Transport scalings for critically-balanced ITG turbulence in tokamaks

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Why do we care?

• Plasma confinement properties depend strongly on quantities such as mean plasma current and ion temperature gradient

• Analytical results for turbulence are rare, and direct numerical simulations are costly

• Scaling laws useful indicators of gross plasma performance and provide guidance for numerical simulations

• Provides guidance for reduced gyrokinetic models (example: intrinsic rotation)
Big picture

Outer scale

Inertial range

Dissipation range

$$E(k_\perp)$$

$$k_\perp^{-\alpha}$$

$$k_\perp^{-\beta}$$

$$(k_\perp \rho_i)_o$$

$$(k_\perp \rho_i)_c$$

$$k_\perp \rho_i$$
Gyrokinetic model

Gyrokinetic variables: \( \mathbf{R}, \quad E = \frac{mv^2}{2}, \quad \mu = \frac{mv^2}{2B} \)

\[
\frac{\partial}{\partial t} \left( h_s - \frac{Z_s e \langle \varphi \rangle_R}{T_s} F_{M,s} \right) + (v_\parallel + v_{M,s}) \cdot \nabla h_s + \frac{c}{B} \{ \langle \varphi \rangle_R , h_s \} = \langle C[h_s] \rangle_R - \langle \mathbf{v}_E \rangle_R \cdot \nabla F_{M,s}
\]

Quasineutrality: \( \sum_s Z_s \left( \int d^3v \; J_0 h_s - \frac{Z_s e \varphi}{T_s} n_s \right) = 0 \)

Assume \( \int d^3v \; J_0 h \sim v_{th}^3 J_0 h \Rightarrow J_0 \frac{h}{F_M} \sim \frac{Z e \varphi}{T} \)
Conjectures

- Five unknowns (3 space scales, potential, distribution function) determined by two equations (GK + QN) and three conjectures:
  - Fluctuation scale lengths in two dimensions of plane perpendicular to B-field are comparable
  - Parallel streaming time and nonlinear turnover time comparable at all scales (critical balance)
  - Parallel length at outer scale set by system size (connection length)
Isotropy

• Conjecture: fluctuation scale lengths in two dimensions of plane perpendicular to B-field are comparable

• Physical idea: linear drive favors structures with $\ell_x \gtrsim \ell_y$. Smaller $\ell_x$ formed through magnetic and zonal flow shear:

$$\ell_x^{-1} \sim (S_{zf} \tau_{nl} + \hat{s} \theta) \ell_y^{-1} \sim \ell_y^{-1} \sim \ell_{\perp}$$
Critical balance

• Conjecture: characteristic time associated with particle streaming and wave propagation along mean field is comparable to nonlinear decorrelation time at each scale

• Physical idea: two points along field correlated only if information propagates between them before turbulence decorrelated in perpendicular plane

\[ \frac{v_{th}}{\ell_\parallel} \sim \tau_{nl}^{-1} \sim \frac{v_{th}}{R} \frac{\rho_i^2}{\ell_\perp^2} \langle \Phi_\ell \rangle \]

\[ \Phi_\ell \equiv \frac{e \varphi_\ell}{\rho_i} \frac{R}{T} \frac{R}{\rho_i} \]

\[ \varphi_\ell \equiv \varphi(r + \ell) - \varphi(r) \]
Outer scale

- Define outer scale as range where injection rate comparable to nonlinear decorrelation time:

\[ \tau_{nl}^{-1} \sim \omega_* \sim \frac{\rho_i v_{th}}{\ell_o L_T} J_{0\ell} \Rightarrow \frac{\ell_o}{\rho_i} \sim \frac{\ell_o}{L_T} J_{0\ell} \]

- Conjecture: characteristic parallel length scale of turbulence at outer scale is the connection length

