University of Oxford
Department of Physics

Oxford Master Course in Mathematica and Theoretical Physics

INTRODUCTION TO GAUGE-STRING DUALITY

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Problem Set V

$U(1)$ Current in Strongly-Coupled $\mathcal{N} = 4$

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$\mathcal{N} = 4$ SYM contains a global $U(1)$-charge (a sub-group of the R-symmetry) with an associated conserved $U(1)$-current J^μ. In the limit of large N and strong coupling, the finite temperature equilibrium state of $\mathcal{N} = 4$ with vanishing $U(1)$-charge is at low energies described by the AdS_5-black brane geometry:

$$\mathrm{d}s_5^2 = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = \frac{L^2}{z^2}\left(\frac{\mathrm{d}z^2}{f} + \left[-f\mathrm{d}t^2 + \mathrm{d}\underline{x}^2\right]\right) \quad , \quad f(z) = 1 - \left(\frac{z}{z_H}\right)^4 .$$

Fluctuations of J^μ are captured by the dynamics of a dual Maxwell field A_m on that bulk background, with action

$$S = -\frac{1}{4g_B^2}\int \mathrm{d}^5x\sqrt{-g}F_{mn}F^{mn} ,$$

and field strength

$$F_{mn} = \partial_m A_n - \partial_n A_m .$$

1 Holographic Expectation Value and Correlator

1.1 Equations of Motion

Question 1 *Show that the bulk equations of motion are Maxwell's equations*

$$\sqrt{-g}\nabla_m F^{mn} = \partial_m\left(\sqrt{-g}g^{mp}g^{nq}F_{pq}\right) = 0 .$$

We can use the symmetry of the system under $U(1)$-gauge transformations,

$$A_m \longrightarrow A_m - \partial_m\lambda ,$$

to set the radial component of A_m to zero, $A_z = 0$. This leaves us with the residual symmetry of z-independent $\lambda(x)$, $x^\mu = (t, \underline{x})$.

Question 2 *Fourier-transforming along the field theory directions,*

$$A_\mu(z,x) = \int \frac{\mathrm{d}^4k}{(2\pi)^4}e^{ik\cdot x}A_\mu(z,k) ,$$

with $k_\mu = (-\omega, \underline{q})$, $k\cdot x = -\omega t + \underline{q}\cdot\underline{x}$, and choosing $\underline{q}$ along the 3-direction, $k_\mu = -\omega, 0, 0, q)$, confirm that Maxwell's equations take the following form:

$$A_3' + \frac{\omega}{qf}A_t' = 0 , \tag{1}$$

$$A_\alpha'' + \left[\frac{f'}{f} - \frac{1}{z}\right]A_\alpha' + \frac{\omega^2 - q^2 f}{f^2}A_\alpha = 0 \quad , \quad \alpha = 1, 2 , \tag{2}$$

$$A_t'' - \frac{1}{z}A_t' - \frac{1}{f}\left[q^2 A_t + \omega q A_3\right] = 0 , \tag{3}$$

$$A_3'' + \left[\frac{f'}{f} - \frac{1}{z}\right]A_3' + \frac{1}{f^2}\left[\omega^2 A_3 + \omega q A_t\right] = 0 , \tag{4}$$

where dashes denote derivatives with respect to z.

The residual gauge symmetry means that these equations of motion do not uniquely determine the evolution of the system: configurations that only differ by residual gauge transformations are physically equivalent and cannot be distinguished. This problem is most easily circumvented by switching to gauge-invariant variables such as the electric field, with transverse components

$$E_\alpha = iF_{t\alpha} = \omega A_\alpha \quad , \quad \alpha = 1, 2$$

and longitudinal component

$$E_L = iF_{t3} = \omega A_3 + qA_t \; .$$

Question 3 *Show that E_α and E_L satisfy the equations of motion*

$$E''_\alpha + \left[\frac{f'}{f} - \frac{1}{z}\right] E'_\alpha + \frac{\omega^2 - q^2 f}{f^2} E_\alpha = 0 \; , \tag{5}$$

$$E''_L + \left[\frac{f'\omega^2}{f\,(\omega^2 - q^2 f)} - \frac{1}{z}\right] E'_L + \frac{\omega^2 - q^2 f}{f^2} E_L = 0 \; . \tag{6}$$

1.2 Near-Boundary Expansion

Question 4 *Show that $z = 0$ is a regular singular point of the equations of motion (5)–(6) and show that the corresponding exponents (the roots of the indicial polynomial) for both equations are $\alpha_+ = 2$ and $\alpha_- = 0$. The near-boundary behaviour of the electric field is therefore*

$$\begin{aligned} E_\alpha &= \mathcal{A}_\alpha + \cdots + \mathcal{B}_\alpha z^2 + \ldots \; , \\ E_L &= \mathcal{A}_L + \cdots + \mathcal{B}_L z^2 + \ldots \; . \end{aligned}$$

