Section S18 ADVANCED QUANTUM MECHANICS

1. Using Fourier transform or any other method, show that the Green's function $G(x,x')$ obeying the equation $\hat{L}\,G(x,x') = \delta(x-x')$ and the boundary conditions $G(x,x') \to C\,\exp(ik|x|)$ for $|x| \to \infty$, where $\hat{L} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E$ is the one-dimensional Schrödinger operator for a free particle with $E > 0$, C is a constant and $k = \sqrt{2mE}/\hbar$, is given by $G(x,x') = \frac{im}{k\hbar^2}\exp(ik|x-x'|)$. [5]

A non-relativistic quantum particle of mass m is incident from the left on the one-dimensional potential $U(x) \leq 0$, where $U(x) \to 0$ for $|x| \to \infty$. Show that the wave function of a stationary scattering state of the particle satisfies the integral equation

$$\psi_s(x) = e^{ikx} - \frac{im}{k\hbar^2}\int_{-\infty}^{\infty} e^{ik|x-x'|}U(x')\psi_s(x')dx'\,.$$

[3]

Considering the asymptotics of $\psi_s(x)$ at $x \to \pm\infty$, define the reflection and transmission coefficients T and R. Show that the reflection coefficient is given by

$$R = \frac{m^2}{k^2\hbar^4}\left|\int_{-\infty}^{\infty} e^{ikx}U(x)\psi_s(x)dx\right|^2\,,$$

where $\psi_s(x)$ is the solution of the integral equation above. [4]

Find the solution of the integral equation for the potential $U(x) = -\alpha\delta(x)$, $\alpha > 0$. Find the transmission and reflection coefficients T and R, and show that $R + T = 1$. [3]

Consider developing a perturbation theory for $\psi_s(x)$ in the form $\psi_s(x) = \psi_s^{(0)}(x) + \psi_s^{(1)}(x) + \ldots$, where $\psi_s^{(0)}(x) = e^{ikx}$ is the solution of the free equation. Find R to leading order in the perturbation theory and compute it for the potential $U(x) = -\alpha\delta(x)$. Compare with the exact solution. Explain the physical meaning of the approximation made. [3]

Now consider stationary states with $E < 0$. Show that the wave function $\psi(x)$ for such states obeying the boundary conditions $\psi(x) \to 0$ for $x \to \pm\infty$ satisfies the integral equation

$$\psi(x) = -\frac{m}{\kappa\hbar^2}\int_{-\infty}^{\infty} e^{-\kappa|x-x'|}U(x')\psi(x')dx'\,,$$

[4]

where $\kappa = \sqrt{-2mE}/\hbar$. Using the equation, find the energy levels in the potential $U(x) = -\alpha\delta(x)$, $\alpha > 0$. How are they related to the singularities of the transmission coefficient in the same potential? [3]

2. Consider a non-relativistic quantum particle of mass m whose wavefunction $\psi(x)$ satisfies the one-dimensional Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + U(x)\psi(x) = E\psi(x),$$

where the potential $U(x) \to 0$ for $|x| \to \infty$. Find the Green's function $G(x, x')$ of the free Schrödinger operator $\hat{L}\,G(x, x') = \delta(x - x')$, where $\hat{L} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E$ with $E < 0$. [3]

Show that the Schrödinger equation for the particle with $E < 0$ can be written as an integral equation

$$\psi(x) = -\frac{m}{\kappa\hbar^2}\int_{-\infty}^{\infty} e^{-\kappa|x-x'|}U(x')\psi(x')dx',$$ [2]

where $\kappa = \sqrt{-2mE}/\hbar$.

Consider a potential of the form

$$U(x) = \begin{cases} -U_0, & |x| \le \frac{a}{2}, \\ 0, & |x| > \frac{a}{2}, \end{cases}$$

where $U_0 > 0$.

a) Find the even and odd parity wave functions corresponding to the stationary states with $E < 0$ (bound states) in this potential. [4]

b) Using the continuity conditions for the wave function $\psi(x)$ and its derivative $\psi'(x)$ at $x = \pm a/2$, show that the conditions determining the bound state energies in the potential $U(x)$ are given by the equations

$$\begin{cases} \sqrt{\frac{U_0}{|E|} - 1}\tan\left[\frac{\kappa a}{2}\sqrt{\frac{U_0}{|E|} - 1}\right] = 1 & \text{(even parity states)}, \\ \sqrt{\frac{U_0}{|E|} - 1}\cot\left[\frac{\kappa a}{2}\sqrt{\frac{U_0}{|E|} - 1}\right] = -1 & \text{(odd parity states)}. \end{cases}$$ [4]

