2S98/2S99

FIRST PUBLIC EXAMINATION

Preliminary Examination in Physics

SECOND PUBLIC EXAMINATION

Honour School of Physics,
Parts A and B: 3 and 4 year Courses

SHORT OPTIONS

TRINITY TERM 2012

Tuesday, 12 June
9.30 am to 11.00 am for candidates offering ONE Short Option
9.30 am to 12.30 pm for candidates offering TWO Short Options

Answer **two** *questions from each option for which you have entered.*

Start the answer to each question in a fresh book.

If you have entered for two Short Options, keep your answers to the two options in different books and at the end hand in two bundles, one for each option.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

INDEX

Some sections start with a relevant rubric.

Section S1 FUNCTIONS OF A COMPLEX VARIABLE

1. (a) Classify the singularity of the function

$$f(z) = e^{-1/z}$$

at $z = 0$ in the complex plane, and write the first three terms of the Laurent series expansion of $f(z)$ about $z = 0$. [7]

(b) Evaluate the integral of the function $f(z)$ round the circle C of centre $z = 0$ and radius 1:

$$\oint_C f(z)\,\mathrm{d}z \ .$$ [8]

(c) Use the result in part (b) to calculate the following real integrals:

$$\int_0^{2\pi} e^{-\cos\theta} \ \cos(\theta + \sin\theta)\,\mathrm{d}\theta \ , \quad \int_0^{2\pi} e^{-\cos\theta} \ \sin(\theta + \sin\theta)\,\mathrm{d}\theta \ .$$ [10]

2. Consider the mapping specified by the function

$$f \ : \ z \mapsto w = z + \frac{1}{z}$$

from the complex z plane to the complex w plane.

(a) Give the subset of the complex z plane in which f is holomorphic, and determine whether the mapping is conformal in the region of holomorphy. [6]

(b) Determine how the mapping transforms angles at the point $z = 1$. [6]

(c) Determine the image through f of the upper half and the lower half of the circle $|z| = 1$. [6]

(d) Determine the image through f of a circle $|z| = \rho$ with $\rho \neq 1$. [7]

3. (a) Give the location and order of the branch points of the function

$$f(z) = \frac{1}{\sqrt{z}} \ .$$ [5]

(b) Set the branch cut along the negative real semiaxis and take the principal branch of $f(z)$. Consider the integral in the complex plane

$$I = \int_\gamma e^z \ f(z) \, \mathrm{d}z \ ,$$

where γ is the straight line parallel to the imaginary axis with real part equal to 1 (see figure).

Using the closed contour Γ depicted in the figure, show that I must equal the integral over the discontinuity of the integrand across the cut. [10]

(c) Use the result in part (b) to prove that $I = 2i\sqrt{\pi}$. [10]

Section S2 ASTROPHYSICS: FROM PLANETS TO THE COSMOS

1. State Kepler's third law of planetary motion and write it in equation form in either S.I. units or other convenient units which you should define. [5]

An eclipsing spectroscopic binary star has a period of 5.45 days. Radial velocity measurements show the two stars moving with equal and opposite amplitudes of $100\,\mathrm{km\,s^{-1}}$. What is the separation of the stars in Astronomical Units? Calculate the masses of the stars in solar units. [10]

The light curve of the binary star during primary eclipse is shown in the figure, which plots flux in relative units against fractional period varying between 0 and 1. Draw a diagram showing the position of the primary and secondary stars at each of t_1, t_2, t_3, and t_4. Calculate the radii of the two stars. [10]

2. Show that the total potential energy Ω of the Sun satisfies:

$$|\Omega| > \frac{GM_\odot^2}{2R_\odot} ,$$

and thus estimate the lifetime of the Sun if gravitational collapse is the only source of energy. [10]

The reaction which converts hydrogen to helium can be summarised as:

$$4\text{p} \rightarrow \text{He} + 2e^+ + 2\nu_e .$$

Use the difference in mass between four protons and one helium nucleus to calculate the efficiency of this reaction. Assuming the present luminosity, and that the Sun moves away from the Main Sequence when it has converted 10% of its mass to helium, calculate the Main Sequence lifetime of the Sun. [$\text{m}_H = 1.0078$ amu. and $\text{m}_{He} = 4.0026$ amu.] [4]

Draw a Hertzsprung-Russell diagram with labelled axes showing the Main Sequence and position of the Sun. Show on the diagram how the Sun evolves away from the Main Sequence. For each stage state the source of energy. [11]

3. Draw sketch diagrams to indicate the emission from the sky at microwave wavelengths that arises from:

(a) dust in the Milky Way,

(b) the motion of the Milky Way,

(c) the structure of the cosmic microwave background.

How do we determine the temperature of the cosmic microwave background? [8]

One side of the microwave sky is 0.105% hotter and the opposite side 0.105% cooler than the average temperature. Calculate the speed of the Milky Way with respect to the background. Is the Milky way moving towards or away from the hotter region? [4]

What is the temperature of the microwave background seen by an observer in a galaxy at a redshift $z = 3$? [4]

The present mass density of the Universe is $2.5 \times 10^{-27}\,\text{kg}\,\text{m}^{-3}$ and the present day energy density in CMB photons is $5 \times 10^5\ \text{eV}\,\text{m}^{-3}$. Compare the energy density in matter with the energy density in the CMB radiation today, and calculate the redshift at which the two energy densities were equal. [9]

Section S3 QUANTUM IDEAS

1. At the end of the 19th century there were some puzzling phenomena that could not be explained using a continuous wave theory of light. Name **two** and explain how the issues associated with these were resolved using the concept of photons. [5]

A photon has energy $E = 18.8\,\mathrm{keV}$, which is equal to the kinetic energy of a neutron. What are the momenta and wavelengths of the photon and neutron? Why are these wavelengths so different? For the neutron to have a de Broglie wavelength equal to the photon wavelength, what speed should the neutron have? [5]