More precisely, as $\alpha_+ - \alpha_- \in \mathbb{N}$ the near-boundary expansion is given by

$$\begin{aligned} E_\alpha &= \mathcal{A}_\alpha + \mathcal{A}^{(1)}_\alpha z + \tilde{\mathcal{A}}^{(2)}_\alpha z^2 \log(\Lambda z) + \mathcal{B}_\alpha z^2 + \mathcal{O}(z^3) \; , \\ E_L &= \mathcal{A}_L + \mathcal{A}^{(1)}_L z + \tilde{\mathcal{A}}^{(2)}_L z^2 \log(\Lambda z) + \mathcal{B}_L z^2 + \mathcal{O}(z^3) \; , \end{aligned}$$

see e.g. [1] for details. Λ is an arbitrary energy scale introduced to make the argument of the logarithm dimensionless. Its meaning will be discussed later.

Question 5 *Show that the equations of motion require*

$$\mathcal{A}^{(1)}_\alpha = \mathcal{A}^{(1)}_L = 0 \; , \qquad \tilde{\mathcal{A}}^{(2)}_\alpha = \frac{k^2}{2}\mathcal{A}_\alpha \; , \qquad \tilde{\mathcal{A}}^{(2)}_L = \frac{k^2}{2}\mathcal{A}_L \; ,$$

where $k^2 = -\omega^2 + q^2$.

1.3 Holographic 1-Point Function

Variation of the On-Shell Action

Question 6 *Consider a generic action*

$$S = \int_{\epsilon}^{z_H} \mathrm{d}z \, L(\Phi, \Phi') \ .$$

By looking at a general variation of S show that, when the equations of motion are satisfied, the following identity holds:

$$\frac{\delta S^{\text{on-shell}}}{\delta \Phi(\epsilon)} = -\frac{\partial L}{\partial \Phi'}(\epsilon) \ .$$

Thus show that in our case

$$\frac{\delta S^{\text{on-shell}}}{\delta A_\mu(\epsilon, x)} = \frac{1}{g_B^2} \sqrt{-g} g^{zz} g^{\mu\nu} A'_\nu(z, x) \Big|_{z=\epsilon} \ ,$$

or in momentum space

$$(2\pi)^4 \, \frac{\delta S^{\text{on-shell}}}{\delta A_\mu(\epsilon, -k)} = \frac{1}{g_B^2} \, \sqrt{-g} g^{zz} g^{\mu\nu} A'_\nu(z, k)\big|_{z=\epsilon} \ .$$

$z = \epsilon$ is the UV-cutoff near the boundary $z = 0$. Using the $U(1)$-constraint (1) verify that

$$\frac{1}{q} \frac{\delta S^{\text{on-shell}}}{\delta A_t(z, -k)} = \frac{1}{\omega} \frac{\delta S^{\text{on-shell}}}{\delta A_t(z, -k)} \ ,$$

and conclude that $S^{\text{on-shell}}$ only depends on the gauge invariant combinations $E_\alpha = \omega A_\alpha$ and $E_L = \omega A_3 + q A_t$. Hence confirm that

$$\begin{aligned}
(2\pi)^4 \, \frac{\delta S^{\text{on-shell}}}{\delta E_\alpha(\epsilon, -k)} &= \frac{1}{\omega^2} \frac{L}{g_B^2} \, \frac{f}{z} E'_\alpha \Big|_{z=\epsilon} \\
&= \frac{1}{\omega^2} \frac{L}{g_B^2} \left[2\mathcal{B}_\alpha + k^2 \mathcal{A}_\alpha \left(\log(\Lambda\epsilon) + \frac{1}{2} \right) \right] + \mathcal{O}(\epsilon) \ , \qquad (7) \\
(2\pi)^4 \, \frac{\delta S^{\text{on-shell}}}{\delta E_L(\epsilon, -k)} &= \frac{L}{g_B^2} \left(\frac{1}{\omega^2 - q^2 f} \right) \frac{f}{z} E'_L \Big|_{z=\epsilon} \\
&= -\frac{1}{k^2} \frac{L}{g_B^2} \left[2\mathcal{B}_L + k^2 \mathcal{A}_L \left(\log(\Lambda\epsilon) + \frac{1}{2} \right) \right] + \mathcal{O}(\epsilon) \ . \qquad (8)
\end{aligned}$$

Note that we have to switch from A_μ to the gauge-invariant electric field in order to make use of the latter's near-boundary expansion, provided by the well-defined equations of motion (5)–(6).

