence are rare, and direct numerical simulations are costly
- Scaling laws useful indicators of gross plasma performance and provide guidance for numerical simulations
- Interesting physics problem

Big picture



Inertial range
Outer scale
Dissipation range
100
10
1
0.1
0.01
$E(k_\perp)$
$k_\perp^{-\alpha}$
$k_\perp^{-\beta}$
$(k_\perp \rho_i)_o$
1
$(k_\perp \rho_i)_c$
$k_\perp \rho_i$

Gyrokinetic model

Gyrokinetic variables: $\mathbf{R},\ E = \frac{mv^2}{2},\ \mu = \frac{mv_\perp^2}{2B}$

$$\frac{\partial}{\partial t}\left(h_s - \frac{Z_s e \langle\varphi\rangle_\mathbf{R}}{T_s} F_{M,s}\right) + \left(\mathbf{v}_\| + \mathbf{v}_{M,s}\right)\cdot\nabla h_s + \frac{c}{B}\left\{\langle\varphi\rangle_\mathbf{R}, h_s\right\}$$
$$= \langle C[h_s]\rangle_\mathbf{R} - \langle \mathbf{v}_E\rangle_\mathbf{R}\cdot\nabla F_{M,s}$$

Quasineutrality: $\sum_s Z_s \left(\int d^3v\ J_0 h_s - \frac{Z_s e\varphi}{T_s} n_s\right) = 0$

Assume $\int d^3v\ J_0 h \sim v_{th}^3 J_0 h \Rightarrow J_0 \frac{h}{F_M} \sim \frac{Ze\varphi}{T}$

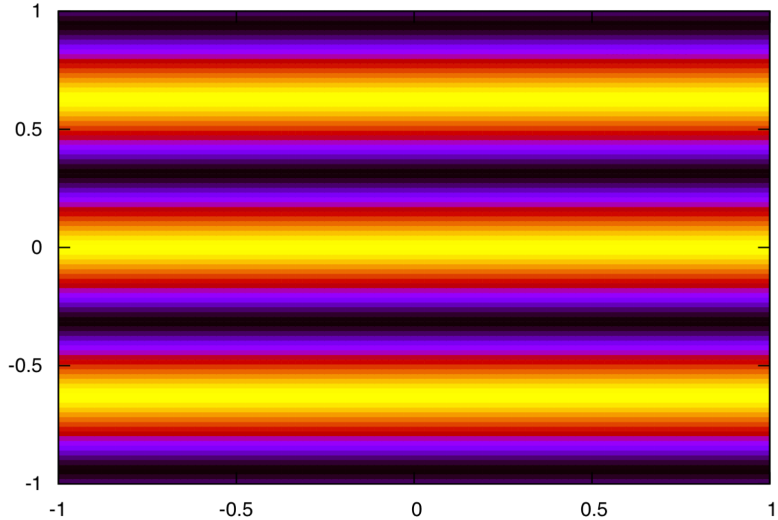
Conjectures

- Fluctuation scale lengths in two dimensions of plane perpendicular to B-field are comparable
- Parallel streaming time and nonlinear turnover time comparable at all scales (critical balance)
- No significant dissipation or driving between outer scale and Larmor scale (inertial range)
- One additional conjecture about drive required to determine outer scale

Isotropy

- Conjecture: fluctuation scale lengths in two dimensions of plane perpendicular to B-field are comparable
- Physical idea: linear drive favors structures with $\ell_x \gtrsim \ell_y$. Smaller ℓ_x formed through magnetic and zonal flow shear:

$$\ell_x^{-1} \sim (S_{\mathrm{zf}}\tau_{nl} + \hat{s}\theta)\ell_y^{-1} \sim \ell_y^{-1} \sim \ell_\perp$$



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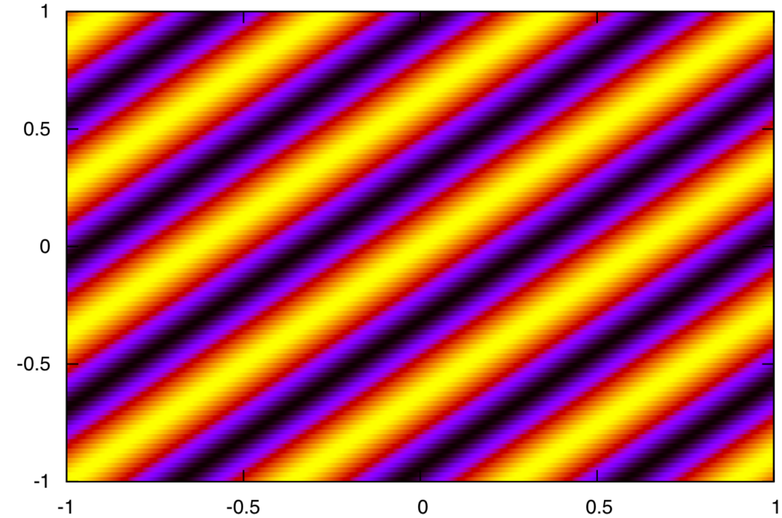