A beam of neutrons is normally incident on the surface of a cubic crystalline medium with lattice spacing d. Show that neutrons are scattered from the crystal planes parallel to the surface at an angle θ from the normal with the following relation

$$d \sin\theta = n\lambda,$$

where n is an integer and λ is the de Broglie wavelength of the incident neutrons. A beam of neutrons with kinetic energy $E_{\mathrm{K}} = 0.188\,\mathrm{eV}$ is observed to scatter in first order at an angle $\theta = 0.117\,\mathrm{rad}$. What is the lattice spacing for this material? [7]

If the neutron beam with kinetic energy $E_{\mathrm{K}} = 0.188\,\mathrm{eV}$ from above has fractional uncertainty $\Delta E_{\mathrm{K}}/E_{\mathrm{K}} = 0.1$, what is the corresponding uncertainty in the de Broglie wavelength? What is the uncertainty $\Delta\theta$ in the first order scattering angle for the neutron beam? If photons with energy $E = 18.8\,\mathrm{keV}$ are used for Bragg scattering, and have the same fractional energy uncertainty as that of the neutron beam, what is the uncertainty in the first order scattering angle for the photons? Comment on your results. [8]

[Can you make the small angle approximation for θ?]

2. Describe the Franck-Hertz experiment and explain how it provides evidence for discrete atomic energy levels. [5]

Discuss the stability of an electron orbiting a nucleus when treated using classical physics. State the assumptions made by Bohr in his model of ionized hydrogen-like atoms with a single electron of mass m_{e} and charge $-e$ orbiting a nucleus with charge Ze and mass M, that enabled him to resolve this problem. In particular, what constraint do these assumptions impose on the de Broglie wavelength of the electron? [5]

Derive the expressions for the Bohr radius and allowed energy levels for such a hydrogen-like ion using Bohr's assumptions. [4]

Give an expression for the wavelength of a photon emitted when the electron in the ion above makes a transition between the energy levels labelled by n and m $(n > m)$. [3]

The following photon wavelengths are observed in absorption at room temperature from an ionized atomic gas with a single electron orbiting the nucleus: $\lambda = 13.5\,\mathrm{nm}, 11.4\,\mathrm{nm}, 10.8\,\mathrm{nm}$. Use this data to determine the effective Rydberg constant and the nuclear charge. [4]

Given that the ground state energy for a helium ion is $-54.4\,\mathrm{eV}$, estimate the average radius of the electron cloud using the uncertainty principle. Compare this with the value predicted by the Bohr model. [4]

3. Write down the time-independent Schrödinger equation (TISE) for a particle of mass m confined to move in the x-direction in a region with potential $V(x)$. Explain briefly the meaning of each term. [3]

What is the physical interpretation of the wave function $\psi(x)$. Write down general expressions for the expectation values of the particle position $\langle x\rangle$ and its square $\langle x^2\rangle$. [3]

Let $\psi_1(x)$ and $\psi_2(x)$ be two orthonormal solutions of the TISE with corresponding energy eigenvalues E_1 and E_2. At time $t=0$, the particle is prepared in the symmetric superposition state

$$\psi^{(+)}(x) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right),$$

and subsequently allowed to evolve in time. What is the average energy of the system as a function of time? What is the minimum time τ for which the system must evolve in order to return to its original state (up to an overall phase factor), when it starts in the state $\psi^{(+)}(x)$? [4]

Determine the probability to find the system in the antisymmetric superposition state

$$\psi^{(-)}(x) = \frac{1}{\sqrt{2}}\left(\psi_1(x) - \psi_2(x)\right),$$

as a function of time when it starts in the state $\psi^{(+)}(x)$. [5]

At time t_1 the particle is found in the antisymmetric superposition state. What is the probability to find the particle in the symmetric superposition state at time $t_1+\tau$, where τ is the time found above? [3]

The particle is confined by an infinite potential well of width a, with $\psi_1(x)$ and $\psi_2(x)$ the first two eigenstates. The particle is initially prepared in the symmetric superposition state. What is the probability to find the particle in the left half of the potential well as a function of time? Determine the frequency at which this probability oscillates in time in terms of the particle mass and potential width. [7]

[$\int_0^{\pi/2} \sin(x)\sin(2x)\mathrm{d}x = 2/3$, $\int_0^{\pi/2} \sin^2(x)\mathrm{d}x = \pi/4$]

Section S4 ENERGY STUDIES

1. Explain why the efficiency of plants averaged over a year for converting solar energy into biomass is about 0.5%. [7]

Estimate how many tonnes of biomass per square kilometre could be grown per year in Europe. [4]

In parts of South America the yield of sugar from sugarcane is 1500 tonne per square kilometre per year. The sugar is fermented into ethanol via the reaction

$$C_6H_{12}O_6 \longrightarrow 2C_2H_5OH + 2CO_2\ .$$

Calculate the area of sugarcane required to produce sufficient ethanol to displace 10^{10} litres of petrol per year. What would be the resulting reduction in carbon emissions, measured in tonnes of carbon per year? (Assume petrol is pure octane.) [8]

Identify **three** advantages and **three** disadvantages of biomass as an energy source. [6]

[Petrol has a heat of combustion of 43.5 MJ kg^{-1} and a density of 0.73 kg per litre. The heat of combustion of ethanol is 30.5 MJ kg^{-1} and of carbohydrate is 16 MJ kg^{-1}. Octane has a composition of C_8H_{18}.]

2. Describe the Rankine cycle and explain why it avoids the practical problems of using a Carnot cycle engine in a thermal power plant. [8]

The condenser in a Rankine cycle power plant operates at 20° C at a pressure of 0.002 MPa. The compressor raises the pressure of the water to 28 MPa and the boiler and superheater produce steam at 600° C and 28 MPa.