Holographic Renormalisation

The AdS/CFT correspondence identifies the (renormalised) on-shell action S^{ren} of the Maxwell field $A_\mu = a_\mu + \dots$ in the AdS_5-black brane geometry with the generating functional W for the dual $U(1)$ current J^μ in $\mathcal{N} = 4$:

$$W = \left\langle \int \frac{\mathrm{d}^4 k}{(2\pi)^4} J^\mu(k) a_\mu(-k) \right\rangle = S^{\text{ren}} .$$

The expectation value of $J^\mu(k)$ in the presence of the external source a_μ is given by

$$\langle J^\mu(k) \rangle_a = (2\pi)^4 \frac{\delta W}{\delta a_\mu(-k)} = (2\pi)^4 \lim_{\epsilon \to 0} \left(\frac{\delta S^{\text{reg}}}{\delta A_\mu(\epsilon, -k)} \right) .$$

As usual for QFTs this expression contains UV-divergencies, which in our example holographically manifest themselves in the $\log(\Lambda\epsilon)$ terms in eqs. (7)–(8). To obtain a finite result suitable counterterms S_{ct} are added to the action, $S^{\text{reg}} = S^{\text{on-shell}} + S_{\text{ct}}$. The counterterms S_{ct} are defined on the UV-cutoff surface $z = \epsilon$ and need to respect the appropriate symmetries. In our case, gauge-invariance together with the requirement that all divergencies be removed determine S_{ct} to be given by the four-dimensional Maxwell action [2, 3]:

$$S_{\text{ct}} = -\frac{L}{4g_B^2} \log(\Lambda\epsilon) \int\limits_{z=\epsilon} \mathrm{d}^4 x \sqrt{-\gamma} F_{\mu\nu} F_{\rho\sigma} \gamma^{\mu\rho} \gamma^{\nu\sigma} .$$

$\gamma_{\mu\nu}$ is the metric induced on the near-boundary cutoff surface $z = \epsilon$,

$$\gamma_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{1}{\epsilon^2} \left[-f(\epsilon) \mathrm{d}t^2 + \mathrm{d}\underline{x}^2 \right] = \frac{1}{\epsilon^2} \eta_{\mu\nu} + \mathcal{O}(\epsilon^4) ,$$

where $\eta_{\mu\nu}$ is the flat Minkowski metric of the dual boundary theory.

Question 7 *Confirm that the variation of S_{ct} is given by*

$$\delta S_{\text{ct}} = \frac{L}{g_B^2} \log(\Lambda\epsilon) \int\limits_{z=\epsilon} \mathrm{d}^4 x \sqrt{-\gamma}\, \delta A_\mu\, \nabla_\nu^{(\gamma)} F^{\nu\mu} ,$$

where $\nabla_\nu^{(\gamma)}$ is the covariant derivative associated with the four-dimensional metric $\gamma_{\mu\nu}$. Hence conclude that

$$(2\pi)^4 \frac{\delta S_{\text{ct}}}{\delta A_\mu(\epsilon, -k)} = -\frac{L}{g_B^2} \log(\Lambda\epsilon) \eta^{\mu\nu} \left[k^2 A_\nu - k_\nu\, k \cdot A \right] + \mathcal{O}(\epsilon)$$

and verify that

$$\begin{aligned}
(2\pi)^4 \frac{\delta S_{\text{ct}}}{\delta E_\alpha(\epsilon, -k)} &= -\frac{L}{g_B^2} \log(\Lambda\epsilon) \frac{k^2}{\omega^2} E_\alpha + \mathcal{O}(\epsilon) , \\
(2\pi)^4 \frac{\delta S_{\text{ct}}}{\delta E_L(\epsilon, -k)} &= \frac{1}{q} (2\pi)^4 \frac{\delta S_{\text{ct}}}{\delta A_t(\epsilon, -k)} \\
&= \frac{1}{\omega} (2\pi)^4 \frac{\delta S_{\text{ct}}}{\delta A_3(\epsilon, -k)} \\
&= \frac{L}{g_B^2} \log(\Lambda\epsilon) E_L + \mathcal{O}(\epsilon) .
\end{aligned}$$

Finally show that

$$\lim_{\epsilon\longrightarrow 0}\left(\frac{\delta\left(S^{\text{on-shell}}+S_{\text{ct}}\right)}{\delta E_\alpha(\epsilon,-k)}\right)=\frac{1}{\omega^2}\frac{L}{g_B^2}\left[2\mathcal{B}_\alpha+\frac{k^2}{2}\mathcal{A}_\alpha\right]\ , \tag{9}$$

$$\lim_{\epsilon\longrightarrow 0}\left(\frac{\delta\left(S^{\text{on-shell}}+S_{\text{ct}}\right)}{\delta E_L(\epsilon,-k)}\right)=-\frac{1}{k^2}\frac{L}{g_B^2}\left[2\mathcal{B}_L+\frac{k^2}{2}\mathcal{A}_L\right]\ , \tag{10}$$