Critical balance

- Conjecture: characteristic time associated with particle streaming and wave propagation along mean field is comparable to nonlinear decorrelation time at each scale
- Physical idea: two points along field correlated only if information propagates between them before turbulence decorrelated in perpendicular plane

$$\frac{v_{th}}{\ell_{||}} \sim \tau_{nl}^{-1} \sim \frac{v_{th}}{R}\frac{\rho_i^2}{\ell_{\perp}^2}\langle\Phi_\ell\rangle$$

$$\Phi_\ell \equiv \frac{e\varphi_\ell}{T}\frac{R}{\rho_i} \qquad \varphi_\ell \equiv \varphi(\mathbf{r}+\ell) - \varphi(\mathbf{r})$$

Outer scale

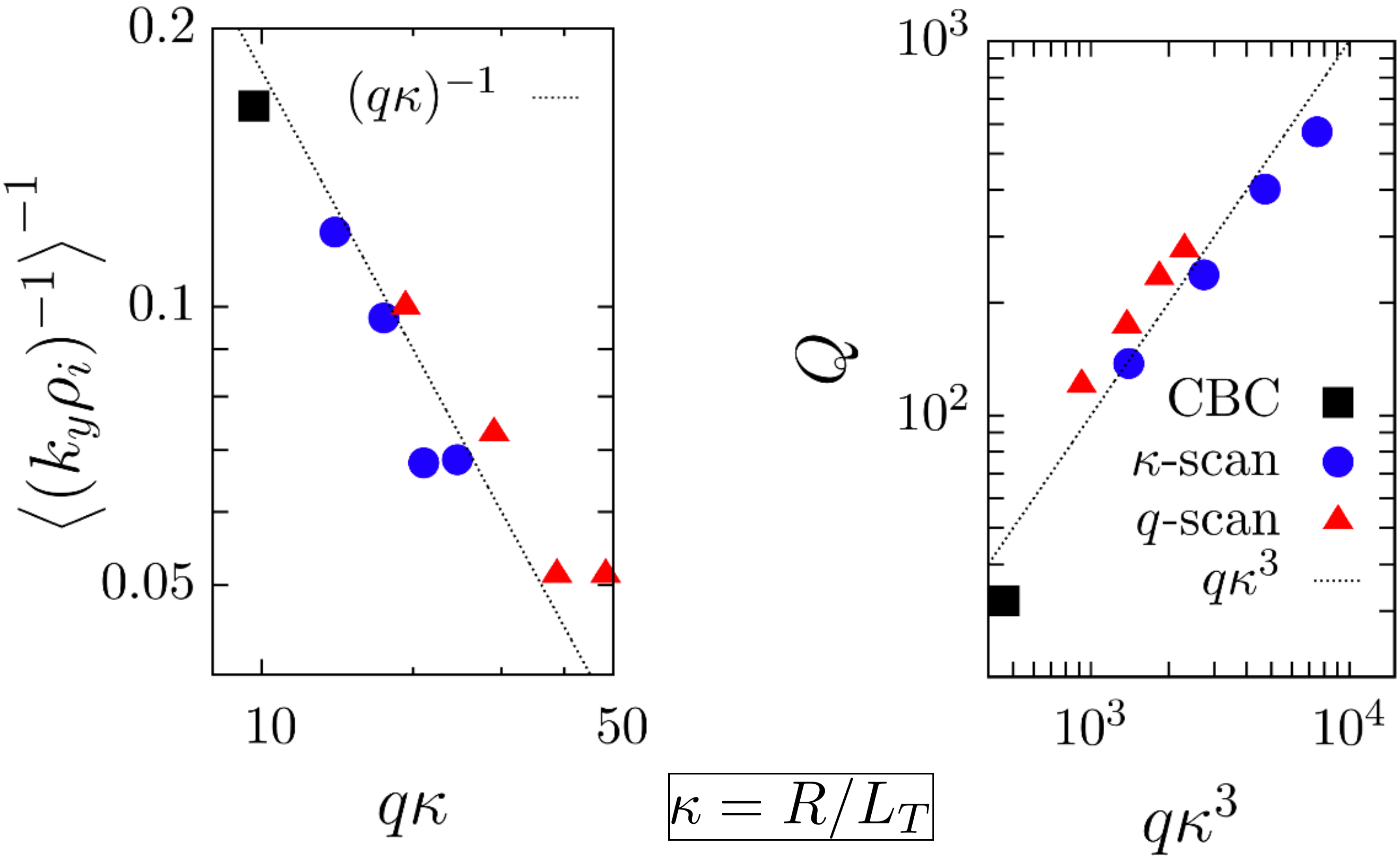
- Define outer scale as range where injection rate comparable to nonlinear decorrelation time:

$$\tau_{nl}^{-1} \sim \omega_*^o \sim \frac{\rho_i v_{th}}{\ell_\perp^o L_T} J_{0\ell} \text{ ==> } \frac{\ell_\perp^o}{\rho_i} \sim \frac{\ell_\parallel^o}{L_T} J_{0\ell}$$

- Conjecture: characteristic parallel length scale of turbulence at outer scale is the connection length
- Physical idea: modes cannot extend much beyond connection length due to stabilizing effect of good curvature

$$\text{==> } \frac{\ell_\perp^o}{\rho_i} \sim \frac{qR}{L_T}, \quad \Phi_o \sim q\left(\frac{R}{L_T}\right)^2 \text{ ==> } Q \sim \frac{\rho_i}{\ell_\perp^o}\Phi_o^2 \sim q\left(\frac{R}{L_T}\right)^3$$

Turbulence scaling tests



Note that Q at large R/L_T much larger than found in previous studies (box size used here for $R/L_T \approx 20$ was $\approx 1000\rho_i$)

Inertial range

- Conjecture: No significant drive or dissipation between outer and dissipation scales
- Flux of free energy (nonlinear invariant) scale-independent in inertial range:

$$W = V^{-1} \sum_s \int d^3r \int d^3v \left(\frac{T_s \delta f_s^2}{F_{M,s}} \right)$$

$$\frac{W_\ell}{\tau_{nl}} \sim \left(\frac{\rho_i}{R} \right)^2 \frac{v_{th}}{R} \frac{\rho_i^2}{\ell_\perp^2} \Phi_\ell^3 \sim \text{constant}$$

==> $\Phi_\ell \sim \Phi_o \left(\frac{\ell_\perp}{\ell_\perp^o} \right)^{2/3} \sim q^{1/3} \left(\frac{R}{L_T} \right)^{4/3} \left(\frac{\ell_\perp}{\rho_i} \right)^{2/3}$

Inertial range

- Use critical balance and expression for Φ_ℓ to get relationship between parallel and perpendicular length scales

$$\frac{\ell_{\|}}{qR} \sim \left(\frac{\ell_\perp}{\rho_i} \frac{L_T}{qR} \right)^{4/3}$$