Using the information in the table below calculate

(a) the dryness fraction of the steam after expansion through the turbine,

(b) the efficiency of the cycle. [10]

	T	P	h_f	h_g	s_f	s_g
Water/Steam	20	0.002	84	2538	0.297	8.667
Dry Steam	600	28		3463		6.282

[Here h_f and h_g are the specific enthalpies and s_f and s_g are the specific entropies of the fluid and gas, respectively, in kJ kg^{-1}, T is in ° C and P is in MPa.]

Describe the principal features and challenges for post-combustion carbon capture and storage (CCS). [7]

3. Comment *critically* with numerical estimates, where appropriate, on **three** of the following statements:

(a) The variability in the output of wind farms means that wind can only contribute a very small fraction of a country's electricity demand.

(b) The German reaction to the Fukushima disaster to plan to do without nuclear power was ill-considered.

(c) It is very important that the UK develop tidal and wave power.

(d) Solar power from the Middle East and North Africa (MENA) region is essential if Europe is to significantly reduce its carbon emissions.

(e) A minimum international carbon price is vital for effective carbon emissions reduction.

(f) Plug-in hybrid electric vehicles are an essential part of a solution to global warming. [25]

Section S7 CLASSICAL MECHANICS

1. State Hamilton's "Principle of Least Action" and explain how it leads to the Euler–Lagrange equation of motion. You may assume without proof the fundamental lemma of the calculus of variations. [8]

A particle has Lagrangian

$$L = \frac{1}{2}\dot{R}^2 + \frac{1}{2}R^2\dot{\phi}^2 + \frac{1}{2}\dot{z}^2 - C\dot{z}\log R\,,$$

where (R, ϕ, z) are cylindrical polar coordinates and C is a constant. Write down the Euler–Lagrange equations for the particle. Using these, or otherwise, find three constants of motion. [6]

The particle is launched from $(R, \phi, z) = (a, 0, 0)$ with $(\dot{R}, \dot{\phi}, \dot{z}) = (0, \Omega, a^2\Omega^2/C)$, where Ω is a constant. Describe its subsequent motion. How does the orbit of the particle differ if it is launched with radial velocity $\dot{R} = \delta$ (where $\delta \ll a\Omega$) instead of $\dot{R} = 0$? [8]

What physical system might this Lagrangian represent? [3]

2. The Poisson bracket of two functions $F(\mathbf{q}, \mathbf{p})$, $G(\mathbf{q}, \mathbf{p})$ is defined as

$$\{F, G\} = \sum_i \left[\frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial G}{\partial q_i}\frac{\partial F}{\partial p_i}\right].$$

Show that

$$\{F, G\} = \sum_i \frac{\partial F}{\partial q_i}\{q_i, G\} + \sum_i \frac{\partial F}{\partial p_i}\{p_i, G\}\,.$$

[4]

A mechanical system has Hamiltonian $H(\mathbf{q}, \mathbf{p})$. Write down Hamilton's equations and show that they can be expressed in Poisson bracket form. State what is meant by the term "constant of motion" and show that a function $C(\mathbf{q}, \mathbf{p})$ is a constant of motion if and only if it satisfies $\{C, H\} = 0$. [6]

Consider the Hamiltonian

$$H = \frac{1}{2}p_R^2 + \frac{p_\phi^2}{2R^2} + \frac{A}{R}\,,$$

where the momenta p_R and p_ϕ are conjugate to the coordinates R and ϕ, respectively, and A is a constant. Show that H, p_ϕ and the function

$$C = p_R\, p_\phi \sin\phi + \frac{p_\phi^2}{R}\cos\phi + A\cos\phi$$

are all constants of motion. [10]

Without further calculation, explain why $D = \{C, p_\phi\}$ is also a constant of motion. How many independent constants of motion are there among H, p_ϕ, C and D? Justify your answer. [5]

3. Prove Noether's theorem: if a Lagrangian $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is invariant under the infinitesmal coordinate transformation $\mathbf{q} \to \mathbf{q} + \epsilon\mathbf{K}(\mathbf{q})$, then the quantity $C = (\partial L/\partial\dot{\mathbf{q}}) \cdot \mathbf{K}$ is a constant of motion. [5]

By writing out the Euler–Lagrange equation verify that

$$L = \frac{1}{2}m\dot{\mathbf{x}}^2 + \frac{1}{2}q\dot{\mathbf{x}} \cdot (\mathbf{B} \times \mathbf{x})$$

is a suitable Lagrangian to use for a particle of mass m moving in a uniform magnetic field $\mathbf{B}$. Show that the equations of motion are invariant under linear translations $\mathbf{x} \to \mathbf{x}+\epsilon\hat{\mathbf{n}}$ in arbitrary directions $\hat{\mathbf{n}}$, but that the Lagrangian L is mapped to $L+\epsilon\mathrm{d}F/\mathrm{d}t$ for some function $F(\mathbf{x})$. Identify this function. Explain why the total derivative term $\epsilon\mathrm{d}F/\mathrm{d}t$ has no effect on the equations of motion. [7]

Generalise Noether's theorem to show that if $L(\mathbf{q}, \dot{\mathbf{q}}, t)$ is mapped to $L(\mathbf{q}, \dot{\mathbf{q}}, t) + \epsilon\mathrm{d}F(\mathbf{q})/\mathrm{d}t$ under the infinitesmal coordinate change $\mathbf{q} \to \mathbf{q} + \epsilon\mathbf{K}(\mathbf{q})$, then there is a corresponding constant of motion. [8]

Hence, or otherwise, identify three constants of motion for the Lagrangian L. [5]

Section S9 FINANCIAL PHYSICS

1. In the context of financial markets, explain what is meant by the term *stylized facts*. [4]

Sketch the pay-off diagrams at expiry for a European call option and a European put option, as a function of the asset price. Compare these with the pay-off diagrams for the long and the short positions in a forward contract. [4]

