Section S18 ADVANCED QUANTUM MECHANICS

1. A non-relativistic quantum particle of mass m is moving in the one-dimensional potential $U(x) = -\alpha\,[\delta(x-a)+\delta(x)+\delta(x+a)]$, where $\alpha > 0$, $a > 0$. Using the Green's function of the Schrödinger operator for a free particle with $E < 0$, $G(x,x') = \frac{m}{\kappa\hbar^2}\exp\left(-\kappa|x-x'|\right)$, where $\kappa = \sqrt{-2mE}/\hbar$, show that the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + U(x)\psi(x) = E\psi(x)$$

for the particle in the potential $U(x)$ with $E < 0$ and the wave function boundary conditions $\psi(x) \to 0$ for $x \to \pm\infty$, corresponding to bound states in the potential $U(x)$, can be written as an integral equation

$$\psi(x) = -\frac{m}{\kappa\hbar^2}\int_{-\infty}^{\infty} e^{-\kappa|x-x'|}U(x')\psi(x')dx'\,.$$

[3]

Using this integral equation and the explicit form of the potential, write down the solution $\psi(x)$ and the set of three algebraic equations determining $\psi(a)$, $\psi(0)$ and $\psi(-a)$. Express these equations in matrix form. [3]

Show that the bound state energies are determined by the equation

$$1 + \eta\lambda - 2e^{2\eta}(1-\eta\lambda) + e^{4\eta}(1-\eta\lambda)^3 = 0\,,$$

where $\eta = \kappa a$, $\lambda = \hbar^2/\alpha m a$. [3]

Analyse the equation for bound states energies in the limit of small and large λ. How many energy levels are there for $\lambda \ll 1$? Find their approximate values in terms of λ and κ. [3]

Find the energy levels and their degeneracy for $\lambda \gg 1$. [2]

Now consider stationary states of the continuous spectrum with $E > 0$ in the same potential $U(x)$. For a particle incident on the potential from the negative x direction with momentum k and obeying the integral equation

$$\psi_s(x) = e^{ikx} - \frac{im}{k\hbar^2}\int_{-\infty}^{\infty} e^{ik|x-x'|}U(x')\psi_s(x')dx'$$

find the solution $\psi_s(x)$ corresponding to stationary scattering states. [2]

By analyzing the asymptotics of the solution for $x \to +\infty$ and writing it in the form $S(k)e^{ikx}$, find the scattering amplitude $S(k)$ in terms of $\psi_s(a)$, $\psi_s(0)$ and $\psi_s(-a)$. [2]

Find the system of three algebraic equations determining $\psi_s(a)$, $\psi_s(0)$ and $\psi_s(-a)$ and write them in matrix form. Use variables λ and $\xi = ka$. [2]

Show that the singularities of $S(k)$ on the imaginary axis of ξ (i.e. for $\xi = i\eta$) are determined by an algebraic equation identical to the one determining the bound states energy levels in the potential $U(x)$. [4]

Do you expect $S(k)$ to have other singularities? If yes, explain their physical significance. [1]

2. A spinless free relativistic particle of mass m obeys the Klein - Gordon equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right]\psi = 0\,,$$

where Δ is the Laplacian in three space dimensions. What is the connection between this equation and the dispersion relation $E^2 = p^2c^2 + m^2c^4$ for the relativistic particle? [2]

A spinless relativistic particle of mass m in an external scalar potential $U(\mathbf{r}, t)$ obeys the equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2} + \alpha\, U(\mathbf{r}, t)\right]\psi = 0\,.$$

By considering the non-relativistic limit of the associated dispersion relation, show that $\alpha = 2m/\hbar^2$. [2]

A spinless relativistic particle of mass m and charge e in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ obeys the equation

$$\left[c^2\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 - \left(i\hbar\frac{\partial}{\partial t} - e\Phi\right)^2 + m^2c^4\right]\psi = 0\,,$$

where $\hat{\mathbf{p}} = -i\hbar\nabla$.