- Physical idea: modes cannot extend much beyond connection length due to stabilizing effect of good curvature

\[ \Rightarrow \frac{\ell_o}{\rho_i} \sim \frac{qR}{L_T} , \Phi_o \sim q \left( \frac{R}{L_T} \right)^2 \Rightarrow Q \sim \frac{\rho_i}{\ell_o} \Phi_o^2 \sim q \left( \frac{R}{L_T} \right)^3 \]
Simulation system

• Use continuum, local, delta-f GK code GS2
• Base case is Cyclone (widely benchmarked)
  – Unshifted, circular flux surface
  – Safety factor is 1.4, magnetic shear=0.8, R/Ln=2.2, R/LT=6.9
  – Electrostatic
  – Modified Boltzmann response for electrons
• Fix R/LT and vary q from 1.4 up to 7.0
• Fix q and vary R/LT from 6.9 to 17.5
Turbulence scaling tests

Note that $Q$ at large $R/L_T$ much larger than found in previous studies (box size used here for $R/L_T \approx 20$ was $\approx 1000 \rho_i$)
Big picture

Inertial range

$E(k_{\perp})$

$k_{\perp}^{-\alpha}$

$k_{\perp}^{-\beta}$

$(k_{\perp} \rho_i)_o$

$1$

$(k_{\perp} \rho_i)_c$

$k_{\perp} \rho_i$
Inertial range

- No significant drive or dissipation between outer and dissipation scales
- Flux of free energy (nonlinear invariant) scale-independent in inertial range:

\[ W = V^{-1} \sum_s \int d^3r \int d^3v \left( \frac{T_s \delta f_s^2}{F_{M,s}} \right) \]

\[ \frac{W_\ell}{\tau_{nl}} \sim \left( \frac{\rho_i}{R} \right)^2 \frac{v_{th}}{R} \frac{\rho_i^2}{\ell_{\perp}^2} \Phi_\ell^3 \sim \text{constant} \]

\[ \implies \Phi_\ell \sim \Phi_o \left( \frac{\ell_{\perp}}{\ell_{\perp}^o} \right)^{2/3} \sim q^{1/3} \left( \frac{R}{L_T} \right)^{4/3} \left( \frac{\ell_{\perp}}{\rho_i} \right)^{2/3} \]
Inertial range

- Use critical balance and expression for $\Phi_\ell$ to get relationship between parallel and perpendicular length scales
  \[
  \frac{\ell_{\|}}{qR} \sim \left( \frac{\ell_{\perp}}{\rho_i} \frac{L_T}{qR} \right)^{4/3}
  \]

- Convert expression for $\Phi_\ell$ into 1D spectrum using Parseval’s theorem

\[
\int dk_y \, \rho_i E(k_y) = V^{-1} \int d^3r \, \Phi^2
\]

\[
E(k_y) \sim k_y \rho_i \left| \Phi_k \right|^2 \sim q^{2/3} \left( \frac{R}{L_T} \right)^{8/3} (k_y \rho_i)^{-7/3}
\]
Inertial range spectra

\[ \int d k_y \, \rho_i E(k_y) = V^{-1} \int d^3 r \, \Phi^2 \]

\[ E(k_y) / (q^3 k^5) \]

\( (k_y \rho_i)^{-7/3} \)

Cyclone: \( \times \)
q-scan: \( \bigcirc \)
\( \kappa \)-scan: \( \triangle \)
Critical balance test

Correlation function:

\[ C(k_y, \Delta \theta) = \frac{\sum_{k_x} \Phi(\theta = 0)\Phi^*(\theta = \Delta \theta)}{\sum_{k_x} |\Phi(\theta = 0)|^2} \]

\[ C(k_y \rho_i, \Delta \theta) \]

\[ \frac{l_\parallel}{qR} \sim \left( \frac{l_\perp}{\rho_i} \right)^{4/3} \]
Inertial range critical balance

\[
\frac{\ell_\parallel}{q R} \sim \left( \frac{\ell_\perp L_T}{\rho_i q R} \right)^{4/3}
\]

\[
\overline{\Delta \theta}(k_y) = \int d(\Delta \theta) C(k_y, \Delta \theta)
\]
Big picture

\[ E(k_{\perp}) \]

\[ k_{\perp}^{-\alpha} \]

\[ k_{\perp}^{-\beta} \]