Derive the Black-Scholes equation, explaining carefully the assumptions and approximations that you make. Describe the resulting hedging strategy for the price of a European call option. [13]

Explain the term *implied volatility*. [4]

2. Discuss the implications of the *independent and identically distributed* (i.i.d.) assumption invoked by standard finance theory when describing price-increments in a market. Discuss how you would determine whether a given price process $y(t)$ constitutes a random walk. [7]

The monthly increment of a particular market index can be described by the following equation:

$$\delta y(t) = \alpha(t-2)\alpha(t-1) + \alpha(t) \ ,$$

where at each timestep t the random variable α takes values -1 or $+1$ with equal probability. Evaluate the following expectation values for the process $\delta y(t)$:

(a) $\mathrm{E}[\delta y(t)]$,

(b) $\mathrm{E}[\delta y(t')\delta y(t)]$ for $t' \neq t$,

(c) $\mathrm{E}[\delta y(t)\delta y(t-1)\delta y(t-2)]$. [10]

Comment on the predictability of the time-series $y(t)$. Derive the magnitude and duration of the worst possible drawdown that can occur for this index. Compare these results with the worst possible drawdown for another index, the monthly increment of which is given by $\delta y(t) = \alpha(t)$. [6]

It has been claimed that trading in derivatives is a cause of financial crashes. Comment on this statement. [2]

3. Discuss the extent to which financial market data show scaling properties. [7]

In a binomial tree model the probability of an upward price increment is p at each time-step and is independent of previous outcomes. The probability of a downward change is $1-p$. Consider shares with a starting price of $x_0 = 100$ currency units, which may rise or fall by 10% at each time-step. What are the possible share values after three time-steps according to this model? Find expressions in terms of p for:

(a) the probabilities of these outcomes, and

(b) the expectation value of the price. [5]

A risk-free investment is available that pays 4% interest at each time-step. What value of the risk-neutral probability does this imply for the binomial model? [3]

We purchase a European put option with a strike price of 108 currency units at time-step 3. What is the payoff of the option for each of the possible final share prices? What is the initial value of the option? [5]

After time-step 2 there are three possible states of the market. At each of these three nodes in the binomial tree, determine how the writer of the European put option could hedge the option. [5]

Section S12 INTRODUCTION TO BIOLOGICAL PHYSICS

1. Describe the hierarchical structure of a folded protein, the forces that govern protein folding and the structural features that distinguish membrane-spanning proteins from soluble proteins. [25]

2. Describe with the aid of sketches the structure, function and mechanism of the enzyme ATP-synthase. [10]

Single molecules of F_1-ATPase isolated from ATP-synthase can be attached to a glass surface and the rotation of the γ-subunit relative to the α- and β-subunits monitored by laser dark-field microscopy. Rotation proceeds in 120° steps, each divided into a larger and a smaller sub-step as shown below.

The distributions of time intervals between steps can be fitted as

$$
\begin{aligned}
P_1(t_1) &= k_1 c e^{-ck_1 t_1} \\
P_2(t_2) &= \left(\frac{k_a k_b}{k_a - k_b}\right)\left(e^{-k_b t_2} - e^{-k_a t_2}\right),
\end{aligned}
$$

where c is the ATP concentration and k_1, k_a and k_b are constants.

What can be inferred about the catalytic cycle of the enzyme from these observations? Your answer should include derivations and sketches of the distributions P_1 and P_2. [15]

3. Prove that the mean-square end-to-end distance travelled in an unbiased 1D random walk of m steps is given by $\langle x_m^2 \rangle = m\Delta$, where Δ is the mean-square step size. [4]

Starting with Fick's law for a particle diffusing in one dimension subject to a conservative force, derive the Einstein relation between the viscous drag coefficient and diffusion constant. [7]

Write down the diffusion equation in 1D and show by direct substitution that a solution exists of the form

$$C(x,t) = A\,t^{-\frac{1}{2}} \exp\left(-\frac{x^2}{4Dt}\right) .$$

For a total number of particles N, $A = N/\sqrt{4\pi D}$. Verify that $\langle x^2 \rangle = 2Dt$ for this solution, where $\langle x^2 \rangle$ is the mean-squared distance of a particle from the origin, and sketch the solution for several values of $t > 0$.

The random walk described above is used as a model of diffusion in 1D. Derive the relationship between the mean-square step size Δ, the mean stepping rate f, and the diffusion coefficient. [14]

Section S16 PLASMA PHYSICS

1. Without detailed proofs, describe what is meant by *Landau damping.* [6]

The one-dimensional distribution of velocities of electrons, $f(v)$, for a plasma obeying Maxwell-Boltzmann statistics is given by

$$f(v) = A \exp\left(-\frac{m_e v^2}{2k_B T}\right) \quad ,$$

where A is a constant of normalization. In contrast to a situation where there is Landau damping of plasma waves, if the velocity distribution function is disturbed such that, at a certain velocity, there are more electrons traveling faster than the phase velocity of a wave than slower than it, then the wave can exponentially grow in amplitude, and the system is unstable to the growth of plasma waves. Consider a plasma with electron number density n_p, temperature T_p, and electron distribution function $f_p(v)$. It is irradiated by a non-relativistic beam of electrons with number density n_b. The mean velocity of the electrons within the the beam is V. In the frame of motion of the electrons within the beam, the temperature of the beam is T_b. The overall distribution function of velocities of the electrons within the beam is given by $f(v_b)$. The distribution functions of the plasma and the beam are such that $V \gg \sqrt{k_B T_p/m_e} \gg \sqrt{k_B T_b/m_e}$. Sketch the form of the total distribution function, $f(v) = f_p(v) + f_b(v)$. [4]

Assume that under the conditions where an unstable wave first exists, it has a phase velocity v_ϕ. With reference to your sketch explain why a good approximation for this velocity will be the value of v where $\mathrm{d}f(v_b)/\mathrm{d}v$ is maximised, and show that this is given by

$$v_\phi = V - \sqrt{\frac{k_B T_b}{m_e}} \quad .$$

[4]

