Section S18 ADVANCED QUANTUM MECHANICS

1. A non-relativistic quantum particle of mass m is moving in the one-dimensional potential

$$U(x) = \begin{cases} -q\,\delta(x-a) & \text{for} \quad x > 0, \\ \infty \quad \text{for} & x < 0, \end{cases}$$

where $q > 0$. Show that the Green's function of a Schrödinger operator for a free particle with $E < 0$ obeying the equation

$$\left(\hat{H} - E\right) G(x, x') = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} G(x, x') - EG(x, x') = \delta(x - x')$$

with the boundary conditions $G(x, x') \to 0$ for $x - x' \to \infty$ and $G(0, x') = 0$ is given by

$$G(x, x') = \begin{cases} \frac{m}{\kappa\hbar^2}\left[e^{\kappa(x-x')} - e^{-\kappa(x+x')}\right], & x < x', \\ \frac{m}{\kappa\hbar^2}\left[1 - e^{-2\kappa x'}\right] e^{-\kappa(x-x')}, & x > x', \end{cases}$$

where $\kappa = \sqrt{-2mE}/\hbar$. [4]

Using the Green's function $G(x, x')$, show that the Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x^2} + U(x)\psi(x) = E\psi(x)$$

for the particle in the potential $U(x)$ with $E < 0$ and the wave function boundary conditions $\psi(x) \to 0$ for $x \to \infty$ and $\psi(0) = 0$, corresponding to the bound states in the potential $U(x)$, can be written as an integral equation. [4]

Show that the bound state energies are determined by the equation $G(a, a) = 1/q$. [4]

Now consider stationary states of the continuous spectrum with $E > 0$ in the same potential $U(x)$. For a particle incident on the potential from the positive x direction and described by the wave function

$$\psi(x) = \begin{cases} A\left(e^{ikx} - e^{-ikx}\right), & 0 < x < a, \\ e^{-ikx} + Be^{ikx}, & x > a, \end{cases}$$

where $k = \sqrt{2mE}/\hbar$, show that the amplitude A is given by

$$A = -\frac{i\lambda\bar{k}}{1 - e^{2i\bar{k}} + i\lambda\bar{k}},$$

where $\bar{k} = ka$, $\lambda = \hbar^2/mqa$. [4]

Find the equation determining the singularities of the amplitude A on the positive imaginary axis of complex $\bar{k}$ and show that it coincides with the equation determining the bound state energies in the potential $U(x)$. [4]

Show that for $\lambda \ll 1$ the amplitude A has poles (singularities) at

$$ka = n\pi\left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{4} + \cdots\right) - i\left(\frac{\lambda n\pi}{2}\right)^2 + \cdots,$$

where $n = \pm 1, \pm 2, ...$ Sketch the corresponding transmission coefficient $|A|^2$ as a function of (real) ka and explain the main features of the sketch. [5]

2. A non-relativistic quantum particle of mass m is scattered by the three-dimensional central potential $U(r) = \alpha/r^2$, $\alpha > 0$. Show that the Schrödinger equation for the wave function $\psi(r,\theta,\varphi) = R(r)Y_{lm}(\theta,\varphi)$,

$$\frac{\hbar^2}{2m}\left[-\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{\hat{L}^2}{r^2}\psi\right]+U(r)\psi = E\psi\,,$$

can be written as the Bessel equation

$$x^2y''(x)+xy'(x)+\left(x^2-\nu^2\right)y(x)=0$$

for $y = r^{1/2}R(r)$ and $x = kr$, where $k = \sqrt{2mE}/\hbar$, $E > 0$, and find ν as a function of l and α. [5]

A solution to the Bessel equation is written in the form $y(x) = AJ_\nu(x) + BY_\nu(x)$, where A and B are constants and the Bessel functions $J_\nu(x)$ and $Y_\nu(x)$ are known to have the following behavior at small and large values of x: $J_\nu \sim x^\nu$, $Y_\nu \sim x^{-\nu}$ at $x \to 0$ and $J_\nu \sim \sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{\pi\nu}{2}+\frac{\pi}{4}\right)$, $Y_\nu \sim \sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{\pi\nu}{2}-\frac{\pi}{4}\right)$ at $x \to \infty$.