and thus

$$\begin{aligned}\langle J^\alpha(k)\rangle &= \frac{1}{\omega}\frac{L}{g_B^2}\left[2\mathcal{B}_\alpha+\frac{k^2}{2}\mathcal{A}_\alpha\right]\ ,\\ \left\langle J^t(k)\right\rangle &= -\frac{q}{k^2}\frac{L}{g_B^2}\left[2\mathcal{B}_L+\frac{k^2}{2}\mathcal{A}_L\right]\ ,\\ \left\langle J^3(k)\right\rangle &= -\frac{\omega}{k^2}\frac{L}{g_B^2}\left[2\mathcal{B}_L+\frac{k^2}{2}\mathcal{A}_L\right]\ .\end{aligned}$$

[Optional] Question 8 *Different choices of the arbitrary energy scale Λ in S_{ct} correspond to different renormalisation group (RG) schemes. Describe how the result for the holographic 1-point functions changes if we choose a different RG-scale Λ' in S_{ct}.*

1.4 Retarded Correlators

Question 9 *Show that the retarded current-current correlators are given by [7]*

$$\begin{aligned}\langle J^\alpha(-k)J^\alpha(k)\rangle_{\text{retarded}} &= \Pi^T(k)\ ,\\ \left\langle J^t(-k)J^t(k)\right\rangle_{\text{retarded}} &= \frac{q^2}{\omega^2-q^2}\Pi^L(k)\ ,\\ \left\langle J^t(-k)J^z(k)\right\rangle_{\text{retarded}} &= \left\langle J^z(-k)J^t(k)\right\rangle_{\text{retarded}}\\ &= \frac{\omega q}{\omega^2-q^2}\Pi^L(k)\ ,\\ \langle J^z(-k)J^z(k)\rangle_{\text{retarded}} &= \frac{\omega^2}{\omega^2-q^2}\Pi^L(k)\ ,\end{aligned} \tag{11}$$

with [5]

$$\Pi^T(k)=\frac{L}{g_B^2}\left[2\frac{\delta\mathcal{B}_\alpha}{\delta\mathcal{A}_\alpha}+\frac{k^2}{2}\right]\ ,\qquad\qquad \Pi^L(k)=\frac{L}{g_B^2}\left[2\frac{\delta\mathcal{B}_L}{\delta\mathcal{A}_L}+\frac{k^2}{2}\right]\ .$$

The result exhibits the general structure of retarded correlators in AdS/CFT:

$$G^R=P(\omega,q)\left[\frac{\delta\mathcal{B}}{\delta\mathcal{A}}+\text{contact terms}\right]\ ,$$

where $P(\omega,q)$ is some simple function of the momentum k^μ completely fixed by Lorentz invariance, i.e. it contains only kinetic and no dynamical information. Contact terms are analytic functions of k^2 corresponding to derivatives of delta-functions in position space which again do not carry dynamical information. The dynamical information is entirely encoded in the dependence of the sub-leading mode $B(A)$ (the expectation value) on the leading mode A (the source), furnished by imposing incoming-wave boundary conditions at the horizon. Moreover, for fluctuations around thermal equilibrium obeying linearised equations of motion this relation becomes linear:

$$\boxed{G^R \sim \frac{\mathcal{B}(\mathcal{A})}{\mathcal{A}}\ .}$$

In order to find $\mathcal{B}(\mathcal{A})$ one must solve the connection problem which relates the modes $\mathcal{A}$, $\mathcal{B}$ of the local solution at the boundary to the incoming and outgoing wave-solutions at the horizon. Only in very few cases can this be done analytically, two of which we will consider in the remainder of this problem set.

For that purpose it is convenient to switch to a dimensionless radial coordinate

$$u = \frac{z^2}{z_H^2} = (\pi T z)^2\ , \qquad\qquad f = 1 - u^2\ ,$$

in terms of which eqs. (5)–(6) read

$$\partial_u^2 E_\alpha + \frac{\partial_u f}{f}\partial_u E_\alpha + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{u f^2} E_\alpha = 0\ , \tag{12}$$

$$\partial_u^2 E_L + \frac{\mathfrak{w}^2 \partial_u f}{f\left(\mathfrak{w}^2 - \mathfrak{q}^2 f\right)}\partial_u E_L + \frac{\mathfrak{w}^2 - \mathfrak{q}^2 f}{u f^2} E_L = 0\ , \tag{13}$$

where

$$\mathfrak{w} = \frac{\omega}{2\pi T}\ , \qquad\qquad \mathfrak{q} = \frac{q}{2\pi T}\ .$$

The near-boundary expansions become

$$E_\alpha = \mathcal{A}_\alpha + \cdots + \frac{\mathcal{B}_\alpha}{(\pi T)^2} u + \dots\ , \tag{14}$$

$$E_L = \mathcal{A}_\alpha + \cdots + \frac{\mathcal{B}_L}{(\pi T)^2} u + \dots\ ,$$

working in a scheme with $\Lambda = 1/z_H$.