By finding the condition for the derivative of the total distribution function with respect to velocity to be non-negative, demonstrate that an instability will develop if

$$\frac{n_b}{n_p} > C \sqrt{\frac{m_e}{k_B T_p}}\, \frac{V T_b}{T_p} \exp(-m_e V^2/2k_B T_p) \quad ,$$

where C is a constant of order unity. [6]

A 10 keV beam of electrons 1-mm in diameter with temperature 100 eV is incident on a plasma with electron number density $10^{26}\mathrm{m}^{-3}$ and temperature 1 keV. Estimate the minimum current in the beam for an instability to develop. [5]

2. For a cold plasma with electron number density n_e find an expression for the plasma frequency, ω_pe. Show that the time-averaged kinetic energy per unit volume of the electrons is equal to the time-averaged energy density of the fluctuating electric field that exists owing to the plasma wave. Explain why you would expect these two quantities to be equal. [8]

Explain what is meant by the *Debye length*, λ_D, and show that for a non-degenerate plasma at a temperature T it is given by

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k_\mathrm{B} T}{n_\mathrm{e} e^2}} \quad .$$

[6]

Show that in a degenerate plasma the ratio of the energy of a plasmon to the Fermi energy is given by

$$\frac{\hbar\omega_\mathrm{pe}}{E_\mathrm{F}} = \sqrt{\frac{4e^2 m_\mathrm{e}}{\hbar^2 \varepsilon_0}}(3\pi^2)^{-2/3} n_\mathrm{e}^{-1/6} \quad .$$

For solid aluminium the ratio of the energy of a plasmon to the Fermi energy is 1.384. What is the electron number density, the Fermi energy, and the energy of a plasmon in aluminium? [6]

In a degenerate plasma, λ_D can be calculated by replacing T by the Fermi temperature, $T_\mathrm{F} = E_\mathrm{F}/k_\mathrm{B}$. Calculate the number of electrons in a Debye sphere for aluminium, and comment on your result. [5]

3. A partially ionized hydrogen plasma in a volume V contains N_{H}, N_{p}, and N_{e} hydrogen atoms, free protons, and free electrons respectively. The respective single-particle partition functions are $Z_{\mathrm{sp_H}}$, $Z_{\mathrm{sp_p}}$ and $Z_{\mathrm{sp_e}}$. The law of mass action for the plasma states that

$$\frac{Z_{\mathrm{sp_e}} Z_{\mathrm{sp_p}}}{Z_{\mathrm{sp_H}}} = \frac{N_{\mathrm{e}} N_{\mathrm{p}}}{N_{\mathrm{H}}} \quad .$$

Using the law of mass action, and stating any further assumptions that you make, derive the Saha equation relating the number of free electrons to the number of hydrogen atoms

$$\frac{N_{\mathrm{e}}^2}{N_{\mathrm{H}}} = \exp\left(-\frac{R}{k_{\mathrm{B}}T}\right) V \left(\frac{2\pi m_{\mathrm{e}} k_{\mathrm{B}} T}{h^2}\right)^{3/2} \quad ,$$

where R is the ionization potential of atomic hydrogen. [10]

Standard models of the early universe indicate that it became transparent to optical photons when it had cooled sufficiently for hydrogen atoms to form. It is estimated that this occurred after 10^5 years, when the number density of electrons was of order 10^9 m^{-3}. Assuming transparency occurred when the fractional ionization was 1%, how hot was the universe when it became transparent? Comment on the magnitude of the thermal energy associated with this temperature compared with the energy required to ionize hydrogen. [9]

In the derivation of the Saha equation, it is usual to ignore the excited electronic states of the hydrogen atom in the determination of $Z_{\mathrm{sp_H}}$. Why is this usually justified? Illustrate your answer by considering the population of the $n = 2$ quantum state of atomic hydrogen at the time when the universe became transparent. [6]

[The ionization potential of hydrogen, R, is 13.6 eV]

Section S18 ADVANCED QUANTUM MECHANICS

1. In natural units ($\hbar = c = 1$) the Dirac equation for a free particle of mass m is given by

$$(\mathrm{i}\gamma_\mu \partial^\mu - m)\psi = 0 \ .$$

Assuming ψ obeys the Klein Gordon equation, show that

$$\gamma_\mu \gamma_\nu + \gamma_\nu \gamma_\mu = 2\eta_{\mu\nu} I$$

where $\eta_{\mu\nu} = (+ \ - \ - \ -)$ and I is the identity matrix. [4]

Under a local $U(1)$ transformation

$$\psi \to \mathrm{e}^{\mathrm{i}e\alpha(x)}\psi \ .$$

Defining the covariant derivative by $D_\mu = \partial_\mu - \mathrm{i}eA_\mu$ show that under the same transformation

$$D_\mu \psi \to \mathrm{e}^{\mathrm{i}e\alpha(x)} D_\mu \psi$$

provided that

$$A_\mu \to A_\mu + \partial_\mu \alpha \ .$$

Hence show that the minimally coupled Dirac equation transforms covariantly under a $U(1)$ gauge transformation. [6]

Show that a wavefunction ψ which obeys the minimally coupled Dirac equation also satisfies the second order equation

$$(-D.D + \frac{\mathrm{e}}{2}\sigma_{\mu\nu}F^{\mu\nu} - m^2)\psi = 0 \ ,$$

where $F^{\mu\nu}$ is the field strength tensor and $\sigma_{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma_\mu, \gamma_\nu]$. [5]