c) Now consider a non-relativistic quantum particle of mass m and wavenumber k incident from the left on the potential $U(x)$. Show that the transmission coefficient $T(E)$, where $E = \hbar^2k^2/2m > 0$, is given by

$$T(E) = \left[\cos^2\zeta a + \frac{(k^2 + \zeta^2)^2}{4k^2\zeta^2}\sin^2\zeta a\right]^{-1},$$

where $\zeta^2 = 2m(E + U_0)/\hbar^2$. [4]

d) Show that the singularities of the transmission coefficient $T(E)$ for complex values of $k = i\sqrt{2m|E|}/\hbar$ correspond to the bound states energies in the potential $U(x)$. *Hint: Use the trigonometric identity* $\tan 2x = 2\tan x/(1 - \tan^2 x)$. [4]

e) What is the maximum value of $T(E)$? At what energies is it attained? Sketch (qualitatively) the function $T(E)$ for $E \ge 0$. [2]

f) Sketch (qualitatively) the location of the singularities of the transmission coefficient $T(E)$ in the complex k plane and in the complex E plane. [2]

3. The Dirac equation in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ is

$$\left[\gamma^\mu \left(\hat{p}_\mu - \frac{e}{c} A_\mu\right) - mc\right] \psi = 0\,,$$

where $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ is the four-component Dirac spinor. The Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, $\hat{p}_\mu = i\hbar\partial_\mu$, and the Dirac matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

where σ^k are Pauli matrices obeying $\sigma_i \sigma_k = \delta_{ik} + i\epsilon_{ikl}\sigma_l$. Consider a relativistic particle with spin 1/2 and charge e moving in a constant magnetic field $\mathbf{B} = \mathrm{curl}\,\mathbf{A} = (0, 0, B)$. Choose the gauge $\mathbf{A} = (-By, 0, 0)$, write down the Dirac Hamiltonian explicitly, then justify and use the *ansatz*

$$\psi = e^{-i\frac{E}{\hbar}t + i\frac{p_x}{\hbar}x + i\frac{p_z}{\hbar}z} \begin{pmatrix} \varphi(y) \\ \chi(y) \end{pmatrix},$$

to write the Dirac equation as a system of coupled equations for the spinors φ and χ. [4]

Show that the spinor φ satisfies the equation

$$\left(\frac{d^2}{dy^2} - \frac{(p_x c + eBy)^2}{\hbar^2 c^2} + \frac{E^2 - m^2 c^4}{\hbar^2 c^2} - \frac{p_z^2}{\hbar^2} + \frac{eB}{\hbar c}\sigma_3\right) \varphi(y) = 0\,.$$

[5]

Show that this equation is identical to the Schrödinger equation for a harmonic oscillator,

$$\left(-\frac{\hbar^2}{2M}\frac{d^2}{d\xi^2} + \frac{M\omega^2\xi^2}{2}\right)\Psi(\xi) = \varepsilon\Psi(\xi)\,,$$

and identify the parameters M and ε. *Hint: change variable to* $\xi = (p_x c + eBy)/\hbar c$. [5]

Using the spectrum of the harmonic oscillator, $\varepsilon = \hbar\omega(n + 1/2)$, and the property $\sigma_3\varphi = \pm\varphi$, show that the energy levels E_{n,p_z} of the particle in a constant magnetic field satisfy the equation ($\mu = |e|\hbar/2mc$):

$$E^2_{n,p_z} = m^2c^4 + 2mc^2\left[\frac{p_z^2}{2m} + \mu B(2n+1) \pm \mu B\right], \qquad n = 0, 1, 2, \ldots.$$

[4]

A spinless relativistic particle of mass m and charge e in an external constant magnetic field $\mathbf{B} = (0, 0, B)$ obeys the stationary Klein-Gordon equation

$$\left[c^2\left(-i\hbar\nabla - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4\right]\phi(x, y, z) = E^2\phi(x, y, z)\,.$$

By reducing this equation to the Schrödinger equation for the harmonic oscillator, show that the energy levels of the particle satisfy

$$E^2_{n,p_z} = m^2c^4 + 2mc^2\left[\frac{p_z^2}{2m} + \mu B(2n+1)\right], \qquad n = 0, 1, 2, \ldots.$$

[5]

Based on these results, can you guess a formula for the energy levels of a relativistic particle of spin s in a constant magnetic field? Discuss the instability occuring for a spin 1 particle in the field $eB > m^2c^4/\hbar c$ at $p_z = 0$. [2]