By considering the boundary condition at the origin, argue that we must set $B = 0$. [4]

Comparing the phase of the sine function in the solution at $r \to \infty$ to the phase of the free particle ($U(r) = 0$), show that the phase shift δ_l is given by

$$\delta_l = -\frac{\pi}{2}\left[\sqrt{(l+1/2)^2+\frac{2m\alpha}{\hbar^2}}-(l+1/2)\right].$$

[5]

Show that for $m\alpha/\hbar^2 \ll 1$, $\delta_l \approx -\pi m\alpha/(2l+1)\hbar^2$, $|\delta_l| \ll 1$, and the scattering amplitude

$$f(k,\theta) = \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)\left(e^{2i\delta_l}-1\right)P_l(\cos\theta)$$

is approximately given by $f(k,\theta) \approx -\pi m\alpha/2\hbar^2 k\sin(\theta/2)$. Compute the corresponding differential cross-section $d\sigma/\mathrm{d}\Omega$ as a function of energy E. *Hint: Expand* $e^{2i\delta_l}$ *and use the Legendre polynomials formula* $(1-2xz+x^2)^{-1/2} = \sum_{l=0}^{\infty} x^l P_l(z)$. [6]

Show that the result $f(k,\theta) \approx -\pi m\alpha/2\hbar^2 k\sin(\theta/2)$ coincides with the one obtained in the first Born approximation for the potential $U(r)$

$$f^{(1)}(k,\theta) = -\frac{2m}{\hbar^2}\int_0^\infty \frac{r'\sin qr'}{q}U(r')dr'\,,$$

where $q = 2k\sin(\theta/2)$. Discuss the validity condition of the Born approximation. [5]

3. A spinless relativistic particle of mass m and charge $e > 0$ in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ obeys the Klein-Gordon equation

$$\left[c^2\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 - \left(i\hbar\frac{\partial}{\partial t} - e\Phi\right)^2 + m^2c^4\right]\psi = 0\,,$$

where $\hat{\mathbf{p}} = -i\hbar\nabla$.

Show that the current density

$$j_\mu = -\frac{i}{2}\left(\psi\partial_\mu\psi^* - \psi^*\partial_\mu\psi\right) - \frac{e}{\hbar c}A_\mu\psi^*\psi\,,$$

where ψ is a solution of the Klein-Gordon equation, satisfies the continuity equation $\partial_\mu j^\mu = 0$. [4]

For the time-independent electromagnetic field, consider solutions of the form $\psi(t, \mathbf{r}) = e^{-i\varepsilon t/\hbar}\varphi(\mathbf{r})$. Show that the stationary Klein-Gordon equation obeyed by $\varphi(\mathbf{r})$ is

$$\left[c^2\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4\right]\varphi = \left(\varepsilon - e\Phi\right)^2\varphi\,.$$

[4]

Introducing $E = \varepsilon - mc^2$, show that in the non-relativistic limit $|E| \ll mc^2$, $|e\Phi| \ll mc^2$, the stationary Klein-Gordon equation reduces to the Schrödinger equation

$$\left[\frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + e\Phi\right]\varphi_0 = E\varphi_0\,,$$

where $\varphi = \varphi_0 + \varphi_1$, with φ_1 denoting a relativistic correction to the solution φ_0 of the Schrödinger equation, $|\varphi_1| \ll |\varphi_0|$. [4]

By writing the stationary Klein-Gordon equation in the form

$$\left[\frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 + e\Phi - E\right]\varphi = \frac{(E - e\Phi)^2}{2mc^2}\varphi\,,$$

and considering the right hand side as a small perturbation in the non-relativistic limit with $\varphi = \varphi_0 + \varphi_1$, show that the Schrödinger equation with the first ($\sim 1/c^2$) relativistic correction is

$$\left[\frac{1}{2m}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^2 - \frac{1}{8m^3c^2}\left(\hat{\mathbf{p}} - \frac{e}{c}\mathbf{A}\right)^4 + e\Phi\right]\varphi = E\varphi\,.$$

[8]

Show that the form of the relativistic correction coincides with the one obtained from the classical Hamiltonian $H = \sqrt{c^2\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4} + e\Phi - mc^2$ after the standard quantum-mechanical substitutions. Do you expect this to happen for the relativistic corrections of higher order in $1/c^2$? [5]