Consider the case $A^\mu = (0, \mathbf{A}(\mathbf{x}))$ and use the basis in which

$$\sigma^{ij} = \sum_k \epsilon^{ijk} \begin{pmatrix} \sigma^k & 0 \\ 0 & \sigma^k \end{pmatrix}$$

where σ^k are the Pauli matrices. By writing $\psi = \mathrm{e}^{-\mathrm{i}(m+E)t}\phi(\mathbf{x})$ find the time independent Schrodinger equations satisfied by the upper and lower components of ϕ in the non-relativistic limit $E \ll m$. Identify all the terms appearing in these equations and explain their meaning. [10]

2. A particle of mass m and wavenumber k is incident from the negative z direction on a potential $V(r)$ which falls off rapidly at large r. Explain why the wavefunction $\psi(r,\theta,\phi)$ takes the form

$$\psi(r,\theta,\phi) = \mathrm{e}^{\mathrm{i}kz} + \frac{\mathrm{e}^{\mathrm{i}kr}}{r} f_k(\theta,\phi)$$

at very large r and show that the differential cross-section is given by

$$\frac{\partial\sigma}{\partial\Omega} = |f_k(\theta,\phi)|^2 \; .$$ [8]

In the first Born approximation

$$f_k(\theta,\phi) = -\frac{m}{2\pi\hbar^2}\int \mathrm{d}^3\mathbf{r}'\, V(\mathbf{r}')\mathrm{e}^{-\mathrm{i}\mathbf{q}.\mathbf{r}'}$$

where $\mathbf{q} = \mathbf{k}_f - \mathbf{k}_i$ and $\mathbf{k}_i$ and $\mathbf{k}_f$ are the wavevectors of the incoming and scattered particle respectively. Find $f_k(\theta,\phi)$ for the potential

$$V(\mathbf{r}) = \begin{cases} V_0, & r \le r_0 \; , \\ 0, & r > r_0 \; . \end{cases}$$ [8]

Show that if $k < k_c$ there are no zeroes in the scattering amplitude and give a rough estimate of the value of k_c. As k increases at what scattering angle does the first zero appear? Show that for $k \gg k_c$ the zeroes are at scattering angles given by

$$\theta_n = 2\arcsin\left(\frac{\pi}{2kr_0}(n+\frac{1}{2})\right), \quad n = 1,2,\ldots$$ [9]

3. In natural units ($\hbar = c = 1$) the wave equation for a massless spin $\frac{1}{2}$ particle in a region with potential $V(\mathbf{x})$ is given by

$$\mathrm{i}\frac{\partial\psi}{\partial t} = -\mathrm{i}\boldsymbol{\sigma}.\nabla\psi + V\psi \; .$$

Show that the four-current $j_\mu = (\psi^\dagger\psi, -\psi^\dagger\boldsymbol{\sigma}\psi)$ is conserved. Under what circumstances is $\boldsymbol{\sigma}.\mathbf{p}$ conserved and what are then the possible values of the helicity $\boldsymbol{\sigma}.\mathbf{p}/|\mathbf{p}|$? [10]

The particle moves in the z direction in a region of constant potential $V(x) = V_0$. Show that

$$\psi(z,t) = \mathrm{e}^{-\mathrm{i}V_0 t}\left(\eta(t-z)u_+ + \rho(t+z)u_-\right) \qquad (1)$$

where η and ρ are arbitrary differentiable functions and find the normalized spinors $u_\pm$. What is the corresponding result if the particle moves in the $\hat{\mathbf{n}}$ direction? [10]

Find j_μ for $\psi(z,t)$ given by (1) in terms of η and ρ and discuss your result. [5]

$$\left[\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right]$$

Section S19 PARTICLE ACCELERATOR SCIENCE

The following formulae may be useful:

Critical photon energy of synchroton radiation:

$$\varepsilon_{\mathrm{c}} = \frac{3}{2}\frac{\hbar c}{\rho}\gamma^3 ,$$

where γ is the relativistic factor of the electron beam and ρ is the dipole radius of curvature.

The energy loss per turn is

$$U_0 = \frac{e^2\gamma^4}{3\varepsilon_0\rho} .$$

1.

(a) Write down the matrices associated with the beam transport in a drift section, in a focussing quadrupole and in a defocusing quadrupole. Describe the consequences of the focusing and defocusing quadrupole matrices on the beam motion.

Under what conditions can the quadrupole be approximated as a thin lens? Write down the thin lens form of the quadrupole matrices. [7]

(b) Consider a transfer line with a quadrupole doublet made, in the order described, of a focussing quadrupole with focal length f, a drift of length L_1, a defocussing quadrupole with focal length f and a drift of length L_2. The quads can be treated in the thin lens approximation. Calculate the transfer matrix of the cell. [8]

(c) Consider now a ring made of a series of cells identical to the one in the previous paragraph. For what values of f, L_1 and L_2 has the cell a stable solution?

What is the phase advance of the full cell if we start from a different point in the cell?

Write down the matrix that expresses the transport for n turns in the cell. [Hint: use the Twiss representation for the one turn map.] [10]

2. Derive the expression for the critical photon energy for the emission of synchrotron radiation from a dipole magnet. [10]

Let us assume we have a storage ring with dipoles with radius $\rho = 10\,\mathrm{m}$ and a 5GeV electron beam. The revolution time is $2\,\mu$s.

(a) What is the critical energy for the synchrotron radiation emitted, and what is it if we have a proton beam? [5]

(b) Calculate the energy loss per turn for such an electron beam. What happens to the beam as it loses energy? [Hint: remember the definition of dispersion]. Will the beam move horizontally? Will the beam move vertically? What is the power lost via synchrotron radiation by a beam of 500 mA? How is this power given back to the beam? [10]

3.

(a) *RF Systems.* Assume that a new 50 GeV (kinetic energy) proton synchrotron is under design. The new accelerator will sit in a new ring tunnel which will have a mean radius of 215 m and will receive an injected beam at 4 GeV (kinetic energy) from a new linear accelerator. Given that the mass of the proton is 0.9383 GeV/c^2:

(i) What is the revolution frequency at 4 GeV, 20 GeV and at 50 GeV?