Show that the current density

$$j_\mu = -\frac{i}{2}\left(\psi\partial_\mu\psi^* - \psi^*\partial_\mu\psi\right) - \frac{e}{\hbar c}A_\mu\psi^*\psi\,,$$

where ψ is a solution of the Klein-Gordon equation, satisfies the continuity equation $\partial_\mu j^\mu = 0$. [6]

For a time-independent electromagnetic field, consider solutions of the form $\psi(t, \mathbf{r}) = e^{-i(mc^2+E)t/\hbar}\varphi(\mathbf{r})$. Show that the stationary Klein-Gordon equation obeyed by $\varphi(\mathbf{r})$ is

$$\left[c^2\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4\right]\varphi = \left(mc^2 + E - e\Phi\right)^2\varphi\,.$$

[2]

For a hydrogen-like atom $e\Phi = -Ze^2/r$, $\mathbf{A} = 0$. Show that the radial part $R(r)$ of the wave function $\varphi = R(r)Y_{lm}$, where $Y_{lm}(\theta, \phi)$ are the standard spherical harmonics, obeys the equation [*Hint: You may use the decomposition* $\Delta\varphi = \frac{1}{r}\frac{d^2(rR)}{dr^2}Y_{lm} - \frac{l(l+1)}{r^2}RY_{lm}$]

$$\frac{1}{r}\frac{d^2(rR)}{dr^2} - V_{eff}R(r) = A\,R(r)\,,$$

where $V_{eff} = \frac{l(l+1) - \alpha_{em}^2 Z^2}{r^2} - \frac{2B}{r}$ and $\alpha_{em} = e^2/\hbar c$. [5]

Find the constants A, B and discuss their non-relativistic limit. [3]

Sketch the effective potential V_{eff}. Explain qualitatively whether you expect bound states to exist for any value of Z. [3]

Do you expect the energy spectrum derived from this equation to be in agreement with experiment? Explain. [2]

3. Consider the following candidate for a wave equation describing a relativistic particle of mass m

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\left(\alpha_1\frac{\partial\psi}{\partial x^1} + \alpha_2\frac{\partial\psi}{\partial x^2} + \alpha_3\frac{\partial\psi}{\partial x^3}\right) + \beta mc^2\psi \equiv H_D\psi\,.$$

Can the coefficients α_i, $i = 1,2,3$ and β be real numbers? Explain. [1]

Assuming that ψ has N components ψ_α, $\alpha = 1,...N$, show that each component satisfies the Klein-Gordon equation,

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2}\right]\psi_\alpha = 0\,,$$

provided that $\alpha_i^2 = I$, $\beta^2 = I$, $\alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}$ and $\alpha_i\beta + \beta\alpha_i = 0$, where I is the identity matrix. [4]

Show that the eigenvalues of α_i and β are equal to ±1. [2]

Show that $\mathrm{Tr}\,\alpha_i = 0$ and $\mathrm{Tr}\,\beta = 0$. [2]

Show that N must be an even number greater than 2. [1]

For $N = 4$, introducing $\psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*,)$, where ψ_α^* is the component complex conjugate to ψ_α, using the wave equation for ψ_α and its Hermitian conjugate, show that the continuity equation holds,

$$\frac{\partial\rho}{\partial t} + div\,\mathbf{j} = 0\,,$$

where $\rho = \psi^\dagger\psi$, $j_k = c\psi^\dagger\alpha_k\psi$. [5]

The Dirac equation in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ is

$$\left[\gamma^\mu\left(p_\mu - \frac{e}{c}A_\mu\right) - mc\right]\psi = 0\,,$$

where $\psi = \begin{pmatrix}\varphi\\ \chi\end{pmatrix}$ is the four-component Dirac spinor. The Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$, $p_\mu = i\hbar\partial_\mu$, and the Dirac matrices are

$$\gamma^0 = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix}, \quad \gamma^k = \begin{pmatrix} 0 & \sigma^k\\ -\sigma^k & 0\end{pmatrix},$$

where σ^k are Pauli matrices obeying $\sigma_i\sigma_k = \delta_{ik} + i\epsilon_{ikl}\sigma_l$. For time-independent external fields, consider stationary solutions of the form $\psi \sim \exp\left[-i(mc^2+E)t/\hbar\right]$. Write down the system of coupled equations for the spinors φ and χ. [3]

Show that in the non-relativistic limit $|E| \ll mc^2$, $|e\Phi| \ll mc^2$, the spinor φ obeys the Pauli equation

$$i\hbar\frac{\partial\varphi}{\partial t} = \left[\frac{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}{2m} + e\Phi - \mu_0\sigma\cdot B\right]\varphi$$

and find the value of the magnetic moment μ_0. [6]

Does the Dirac equation uniquely predict the value of μ_0? Explain. [1]