2 Quasinormal Modes: The Complex Eigenmodes of Black Holes

Perturbations of a black-hole (or black-brane) background that take the form of planar waves $\sim e^{-i\omega t + i\underline{q}\cdot\underline{x}}$ are subject to special boundary conditions: At the boundary of space-time they simply need to vanish (no external source), but at the horizon they must represent incoming-waves to ensure that no information escapes the black hole. Consequently, perturbations are damped as they dissipate energy into the black-hole, and admit a discrete spectrum *complex* frequencies $\omega(\underline{q})$ called *quasinormal modes* (as opposed to the *real* normal modes of dissipationless systems).

In asymptotically AdS-spacetimes, quasinormal modes (QNMs) are thus non-trivial plane-wave solutions to field fluctuations satisfying incoming boundary conditions at the horizon, and Dirichlet boundary conditions at the boundary, $\mathcal{A} = 0$, $\mathcal{B}(\mathcal{A}) \neq 0$. Hence, QNMs of a field in asymptotically AdS_{d+1} correspond exactly to the poles of the retarded correlator of the dual CFT_d-operator.

Our system in the limit of vanishing spatial momentum $q \to 0$ is one of the few cases in which the QNM spectrum can be computed analytically. For $q = 0$, rotational invariance is restored and $E_L = E_\alpha \equiv E$ satisfy the same equation of motion

$$\partial_u^2 E + \frac{\partial_u f}{f}\partial_u E + \frac{\mathfrak{w}^2}{u f^2} E = 0 \,. \tag{15}$$

Eq. (15) has three RSPs at $a = -1$, $b = 0$ (boundary), and $c = 1$ (horizon) with exponents

$$(\alpha, \alpha') = (-\frac{\mathfrak{w}}{2}, \frac{\mathfrak{w}}{2}) \,, \qquad (\beta, \beta') = (1, 0) \,, \qquad (\gamma, \gamma') = (-\frac{i\mathfrak{w}}{2}, \frac{i\mathfrak{w}}{2}) \,,$$

respectively.

2.1 Riemann's Differential Equation

Question 10 *Show that any ODE with exactly three RSPs $(a; b; c)$ with corresponding exponents $(\alpha, \alpha'; \beta, \beta'; \gamma, \gamma')$ takes the form of Riemann's differential equation:*

$$\begin{aligned} 0 = {} & \frac{\mathrm{d}^2 E}{\mathrm{d}u^2} + \left\{ \frac{1-\alpha-\alpha'}{u-a} + \frac{1-\beta-\beta'}{u-b} + \frac{1-\gamma-\gamma'}{u-c} \right\} \frac{\mathrm{d}E}{\mathrm{d}u} \\ & + \left\{ \frac{\alpha\alpha'(a-b)(a-c)}{u-a} + \frac{\beta\beta'(b-a)(b-a)}{u-b} + \frac{\gamma\gamma'(c-a)(c-b)}{u-c} \right\} \frac{E}{(u-a)(u-b)(u-c)} \,. \end{aligned} \tag{16}$$

To that end confirm that a generic second-order ODE

$$\frac{\mathrm{d}^2 E}{\mathrm{d}u^2} + P(u)\frac{\mathrm{d}E}{\mathrm{d}u} + Q(u)E = 0 \,, \tag{17}$$

has three finite RSPs $(a; b; c)$ if and only if

$$P = \frac{\sum_{n\geq 0} p_n u^n}{(u-a)(u-b)(u-c)} \,, \qquad Q = \frac{\sum_{n\geq 0} q_n u^n}{(u-a)^2 (u-b)^2 (u-c)^2} \,. \tag{18}$$

Switching the independent variable to $t = 1/u$,

$$\partial_u = -t^2 \partial_t \ , \qquad \partial_u^2 = t^4 \partial_t^2 + 2t^3 \partial_t \ ,$$

show that $u = \infty \Leftrightarrow t = 0$ *is a regular point of eq.* (17) *if and only if*

$$p_2 = 2 \ , \qquad p_{n\geq 3} = 0 \ , \qquad q_{n\geq 3} = 0 \ .$$

Conclude that P *and* Q *can written as*

$$\begin{aligned} P &= \frac{1-\alpha-\alpha'}{u-a} + \frac{1-\beta-\beta'}{u-b} + \frac{1-\gamma-\gamma'}{u-c} \ , \\ Q &= \left[\frac{\alpha\alpha'\,(a-b)\,(a-c)}{u-a} + \frac{\beta\beta'\,(b-c)\,(b-a)}{u-a} + \frac{\gamma\gamma'\,(c-a)\,(c-b)}{u-c} \right] \frac{1}{(u-a)\,(u-b)\,(u-c)} \ , \end{aligned}$$

with $\alpha+\alpha'+\beta+\beta'+\gamma+\gamma' = 0$*, Finally show that* (α,α')*,* (β,β')*, and* (γ,γ')*, are indeed the exponents at the RSPs* a*,* b*, and* c *respectively.*