(ii) If the harmonic number is 32, what is the RF frequency at 4 GeV and at 50 GeV for a synchronous phase $\varphi_s = 0°$? [5]

(b) *Beam Diagnostics.* Beam diagnostics are used regularly to monitor the beam lifetime in a storage ring.

(i) Define the beam lifetime in a storage ring.

(ii) What effects cause the circulating beam to decay in intensity with time?

(iii) What beam diagnostic technique would be needed to monitor the beam intensity? [8]

(c) *Applications of Accelerators.* Describe the basic features of two electron-positron linear colliders such as the International Linear Collider (ILC) and the Compact Linear Collider (CLIC). Include in your description

(i) the lay-out of each type of machine;

(ii) their energy and luminosity;

(iii) the concept of acceleration for each machine. [12]

Section S25 PHYSICS OF CLIMATE CHANGE

1. Define the terms *spectral irradiance* and *optical depth* in the context of atmospheric radiative transfer and give an expression for the rate of change of optical depth with height for an atmosphere containing a single radiatively active gas, taking care to define all terms. [7]

Diffuse outgoing spectral irradiance at a particular wavelength at the top of this atmosphere is given by

$$I_0 = \pi B_0 + \pi \int_0^{\tau_s} \frac{dB(\tau)}{d\tau} e^{-\tau} d\tau \ ,$$

where τ is the optical depth and $B(\tau)$ is the Planck function at that wavelength corresponding to the atmospheric temperature at optical depth τ. Solve this equation for an atmospheric temperature profile in which $B = B_0$ for $0 \leq \tau < \tau_1$ and $B = B_1$ for $\tau_1 < \tau \leq \tau_s$, where τ_s is the optical depth at the planet's surface. (Assume the surface acts as a black body.) Hence explain the significance of the terms on the right-hand side. [8]

Derive expressions for I_0 for the cases: (a) $B(\tau) = B_0 + \alpha\tau$; (b) $B(\tau) = B_0 + \beta\tau^2$, assuming in both cases that the atmosphere is optically thick, so you can neglect direct transmission from surface to space. [You may use the integral $\int xe^{-x}dx = -(1+x)e^{-x} +$ const.]

How does I_0 change in each case if the concentration of the radiatively active gas is suddenly increased by a factor of two? [10]

2. Explain what is meant by *hydrostatic equilibrium*. Derive an expression for the rate of change of temperature with height in a dry atmosphere in hydrostatic equilibrium, or *dry adiabatic lapse rate*, Γ_d, making clear the assumptions you make. Observed lapse rates in the atmosphere are typically 6-7 K km^{-1}: explain qualitatively the origins of the discrepancy. [8]

A model of the Greenland ice cap represents the rate of mass loss due to melting as $-\alpha T^{-1}$, where T is the top-of-ice-cap temperature in degrees Celsius and α is a constant, and the rate of mass gain due to snow accumulation, normalised by the rate at the surface (assumed constant), as $1 - \beta z$, where z is the top-of-ice-cap altitude. Therefore

$$\frac{dM}{dt} = 1 - \beta z + \frac{\alpha}{T} \ ,$$

where M is the ice mass per unit area. Assuming a fixed lapse rate Γ, draw a sketch of melting and snow accumulation as functions of z. Hence show that this model can support a stable ice-cap provided a condition on α, β and T_s, the sea-surface temperature in the vicinity of Greenland, is satisfied, and find that condition. Find an expression for the minimum height of the ice-sheet for it to be stable. [12]

The ice-sheet is initially stable with $z = 4$ km, $T_s = 275$ K, $\beta = 0.2$ km^{-1} and $\Gamma = 6$ K km^{-1}. Find α and discuss what happens as T_s slowly increases, assuming no change in snow accumulation as a function of altitude. [5]

3. Estimate the change in atmospheric carbon dioxide (CO_2) concentration (in parts per million by volume, or ppmv) resulting from complete combustion of one gigatonne of fossil carbon (10^9 tonnes, or 1 GtC), assuming all the CO_2 generated remains in the atmosphere. [Take the molar mass of carbon to be 12×10^{-3} kg.] [4]

An idealised model of the response of atmospheric CO_2 concentrations, $C(t)$ (in ppmv), to fossil CO_2 emissions, $E(t)$, (in GtC/year), is given by

$$\frac{dC}{dt} = b_1 E - b_0(C - C_0) \qquad (1)$$

where

$$b_1 = 0.25\,\text{ppmv/GtC}\ , \qquad b_0 = 0.005/\text{year}\ , \qquad C_0 = 279\ \text{ppmv}\ ,$$

and time is measured in years. Give a physical interpretation of the two terms on the right-hand side of equation (1) with reference to your answer to the previous section. Anthropogenic emissions are maintained at $E_0 = 0.02\,\text{GtC/year}$ for thousands of years prior to 1800. What is the concentration of CO_2 in 1800? [4]

Starting in 1800, anthropogenic CO_2 emissions increase exponentially, $E = E_0 e^{gt}$, where $g = 0.03$/year and t is the number of years since 1800. Verify by direct substitution that subsequent CO_2 concentrations are given by

$$C = C_0 + \frac{b_1 E_0}{b_0}\left[\frac{g e^{-b_0 t} + b_0 e^{gt}}{g + b_0}\right] .$$

Compute the atmospheric CO_2 concentration in 2000. [6]

Describe with the aid of a sketch the evolution of atmospheric CO_2 concentrations from 1800 to 2500 in this simple model under two scenarios: (a) when emissions are set to zero in 2000 and (b) when emissions are held constant from 2000 onwards. Indicate relevant time-constants and asymptotic behaviour. If the equilibrium warming response to doubling CO_2 is 3K and the thermal adjustment time is less than 20 years, what approximate peak warming would you expect under each scenario? [9]