Dissipation range

\[ (k_{\perp} \rho_i)_o \]

\[ 1 \]

\[ (k_{\perp} \rho_i)_c \]
Dissipation scale

- In analogy with Reynolds number, define

\[ D_0 \equiv (\nu_i \tau_{\rho_i})^{-1} \sim q^{1/3} \left( \frac{R}{L_T} \right)^{4/3} \left( \frac{v_{th}}{\nu_i R} \right) \]

- At dissipation scale, dissipation rate comparable to nonlinear decorrelation rate

\[ \nu_i \left( \frac{v_{th}}{\delta v} \right)^2 \sim \tau_{nl}^{-1} \Rightarrow D_0 \sim \left( \frac{v_{th}}{\delta v} \right)^2 \frac{\tau_{nl}}{\tau_{\rho_i}} \sim \left( \frac{v_{th}}{\delta v} \right)^2 \left( \frac{\ell_\perp}{\rho_i} \right)^2 \frac{\Phi_{\rho_i}}{\Phi_{\ell}} \]

- Dissipation scale assumed below ion Larmor scale
- Perpendicular space and velocity scales related via phase mixing:

\[ \frac{\delta v}{v_{th}} \sim \frac{\ell_\perp}{\rho_i} \]
Perpendicular phase mixing

- Drift velocity = $F[\langle \Phi \rangle]$
- Particles with Larmor orbits separated by turbulence wavelength ‘see’ different averaged potential
- Drift velocities decorrelated, thus phase mixing

$$\frac{k_\perp \delta v_\perp}{\Omega} \sim 1$$

$$\Rightarrow \frac{\delta v_\perp}{v_{th}} \sim (k_\perp \rho_i)^{-1}$$

Schekochihin et al., PPCF 2008
Dissipation scale

\[ D_0 \sim \left( \frac{v_{th} \ell_\perp}{\delta v \rho_i} \right)^2 \frac{\Phi_{\rho_i}}{\Phi_\ell} \quad \frac{\delta v}{v_{th}} \sim \frac{\ell_\perp}{\rho_i} \]

- Carrying out inertial range analysis (as before, but with \( J_0(k_\perp \rho_i) \sim (k_\perp \rho_i)^{-1/2} \)) gives*

\[ \Phi_\ell \sim \left( \frac{\ell_\perp}{\rho_i} \right)^{7/6} \Rightarrow (k_\perp \rho_i)_c \sim D_0^{3/5} \]

\[ (k_\perp \rho_i)_c \sim q^{1/5} \left( \frac{R}{L_T} \right)^{4/5} \left( \frac{v_{th}}{\nu_i R} \right)^{3/5} \]

*Schekochihin et al., PPCF 2008
Application: intrinsic rotation

- Significant ‘intrinsic’ rotation observed in experiments with no obvious momentum injection
- Rotation profiles depend on heating mechanism and can reverse sign – can’t be explained solely by ‘pinch’ from edge of plasma
- Lowest-order GK equation gives no momentum flux for up-down symmetric plasma without flow
- Must include higher-order terms to calculate flux generating intrinsic rotation
- Horrible mess – best to simplify…
Higher-order GK equation