By a change of dependent variable and a Möbius transformation of the independent variable that moves the RSPs to $(0, 1, \infty)$, eq. (16) can be transformed into the hypergeometric equation. If $\gamma \neq \gamma'$ then two independent solutions of eq. (16) are

$$E_1 = \left(\frac{u-c}{u-b}\right)^{\gamma} \left(\frac{u-a}{u-b}\right)^{\alpha} {}_2F_1\left(\gamma+\beta+\alpha, \gamma+\beta'+\alpha; 1+\gamma-\gamma'; \frac{(a-b)\,(u-c)}{(a-c)\,(u-b)}\right) \ , \tag{20}$$

$$E_2 = \left(\frac{u-c}{u-b}\right)^{\gamma'} \left(\frac{u-a}{u-b}\right)^{\alpha} {}_2F_1\left(\gamma'+\beta+\alpha, \gamma'+\beta'+\alpha; 1+\gamma'-\gamma; \frac{(a-b)\,(u-c)}{(a-c)\,(u-b)}\right) \ , \tag{21}$$

Question 11 *Show that the solution to eq.* (15) *representing an incoming-wave at the horizon is [6]*

$$E = C\,(1-u)^{-\frac{i\mathfrak{w}}{2}}\,(1+u)^{-\frac{\mathfrak{w}}{2}}\,u^{\left(\frac{1+i}{2}\right)\mathfrak{w}}{}_2F_1\left(1 - \left(\frac{1+i}{2}\right)\mathfrak{w}, -\left(\frac{1+i}{2}\right)\mathfrak{w}; 1 - i\mathfrak{w}, -\frac{1-u}{2u}\right) \ .$$

2.2 Current-Current Correlator at $q = 0$

In the limit $q \to 0$ the only non-trivial retarded current-current correlators from eq. (11) are

$$\left\langle J^i(-\omega) J^i(\omega) \right\rangle_{\text{retarded}} = \Pi(\omega)\delta^{ij} \ , \qquad i, j = 1, 2, 3 \ ,$$

with

$$\Pi(\omega) = \frac{L}{g_B^2}\left[2\frac{\mathcal{B}}{\mathcal{A}} - \frac{\omega^2}{2}\right] \ .$$

Question 12 *Using the known transformation rules for the hypergeometric function one can show that close to the boundary*

$$E = C u^{\left(\frac{1+i}{2}\right)\mathfrak{w}} \frac{\Gamma(1-i\mathfrak{w})}{\Gamma(1-\left[\frac{1+i}{2}\right]\mathfrak{w})\Gamma(1+\left[\frac{1-i}{2}\right]\mathfrak{w})} (2u)^{-\left(\frac{1+i}{2}\right)\mathfrak{w}}$$
$$\left\{1+2u\left(\frac{\mathfrak{w}^2}{2}\right)\left[-\log(2u)+\psi(2)+\psi(1)-\psi\left(1-\left[\frac{1+i}{2}\right]\mathfrak{w}\right)-\psi\left(\left[\frac{1-i}{2}\right]\mathfrak{w}\right)\right]+\mathcal{O}\left(u^2\right)\right\} .$$

$\Gamma(z)$ *is the gamma function, and* $\psi(z)$ *is its logarithmic derivative,* $\psi(z)=\Gamma'(z)/\Gamma(z)$*. Both functions have simple poles at* $-z\in\mathbb{N}_0$*. Show that the retarded correlator is [6]*

$$\Pi(\omega) = -2\frac{L\left(\pi T\right)^2}{g_B^2}\mathfrak{w}^2\left[\psi\left(-\left[\frac{1+i}{2}\right]\mathfrak{w}\right)+\psi\left(\left[\frac{1-i}{2}\right]\mathfrak{w}\right)\right] + \textit{contact terms}$$

and that the QNM spectrum is given by

$$\omega(q=0) = n\left(\pm 1 - i\right) , \qquad n\in\mathbb{N} .$$

Explain why the fact that all QNMs lie in the lower half of the complex plane is necessary for the black-brane background to be stable.