Give an example of an important aspect of the behaviour of the real carbon cycle that is omitted from this simple model. [2]

Section S26 STARS AND GALAXIES

1. Starting from the equation of hydrostatic equilibrium, show that the relation between the total thermal energy U and the total gravitational energy Ω of a star is

$$3(\gamma - 1)U + \Omega = 0$$

for an ideal gas, where γ is the ratio of the heats capacities, $\gamma = C_P/C_V$. [5]

Assuming a star is composed of fully-ionized gas ($\gamma = 5/3$), give the relation between Ω and U, and comment on whether this results in a positive or negative heat capacity. Use your result to explain why nuclear burning in most stars is self-regulating (i.e. stable and not resulting in nuclear runaway), and to explain the pre-main-sequence evolution of a gas cloud into a star. [6]

Estimate for the Sun the thermal timescale for it to radiate away its thermal energy in the absence of nuclear burning. Discuss briefly why the ratio of heat capacities, γ, may vary with position within a star. What will happen if $\gamma \leq 4/3$ in the outer layers of a star? [6]

Show that the Schwarzschild condition for stability against convection at any point within a star, whose material behaves as an ideal gas, is given by

$$\frac{P}{T}\frac{\mathrm{d}T}{\mathrm{d}P} < \frac{\gamma - 1}{\gamma},$$

where P and T are the pressure and temperature respectively at that point. [5]

Explain why convection is important in (a) the outer regions of the Sun and (b) near the cores of red giant stars. [3]

2. Explain why the absorption line strengths in the spectra of different stars can vary so greatly, despite most stars having very similar chemical composition. Outline the Harvard spectral classification scheme for stars of different colour, and describe the key spectral features of stars in each of the spectral classes A0, G2 and M5. From the spectrum, how would you discriminate between a red giant star and a dwarf star of the same colour? [6]

Describe briefly the evolution of stars of mass (a) 1 $M_\odot$, (b) 5 $M_\odot$ and (c) 15 $M_\odot$, including in your description an indication of the dominant energy generation mechanism and an estimate of its duration, and a discussion of the final stages of the life of the star. [6]

Sketch the Hertzsprung-Russell (Temperature – Luminosity) diagrams for (i) a young open cluster and (ii) an old globular cluster. You should label on your plot the key features of the stellar populations. [4]

For massive stars on the main sequence, using simple scaling relations and stating any assumptions you make, derive the relation between luminosity (L) and stellar mass (M), $L \propto M^k$ (where k is a constant to be determined).

[You may assume that within massive stars radiative transfer dominates the energy transport, and that the opacity is approximately independent of temperature and density for massive stars.] [5]

The Hyades is an open cluster, with a trigonometric parallax determined by the Hipparcos satellite of 0.02 arc seconds. The brightest main sequence star has an apparent magnitude of 5.3 in the V-band and a spectral type of A2V. Calculate the luminosity (in $L_\odot$) and estimate the mass (in $M_\odot$) of stars at the main sequence turn-off, and hence estimate the age of this cluster, stating any assumptions you make.

[You may take the absolute bolometric magnitude of the Sun to be $M_{\rm bol} = 4.6$, and the bolometric correction for the V-band for a A2V star is -0.1 mag. You may also assume if necessary that fusing 4 protons to form ^{4}He liberates 26.7 MeV of energy.]. [4]

3. Sketch what our own Milky Way would appear like if viewed (a) face on and (b) edge-on, indicating the main features and approximate sizes. Indicate regions where recent star formation may have occurred, and also the regions where very old stars may be found. [4]

A spiral galaxy is observed. The thin disk is seen to extend 30 arc seconds in diameter in one direction (the major axis) and 7.5 arc seconds on the minor axis. A long slit is used to take a spectrum along the major axis, and this reveals the Hα emission line of hydrogen (the $n = 3 \to 2$ transition with $\lambda_{\rm rest} = 656.3\,{\rm nm}$). What astrophysical mechanism results in this line emission? The observed line emission is spatially extended, with a wavelength of $\lambda_{\rm centre} = 689.1\,{\rm nm}$ at centre of the galaxy, and it asymptotes to $\lambda_{\rm min} = 688.5\,{\rm nm}$ and $\lambda_{\rm max} = 689.7\,{\rm nm}$ towards the edges. Calculate the average redshift of the galaxy and its maximum rotation velocity, $v_{\rm max}$. [4]

Taking the value of the Hubble constant as $71\,{\rm km\,s^{-1}\,Mpc^{-1}}$, find the distance to the galaxy described above, and calculate the radius of the observable disk. Use your results to estimate the total mass of the galaxy. If the total apparent magnitude of this galaxy in the visible is $m_V = 16$, calculate its luminosity in units of $L_\odot$ (you may take the absolute magnitude of the Sun to be $M_V = 4.8$ magnitudes). Compare your results for the dynamical mass of this galaxy with an estimate of the mass in stars, and comment on your result. [8]

Another spiral galaxy is observed with $v_{\rm max} = 100\,{\rm km\,s^{-1}}$. Using your results above, estimate the luminosity of this galaxy, justifying any scaling relation you use. [3]

Our own Galaxy, the Milky Way, exhibits a flat rotation curve around the solar orbit (which has a radius of $r = 8\,{\rm kpc}$), with circular velocity $v_{\rm c} = 220\,{\rm km\,s^{-1}}$. Assuming a simple model where the matter distribution is spherically-symmetric, calculate the matter density $\rho(r)$, as a function of distance r from the Galactic centre. If there exists a sharp cut-off radius, r_*, beyond which the matter density falls to zero, find the escape velocity for stars at $r < r_*$ as a function of r_*, r and v_c. If the star in the solar neighbourhood with the greatest observed radial velocity of $430\,{\rm km\,s^{-1}}$ (relative to the Galactic centre), estimate r_* for this simple model stating any assumptions you make. [6]