Section S18 ADVANCED QUANTUM MECHANICS

1. A non-relativistic quantum particle of mass m and wavenumber k is incident from the negative x direction on the one-dimensional potential well

$$U(x) = \begin{cases} -U_0, & |x| \le \frac{a}{2} \,, \\ 0, & |x| > \frac{a}{2} \,. \end{cases}$$

In the region $x > a/2$, the particle is described by the wave function $\psi(x) = S(E)e^{ik(x-a)}$, where $E = \hbar^2 k^2/2m$.

Show that the transmission amplitude $S(E)$ is given by

$$S(E) = \frac{k\kappa}{k\kappa\cos\kappa a - \frac{i}{2}\,(k^2+\kappa^2)\sin\kappa a}\,,$$

where $\kappa = \sqrt{2m(E+|U_0|)}/\hbar$. [6]

Find the transmission probability $T(E)$. [4]

Show that the transmission amplitude has singularities (zeros of the denominator of $S(E)$) in the complex k plane determined by the equations

$$\tan\frac{\kappa a}{2} = -\frac{ik}{\kappa}\,, \qquad \cot\frac{\kappa a}{2} = \frac{ik}{\kappa}\,.$$

[5]

Find the even and odd parity wave functions corresponding to the stationary states with $E < 0$ (bound states) in the potential well. [4]

Find equations determining the bound state energies in the potential well and show that these energies coincide with the singularities of the transmission amplitude. [4]

Find the energies E for which the transmission probability $T(E)$ reaches its maximum value. Sketch the function $T(E)$ for $E \ge 0$.

Treating E formally as a complex variable, sketch (qualitatively) the location of the poles of the transmission amplitude in the complex k plane and the complex E plane. [2]

2. A spinless relativistic particle of mass m in an external scalar field $\Phi(\mathbf{r},t)$ obeys the equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2} + \frac{2m}{\hbar^2}\Phi(\mathbf{r},t)\right]\psi = 0\,.$$

Show that in the non-relativistic limit $\Phi(\mathbf{r},t)$ has the meaning of the usual potential energy. [5]

Show that the wave function $R(r) = r\psi(r)$ of the s-wave spinless particle in the external field

$$\Phi(r) = \begin{cases} -U_0, & r \le a\,, \\ 0, & r > a \end{cases}$$

satisfying the boundary condition $R(0) = 0$ is

$$R(r) = \begin{cases} A\sin\left(r\sqrt{\frac{2mU_0}{\hbar^2} - \kappa^2}\right), & r \le a\,, \\ B\,e^{-\kappa r}, & r > a\,, \end{cases}$$

where $\kappa = \sqrt{m^2c^4 - \epsilon^2}/\hbar c > 0$. [Hint: You may use $\Delta\psi = \frac{1}{r}\frac{d^2(r\psi)}{dr^2}$.] [5]

Show that the discrete energy spectrum ϵ_n is determined by the equation

$$\tan\sqrt{\frac{2mU_0a^2}{\hbar^2} - \kappa_n^2a^2} = -\frac{1}{\kappa_n a}\sqrt{\frac{2mU_0a^2}{\hbar^2} - \kappa_n^2a^2}\,,$$

where $\kappa_n = \sqrt{m^2c^4 - \epsilon_n^2}/\hbar c$.

What is the spectrum of an antiparticle in this field? [6]

Find the algebraic equation determining the critical value $U_{0,crit}$ of the external field corresponding to $\epsilon_{n=0} = 0$. What physical processes one may expect to occur for external fields exceeding the critical value? Is the one-particle equation adequate in this case? Explain. [5]

A spinless relativistic particle of mass m and charge e in an external electrostatic field ϕ obeys the equation

$$\left(-\hbar^2c^2\Delta + m^2c^4\right)\psi = \left(\mathrm{i}\hbar\partial_t - e\phi\right)^2\psi\,.$$

For energies close to the rest energy, i.e. for $\epsilon = mc^2 + E$, where $|E| \ll mc^2$, show that in sufficiently strong fields, the force experienced by the particle is attractive irrespective of the sign of the particle's charge. [Hint: Reduce the equation to the Schrödinger equation with the appropriate effective potential.] [4]

3. The Dirac equation in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ is

$$\left[\gamma^\mu \left(p_\mu - \frac{e}{c} A_\mu\right) - mc\right] \psi = 0\,,$$

where $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ is the four-component Dirac spinor. The Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$, $p_\mu = \mathrm{i}\hbar\partial_\mu$, and the Dirac matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

where I is the identity matrix, and σ^k are the Pauli matrices obeying $\sigma_i\sigma_k = \delta_{ik} + \mathrm{i}\epsilon_{ikl}\sigma_l$.

Assuming the external field is time-independent, consider stationary solutions of the Dirac equation with the time dependence of the form $\psi \sim \exp(-\mathrm{i}\epsilon t/\hbar)$.

Write down the system of coupled equations for the two-component spinors φ and χ. [6]

Consider the positive energy solution with $\epsilon = mc^2 + E$. Show that in the non-relativistic limit, where $|E| \ll mc^2$, $|e\Phi| \ll mc^2$, the spinor φ obeys the Pauli equation

$$\mathrm{i}\hbar\frac{\partial\varphi}{\partial t} = \left[\frac{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}{2m} + e\Phi - \mu\boldsymbol{\sigma}\cdot\mathbf{B}\right]\varphi$$

and find the value of the magnetic moment μ. [8]

Is the value of μ universal for all charged particles with spin 1/2?

Do you expect the theoretical prediction for μ following from the Dirac equation to be exact? Explain.

Is the value of the magnetic moment fixed uniquely by the Dirac equation? [Hint: Consider a non-minimal coupling to an electromagnetic field.] [3]

Consider further the case of $\mathbf{A} = 0$, and let $e\Phi = U(r)$. By expanding the Dirac equation to the next order in $|E|/mc^2 \ll 1$, $|U|/mc^2 \ll 1$, show that the spinor φ obeys the equation

$$\mathrm{i}\hbar\frac{\partial\varphi}{\partial t} = \left(\frac{\mathbf{p}^2}{2m} + U(r) + H_1\right)\varphi\,,$$

where the perturbation operator is given by

$$H_1 = -\frac{\mathbf{p}^4}{8m^3c^2} + \frac{1}{2m^2c^2}\frac{1}{r}\frac{dU(r)}{dr}\,\mathbf{L}\cdot\mathbf{S} - \frac{\hbar^2}{4m^2c^2}\frac{dU}{dr}\frac{d}{dr}\,.$$

What is the physical meaning of terms in H_1? [8]

[You may use the following identity without proof:

$$(\boldsymbol{\sigma}\cdot\mathbf{p})f(\boldsymbol{\sigma}\cdot\mathbf{p}) = f\mathbf{p}^2 - \hbar^2(\partial_i f)\partial_i - \mathrm{i}\hbar^2\sigma^i\epsilon_{ijk}(\partial_j f)\partial_k\,.\quad]$$