3 Linear Response and AC-Conductivity

3.1 QM Perturbation Theory and Kubo Formulae

Question 13 *Consider adding a time-dependent perturbation $\delta H(t)$ to a static Hamiltonian H_0 at $t \to -\infty$, $H(t) = H_0 + \delta H(t)$. Working in a the interaction picture,*

$$\begin{aligned} |\psi(t)\rangle &= e^{iH_0 t}\, |\psi(t)\rangle_{\mathrm{S}} \ , \\ O(t) &= e^{iH_0 t}\, O_{\mathrm{S}}\, e^{-iH_0 t} \ , \end{aligned}$$

(the subscript S *denotes the Schrödinger picture), show that the density matrix $\rho(t)$ of the system evolves as*

$$\rho(t) = U(t)\rho_0 U^{-1}(t) \ , \qquad U(t) = \mathcal{T}\exp\left(-i\int_{-\infty}^{t} \mathrm{d}t'\, \delta H(t')\right) \ ,$$

where $\mathcal{T}$ is the time-ordering operator. Hence confirm that to linear order in δH

$$\langle O(t)\rangle = \langle O(t)\rangle_{\rho_0} + i\int_{-\infty}^{t} \mathrm{d}t' \left\langle \left[\delta H(t'), O(t)\right]\right\rangle_{\rho_0} + \mathcal{O}\left(\delta H^2\right) \ .$$

Thus show that the linear response of an operator O to an external source $\varphi(x)$,

$$\delta H(t) = -\int \mathrm{d}\underline{x}\, O(t,\underline{x})\varphi(t,\underline{x}) \ ,$$

is captured by the general Kubo formula

$$\delta\left\langle O(t,\underline{x})\right\rangle = -\int \mathrm{d}t\mathrm{d}\underline{x}\, G^R(t-t',\underline{x}-\underline{x}')\varphi(t',\underline{x}') \ , \tag{22}$$

where

$$G^R(t-t',\underline{x}-\underline{x}') = -i\theta(t-t')\left\langle O(t,\underline{x}), O(t',\underline{x}')\right\rangle$$

is the retarded correlator.

3.2 AC-Conductivity

Question 14 *Ohm's law is a simple application of linear response theory. Show that the alternating-current (AC) conductivity $\sigma^{ij}(\omega)$ for a spatially homogeneous electric field*

$$F_{tj}(\omega) = -i\omega A_j(\omega)$$

is given by

$$\sigma^{ij}(\omega) = \frac{\left\langle J^i(-\omega)J^j(\omega)\right\rangle_{\mathrm{retarded}}}{i\omega} \ .$$

This is a specific example of a Kubo formula expressing a macroscopic transport coefficient (σ^{ij}) in terms of retarded correlators of microscopic currents. The R-charge AC-conductivity of neutral $\mathcal{N}=4$ at strong coupling is hence

$$\sigma^{ij}(\omega) = \frac{\Pi(\omega)}{i\omega}\delta^{ij} \ .$$

4 Hydrodynamics and R-Charge Diffusion

Hydrodynamics is an effective description of field theories approaching thermal equilibrium. It is based on two assumptions: Firstly, close to equilibrium, i.e. at late times, only fluctuations of small frequencies and small wave-vectors are expected to have survived. This means that all hydrodynamic fluctuations admit a series expansion in their momenta, or equivalently in gradients in position space. Secondly, the dynamics of a hydrodynamic system are assumed to be entirely captured by the conservation equations for the global charges which will characterise the equilibrium state at $t \to \infty$.

For instance, if we restrict ourselves to hydro fluctuations of a global $U(1)$-charge around a neutral equilibrium, their dynamics are governed by a single equation of motion, namely the conservation of the associated $U(1)$-current $J^\mu = (n, J^i)$, $\partial_\mu J^\mu = 0$. In the hydro regime the system must therefore be described by a single degree of freedom which we can take to be the fluctuation of the charge density n. Following the assumptions of hydrodynamics, it must be possible to write the spatial components of the current in terms of n. To first order in momenta (or gradients), the only expression compatible with Lorentz invariance is

$$J^i(x) = -D\partial^i n(x) + \mathcal{O}\left(\partial^2\right) \ ,$$

known as Fick's law.

4.1 Diffusion

Question 15 *Show that an initial hydro charge fluctuation $n(t=0, \underline{x}) \equiv n_0((\underline{x})$ is diffusing over time $t > 0$,*

$$n(t, \underline{x}) = \int \frac{\mathrm{d}q^3}{(2\pi)^3} e^{-Dq^2 t} n_0(\underline{q}) \ , \tag{23}$$

with diffusion constant D.