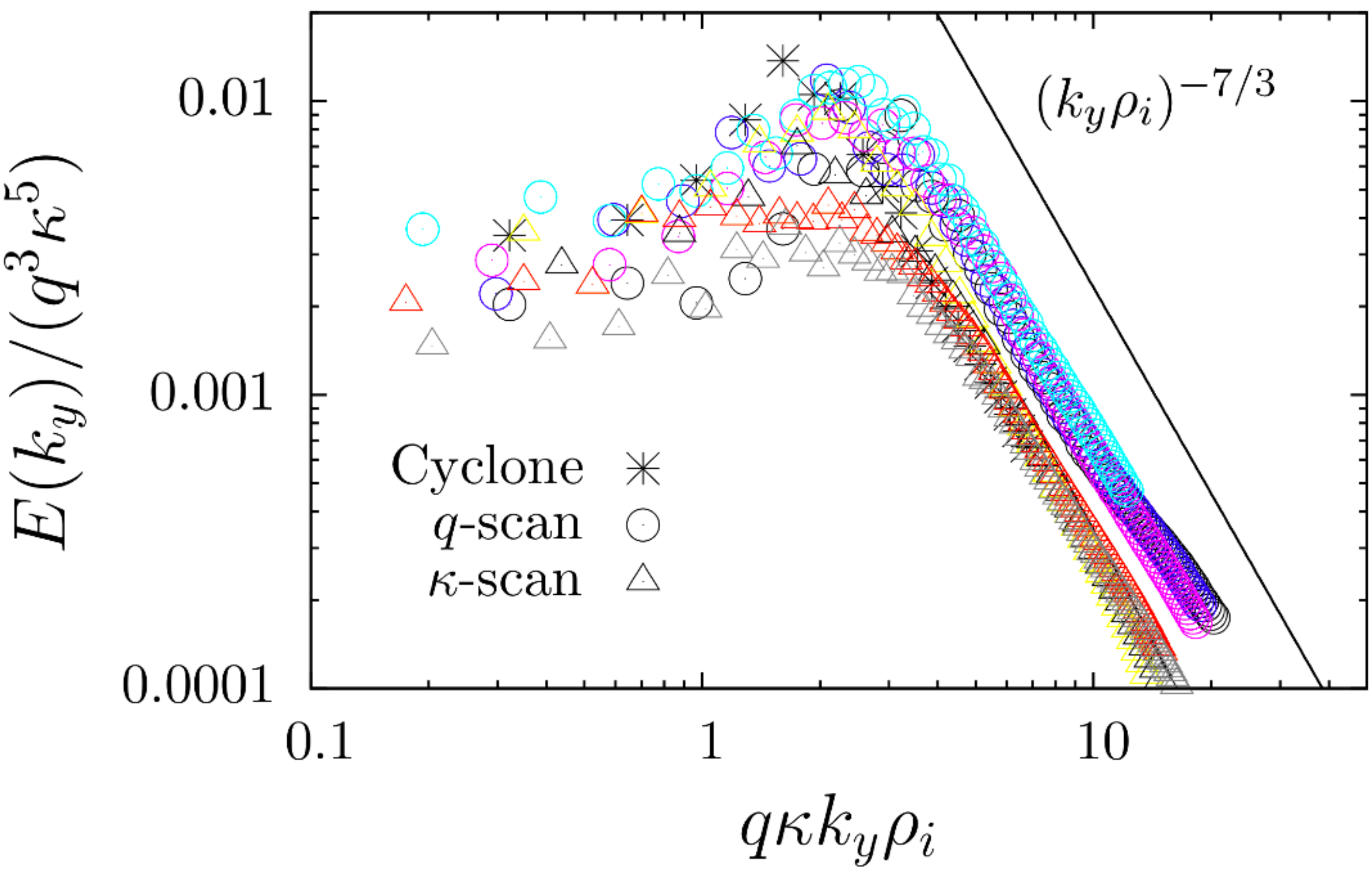
- Convert expression for Φ_ℓ into 1D spectrum using Parseval's theorem

$$\int dk_y \ \rho_i E(k_y) = V^{-1} \int d^3 r \ \Phi^2$$

$$E(k_y) \sim k_y \rho_i \left|\Phi_k\right|^2 \sim q^{2/3} \left(\frac{R}{L_T} \right)^{8/3} (k_y \rho_i)^{-7/3}$$