Using our scalings + \( B_p/B << 1 \) and \( \nu \tau_{nl} << 1 \):

\[
\frac{dg_s}{dt} + \mathbf{v}_\parallel \cdot \nabla \left( g_s - Z_s e \langle \varphi \rangle \frac{\partial F_{0s}}{\partial E} \right) + \langle \mathbf{v}_E^\perp \rangle \cdot \nabla F_{0s} \\
+ (\mathbf{v}_{Cs} + \mathbf{v}_{Ms} + \langle \mathbf{v}_E^\perp \rangle) \cdot \nabla_\perp \left( g_s - Z_s e \langle \varphi \rangle \frac{\partial F_{0s}}{\partial E} \right) \\
= - \mathbf{v}_{Ms} \cdot \nabla \theta \frac{\partial}{\partial \theta} \left( g_s - Z_s e \langle \Phi \rangle \frac{\partial F_{0s}}{\partial E} \right) - \langle \mathbf{v}_E^\parallel \rangle \cdot \nabla_\perp g_s \\
- \langle \mathbf{v}_E^\perp \rangle \cdot \nabla \theta \frac{\partial g_s}{\partial \theta} - \langle \mathbf{v}_E^\parallel \rangle \cdot \nabla F_{0s} - \langle \mathbf{v}_E^\perp \rangle \cdot \nabla F_{1s} \\
+ Z_s e \left( \mathbf{v}_\parallel \cdot \nabla \langle \Phi \rangle + \mathbf{v}_{Ms} \cdot \nabla_\perp \langle \Phi \rangle \right) \left( \frac{\partial g_s}{\partial E} + \frac{\partial F_{1s}}{\partial E} \right) \\
+ \psi\text{-profile variation} \quad (R,E,\mu) \text{ variables}
\]
Neglecting neoclassical contributions, momentum flux given by

\[ \Pi = \Pi_{-1}^{tb} + \Pi_0^{tb} + \frac{m_s c}{2 Z_s e} \left\langle R^2 \right\rangle_{\psi} \frac{\partial p_s}{\partial t} \]

\[ \Pi_{-1}^{tb} = \frac{1}{\left\langle ||\nabla \psi|| \right\rangle_{\psi}} \left\langle \int d^3 v \ m_s R^2 v \cdot \nabla \phi \left( \frac{v}{E} \cdot \nabla \psi \right) \delta f(R) \right\rangle_{\psi} \]

\[ \Pi_0^{tb} = \frac{m_s^2 c^2}{2 Z_s e} \frac{1}{V' \left\langle ||\nabla \psi|| \right\rangle_{\psi}} \frac{\partial}{\partial \psi} \left( V' \left\langle \hat{b} \times \nabla \varphi \cdot \nabla \psi \frac{I^2}{B^3} \int \, d^3 v \ v_{\parallel}^2 \delta f \right\rangle_{\psi} \right) \]

\[ - \frac{1}{\left\langle ||\nabla \psi|| \right\rangle_{\psi}} \frac{m_s^2 c}{2 Z_s e} \left\langle \frac{I^2}{B^2} \int \, d^3 v \ C_{ii}[v_{\parallel}^2] F_{2}^{tb} \right\rangle_{\psi} \]
Preliminary results

JET shot 19649 (L-mode) at $\rho = 0.16$, no equilibrium flow

Small momentum flux generated by including neoclassical correction to $F_0$ in GK equation:

Momentum flux ($\Pi^{tb}_{-1}$)

Fluxes
Symmetry breaking

JET shot 19649 (L-mode) at $\rho=0.16$, no equilibrium flow

$k_\psi$ symmetry broken

**No low-flow**

**low-flow**
Radial momentum flux profiles

- Momentum flux sign reversal allows for both co- and counter-rotating regions of plasma
- Different flux contributions of same order and possibly different sign

JET shot 19649
Conclusions

• Simple scalings for turbulence spatial scales and amplitudes derived and numerically confirmed. Predictions for scalings of:
  – Turbulence amplitude, heat flux, peak space scale, spectrum, decorrelation time, cutoff scale
• Critical balance robustly satisfied – plasma turbulence three dimensional
• Scalings allow for B/Bp expansion of higher order GK Eq., making it tractable to solve numerically. This allows us to address problem of intrinsic rotation.