Question 16 *As the solution (23) is defined for $t > 0$ only it cannot be Fourier-transformed in time. Applying the Laplace-transform instead,*

$$n(z, \underline{q}) = \int_0^\infty \mathrm{d}t\, e^{izt} n(t, \underline{q}) \ , \qquad \mathrm{Im}(z) > 0 \ , \tag{24}$$

show that

$$n(z, \underline{q}) = \frac{n_0(\underline{q})}{-iz + D\underline{q}^2} \ .$$

We shall now compare the hydro result (24) with the result from the general Kubo formula (22) to obtain the retarded correlator in the hydro regime.

To that end let us prepare the system in the following way: we adiabiatically turn on a source $\mu(t, \underline{q})$ at $t \to -\infty$ resulting in a charge fluctuation $n_0(\underline{q})$ at $t = 0$, switch it off at once at $t = 0$, and let it evolve hydrodynamically for $t > 0$:

$$\mu(t, \underline{q}) = \theta(-t) e^{\epsilon t} \mu_0(\underline{q}) \ .$$

As the system is in thermal equilibrium for $t \leq 0$, $n_0(\underline{q})$ and $\mu_0(\underline{q})$ are simply related by the static susceptibility χ,

$$n_0(\underline{q}) = \chi \mu_0(\underline{q}) \ .$$

Question 17 *Evaluating (22) at $t = 0$ show that the Laplace-transform of the retarded correlator satisfies*

$$G^R(z = i\epsilon, \underline{q}) = -\chi \ .$$

Further show that

$$\langle n(t, \underline{q} \rangle = -\mu_0(\underline{q}) \int \frac{\mathrm{d}\omega}{2\pi} G^R(\omega, \underline{q}) \frac{e^{-i\omega t}}{\epsilon + i\omega} \ ,$$

where

$$G^R(t - t', \underline{q}) = \int \frac{\mathrm{d}\omega}{2\pi} G^R(\omega, \underline{q}) e^{-i\omega(t-t')} \ .$$

Recalling that $G^R(t-t', \underline{q}) = 0$ for $t < t'$, argue that $G^R(\omega, \underline{q})$ must be analytic in the upper-half plane. This implies that Fourier- and Laplace transform are identical in the upper-half plane. Thus show that

$$\langle n(z + i\epsilon, \underline{q}) \rangle = -\frac{G^R(z + i\epsilon, \underline{q}) + \chi}{iz} \mu_0(\underline{q}) \ .$$

Hence conclude that in the hydro regime

$$G^R(\omega, \underline{q}) = -\frac{D\underline{q}^2}{D\underline{q}^2 - i\omega} \chi$$

with a diffusion pole at

$$\omega = -iD\underline{q}^2 \ .$$

4.2 Holographic R-Charge Diffusion

The goal of the final part of this problem set is to compute the R-charge diffusion constant D of large-N, strongly-coupled $\mathcal{N} = 4$ by identifying the hydro diffusion pole in the retarded current-current correlator.

Question 18 *Argue that in the hydro regime the solution to eqs. (12)–(13) can be written as*

$$E_{\alpha/L} = C_{\alpha/L} f(u)^{-i\mathfrak{w}/2} \frac{1}{\mathfrak{w}} \left[\mathfrak{w} + \mathcal{F}^{(0,1)}_{\alpha/L} \mathfrak{q} + \mathcal{F}^{(2,0)}_{\alpha/L} \mathfrak{w}^2 + \mathcal{F}^{(1,1)}_{\alpha/L} \mathfrak{w}\mathfrak{q} + \mathcal{F}^{(0,2)}_{\alpha/L} \mathfrak{q}^2 + \mathcal{O}\left(\partial^3\right) \right] ,$$

where the $\mathcal{F}^{(n,m)}_{\alpha/L}$ are regular at the horizon and vanish at the boundary. Solving eqs. (12)–(13) order by order in $\mathfrak{w}$, $\mathfrak{q}$ subject to these boundary conditions yields [7]

$$\begin{aligned}
\mathcal{F}^{(0,1)}_{\alpha} = \mathcal{F}^{(1,1)}_{\alpha} = \mathcal{F}^{(0,2)}_{\alpha} &= 0 , \\
\mathcal{F}^{(2,0)}_{\alpha} &= i \log \frac{1+u}{2} , \\
\mathcal{F}^{(0,1)}_{L} = \mathcal{F}^{(1,1)}_{L} &= 0 , \\
\mathcal{F}^{(2,0)}_{L} &= i \log \frac{1+u}{2} , \\
\mathcal{F}^{(0,2)}_{L} &= \frac{i}{\mathfrak{w}} (1-u) .
\end{aligned}$$

Thus show that the transverse current-current correlator is analytic in the hydro regime, while the longitudinal current-current correlator has a diffusion pole. Confirm that the R-charge diffusion constant of strongly-coupled $\mathcal{N}=4$ at $N \to \infty$ is given by [8]

$$D = \frac{1}{2\pi T} .$$

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