Inertial range spectra

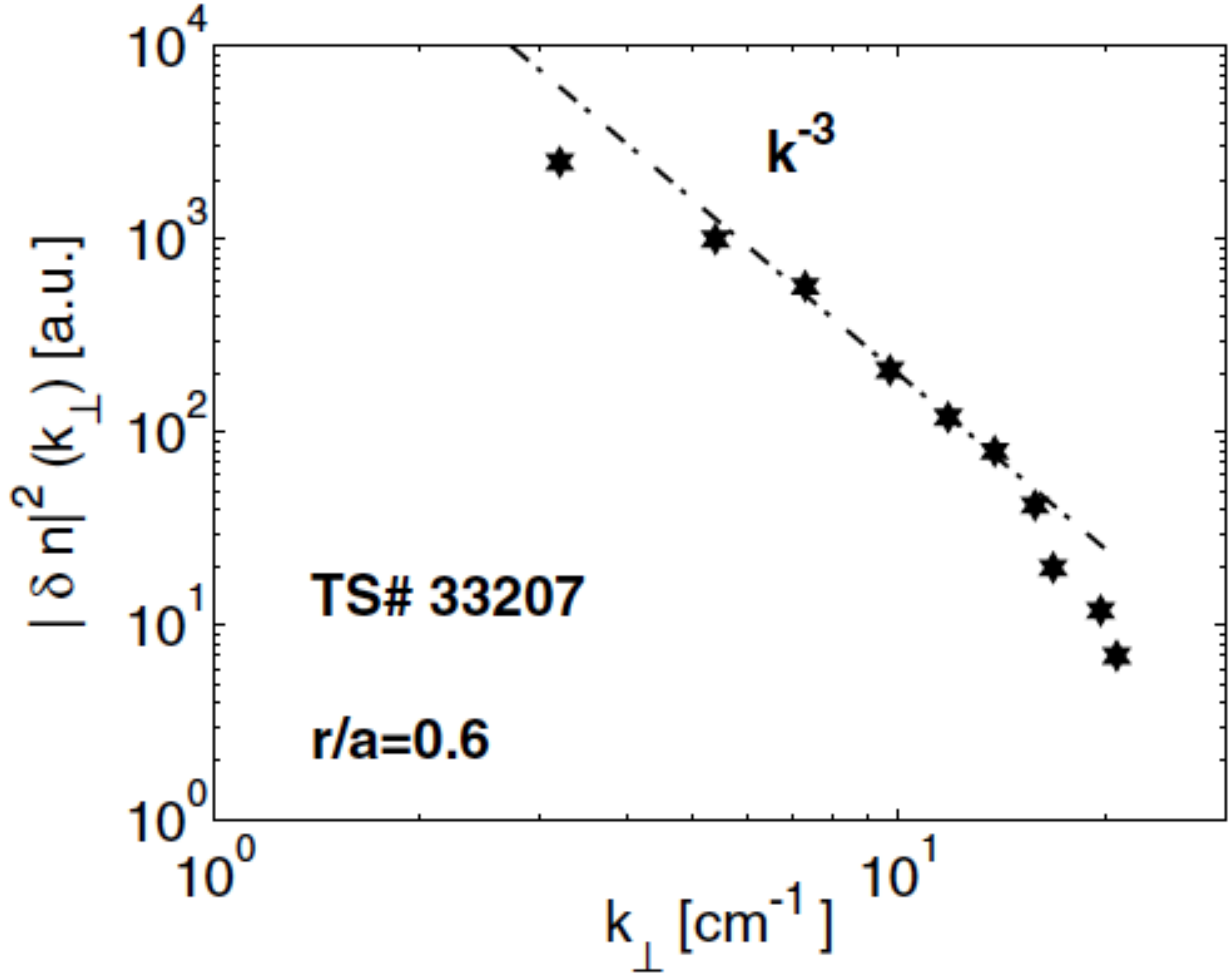
$$\int dk_y \; \rho_i E(k_y) = V^{-1} \int d^3 r \; \Phi^2$$



Inertial range spectra

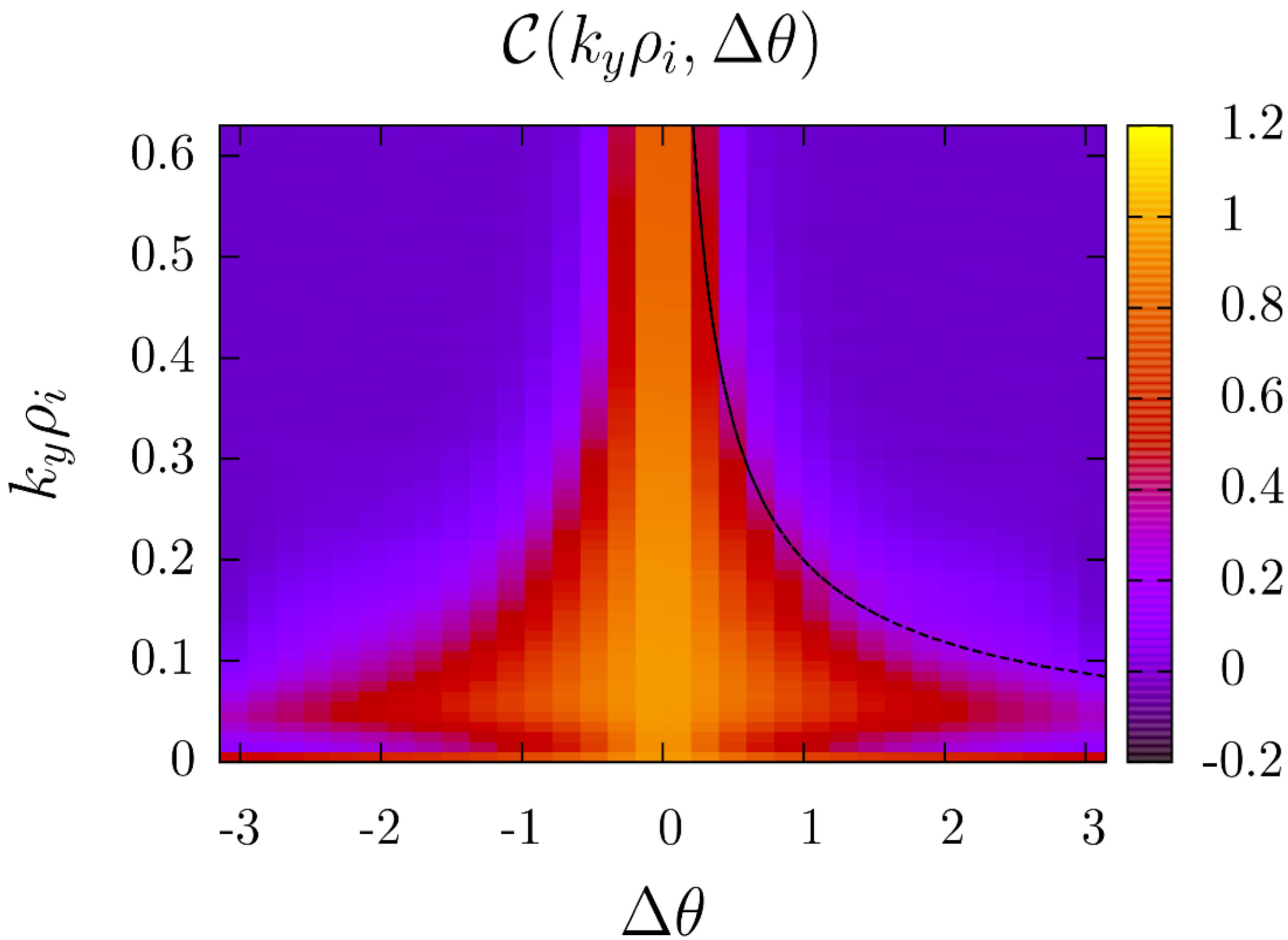
$$E(k_\perp) \sim k_\perp \left|\Phi_k\right|^2 \sim k_\perp^{-7/3} \Rightarrow \left|\Phi_k\right|^2 \sim k_\perp^{-10/3}$$

Tore Supra
fluctuation
spectrum



Critical balance test

Correlation function: $$\mathcal{C}(k_y, \Delta\theta) = \frac{\sum_{k_x} \Phi(\theta=0)\Phi^*(\theta=\Delta\theta)}{\sum_{k_x} |\Phi(\theta=0)|^2}$$

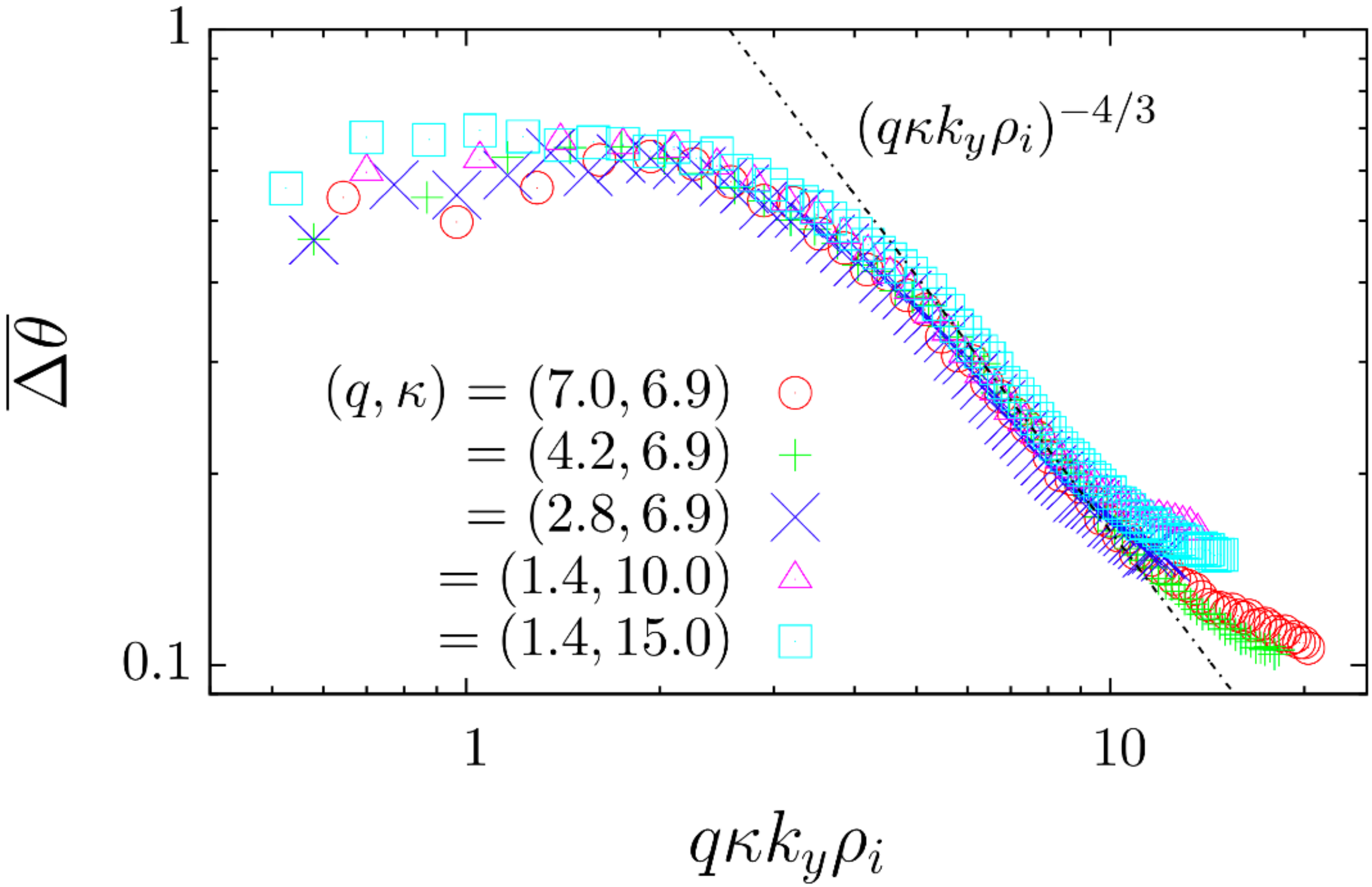


$$\frac{\ell_\parallel}{qR} \sim \left(\frac{\ell_\perp}{\rho_i}\right)^{4/3}$$

Inertial range critical balance

$$\frac{\ell_\parallel}{qR} \sim \left(\frac{\ell_\perp}{\rho_i}\frac{L_T}{qR}\right)^{4/3}$$

$$\overline{\Delta\theta}(k_y) = \int d(\Delta\theta)\mathcal{C}(k_y, \Delta\theta)$$



Dissipation scale

- In analogy with Reynolds number, define

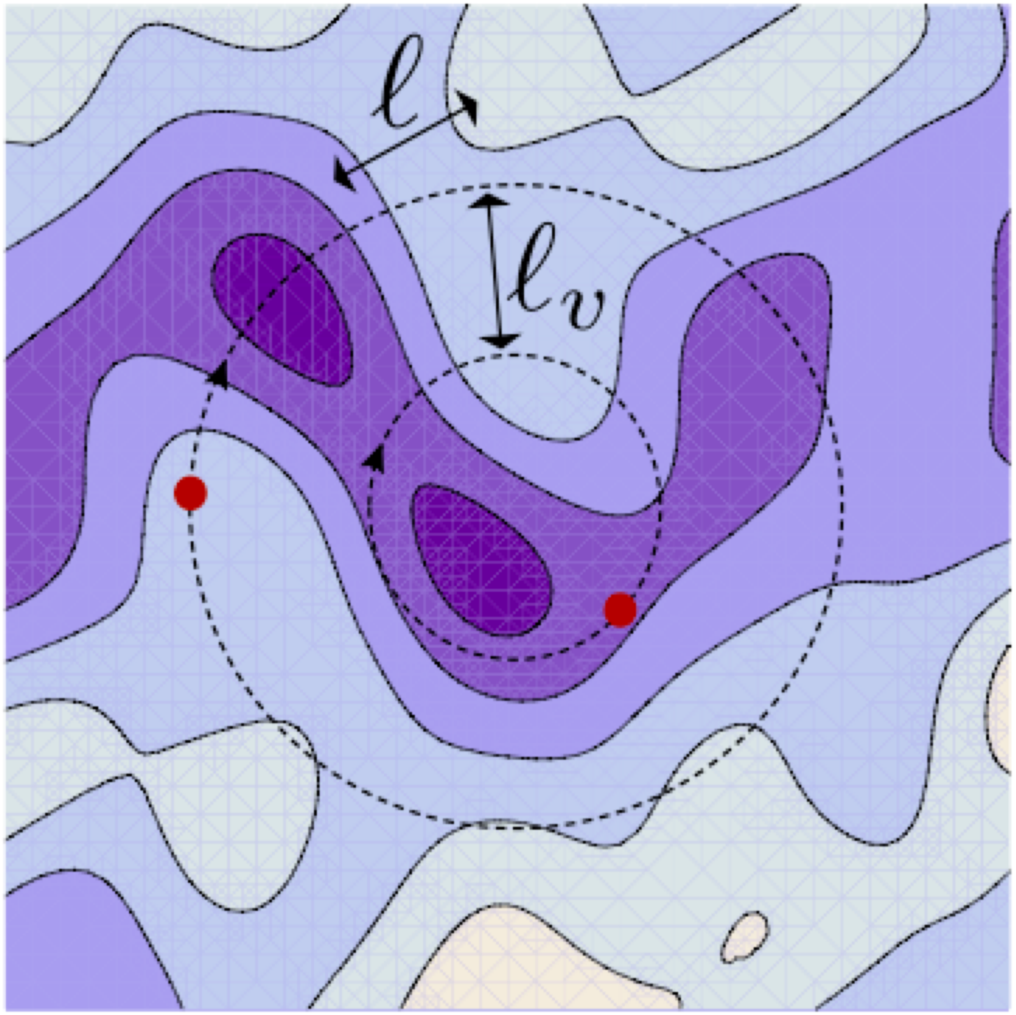
$$\mathrm{Do} \equiv (\nu_i \tau_{\rho_i})^{-1} \sim q^{1/3} \left(\frac{R}{L_T}\right)^{4/3} \left(\frac{v_{th}}{\nu_i R}\right)$$

- At dissipation scale, dissipation rate comparable to nonlinear decorrelation rate

$$\nu_i \left(\frac{v_{th}}{\delta v}\right)^2 \sim \tau_{nl}^{-1} \Rightarrow \mathrm{Do} \sim \left(\frac{v_{th}}{\delta v}\right)^2 \frac{\tau_{nl}}{\tau_{\rho_i}} \sim \left(\frac{v_{th}}{\delta v}\right)^2 \left(\frac{\ell_\perp}{\rho_i}\right)^2 \frac{\Phi_{\rho_i}}{\Phi_\ell}$$

- Dissipation scale assumed below ion Larmor scale
- Argue that perpendicular space and velocity scales are related via phase mixing…

Perpendicular phase mixing



Schekochihin *et al*., PPCF 2008

- Drift velocity = $F[\langle\Phi\rangle]$
- Particles with Larmor orbits separated by turbulence wavelength 'see' different averaged potential
- Drift velocities decorrelated, thus phase mixing

$$\frac{k_\perp \delta v_\perp}{\Omega} \sim 1$$

$$\Rightarrow \frac{\delta v_\perp}{v_{th}} \sim (k_\perp \rho_i)^{-1}$$

Dissipation scale

$$\mathrm{Do} \sim \left(\frac{v_{th}}{\delta v} \frac{\ell_\perp}{\rho_i} \right)^2 \frac{\Phi_{\rho_i}}{\Phi_\ell} \qquad \frac{\delta v}{v_{th}} \sim \frac{\ell_\perp}{\rho_i}$$

- Carrying out inertial range analysis (as before, but with $J_0(k_\perp \rho_i) \sim (k_\perp \rho_i)^{-1/2}$) gives

$$\Phi_\ell \sim \left(\frac{\ell_\perp}{\rho_i} \right)^{7/6} \Rightarrow (k_\perp \rho_i)_c \sim \mathrm{Do}^{3/5}$$

$$(k_\perp \rho_i)_c \sim q^{1/5} \left(\frac{R}{L_T} \right)^{4/5} \left(\frac{v_{th}}{\nu_i R} \right)^{3/5}$$

Back to big picture