University of Oxford
Department of Physics

Oxford Master Course in Mathematica and Theoretical Physics

INTRODUCTION TO GAUGE-STRING DUALITY

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Problem Set IV

The BTZ Black Hole and Causal CFT_2 Correlators

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Consider a free scalar field Φ with mass m in the $(2+1)$-dimensional non-extremal BTZ black-hole background. It is dual to a scalar operator in a strongly-coupled (1+1)-dimensional CFT with large central charge.

BTZ Geometry

The non-extremal BTZ black hole is a solution to Einstein's equations in $(2+1)$-dimensions with negative cosmological constant $\Lambda = -1/L^2$. The standard form of the metric is

$$\mathrm{d}s^2 = \frac{1}{\mathcal{F}(\rho)}\mathrm{d}\rho^2 - \mathcal{F}(\rho)\mathrm{d}\tau^2 + \rho^2\left(\mathrm{d}\theta - \frac{\rho_+\rho_-}{L\rho^2}\mathrm{d}\tau\right)^2 ,$$

where

$$\mathcal{F}(\rho) = \frac{\left(\rho^2 - \rho_+^2\right)\left(\rho^2 - \rho_-^2\right)}{L^2\rho^2} ,$$

with radial coordinate $\rho \in \mathbb{R}^+$, angle $\theta \sim \theta + 2\pi$, and time coordinate $\tau \in \mathbb{R}$. The metric has an inner/outer horizon at ρ_-/ρ_+. For our purposes it is convenient to introduce coordinates $r \in (\rho_+, \infty)$, $t \in \mathbb{R}$, $x \in \mathbb{R}$, defined by

$$\begin{aligned} \rho^2 &= \left(\rho_+^2 - \rho_-^2\right)\left(\frac{r}{r_H}\right)^2 + \rho_-^2 , \\ t &= \frac{\rho_+}{r_H}\tau - \frac{\rho_-}{r_H}L\,\theta , \\ x &= -\frac{\rho_-}{r_H}\tau + \frac{\rho_+}{r_H}L\,\theta , \end{aligned}$$

that cover the region outside the outer horizon $\rho > \rho_H$. In terms of (t, x, r) the BTZ metric reads

$$\mathrm{d}s^2 = \frac{\mathrm{d}r^2}{F(r)} - F(r)\mathrm{d}t^2 + \frac{r^2}{L^2}\mathrm{d}x^2$$

with

$$F(r) = \frac{r^2}{L^2}\left(1 - \frac{r_H^2}{r^2}\right) .$$

The outer horizon is at $r = r_H$.

Close to the boundary $r \to \infty$ the metric reduces to AdS_3 with AdS-radius L,

$$\mathrm{d}s^2 \to L^2\frac{\mathrm{d}r^2}{r^2} + \frac{r^2}{L^2}\left[-\mathrm{d}t^2 + \mathrm{d}x^2\right] ,$$

where $\left[-\mathrm{d}t^2 + \mathrm{d}x^2\right]$ is the flat metric of the dual $(1+1)$-dimensional CFT that lives on the boundary of the $(2+1)$-dimensional BTZ bulk geometry.

Question 1 *Confirm that the temperature of the BTZ black hole is given by*

$$T = \frac{1}{2\pi}\frac{r_H}{L^2} .$$

According to the AdS/CFT dictionary the BTZ bulk geometry is therefore dual to a $(1+1)$*-dimensional CFT in thermal equilibrium at the same temperature* T*.*

Question 2 *In a CFT, i.e. a scale-invariant field theory, at fine temperature T, T is the only energy scale of the system. Hence all field theory quantities can be expressed in units of $2\pi T = r_H/L^2$. This is achieved by rescaling the bulk coordinates as*

$$\frac{r_H}{L^2}t \to t\ , \qquad \frac{r_H}{L^2}x \to x\ , \qquad \frac{r}{r_H} \to \frac{1}{z}\ .$$

The new dimensionless radial coordinate z was introduced for convenience. Show that this coordinate transformation brings the metric into the form

$$\mathrm{d}s^2 = g_{mn}\mathrm{d}x^m\mathrm{d}x^n = \frac{L^2}{z^2}\left(\frac{\mathrm{d}z^2}{f(z)} - f(z)\mathrm{d}t^2 + \mathrm{d}x^2\right)\ , \tag{1}$$

with

$$f(z) = 1 - z^2\ .$$

The Klein-Gordon Equation

Now consider a massive scalar ϕ in the fixed BTZ background with action

$$S = \eta\int \mathrm{d}^3x\sqrt{-g}\left(g^{mn}\nabla_m\phi\nabla_n\phi + m^2\phi^2\right)\ .$$

Question 3 *Show that the Fourier transform*

$$\varphi(z,k) = \int \mathrm{d}t\mathrm{d}x e^{i(\omega t - qx)}\phi(z,t,x)$$

with momentum $k^\mu = (\omega, q)$ satisfies the equation of motion

$$\varphi'' - \frac{1+z^2}{zf}\varphi' + \left(\frac{\omega^2}{f^2} - \frac{q^2}{f} - \frac{m^2L^2}{z^2f}\right)\varphi = 0\ , \tag{2}$$

where dashes denote derivatives with respect to z.

From now on we shall set the AdS-radius to $L = 1$ (or equivalently let $mL \to m$), bearing in mind that all bulk couplings such as m are measured in units of L.

Near-Boundary and Near-Horizon Analysis

A second-order ordinary differential equation (ODE) for the dependent variable y and independent variable x has a regular singular point (RSP) at x_0 if it takes the form

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{p(x)}{x-x_0}\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{q(x)}{(x-x_0)^2}y = 0\ , \tag{3}$$

where $p(x)$ and $q(x)$ are both regular at $x = x_0$, i.e. locally they admit a Taylor series expansion around x_0 :

$$p(x) = \sum_{n=0}^{\infty} p_n\left(x-x_0\right)^n\ , \qquad q(x) = \sum_{n=0}^{\infty} q_n\left(x-x_0\right)^n\ .$$

Question 4 *Show that boundary and horizon are RSPs of the scalar equation of motion* (2).

Question 5 *Show that*

$$y(x) = (x - x_0)^\alpha \sum_{n=0}^{\infty} c_n \, (x - x_0)^n \ , \qquad\qquad c_0 \neq 0 \ ,$$

is a solution of (3) *in the neighbourhood of* x_0 *if the following two conditions are satisfied:*

- $\alpha = \alpha_+$ *is given by the larger of the two roots* $\{\alpha_+, \alpha_-\}$*,* $\mathrm{Re}(\alpha_+) \geq \mathrm{Re}(\alpha_-)$*, of the* indicial polynomial $P(\alpha) := \alpha^2 + (p_0 - 1)\,\alpha + q_0$.
- *The coefficients* $b_{n\geq 1}$ *are fixed in terms of* $b_0 =: B$ *by the recursive relation*

$$c_n(\alpha) = -\frac{1}{P(\alpha+n)} \sum_{k=0}^{n-1} \left[(\alpha + k)\, p_{n-k} + q_{n-k} \right] c_k(\alpha) \ ,$$

where $\alpha = \alpha_+$*.*

If $\alpha_+ - \alpha_- \notin \mathbb{N}_0$ the same procedure yields a second independent solution with $\alpha = \alpha_-$ (what goes wrong if $\alpha_+ - \alpha_- \in \mathbb{N}_0$?), so that the general solution in the neighbourhood of x_0 is

$$y = A\,(x - x_0)^{\alpha_-} \left[1 + \sum_{n\geq 1} a_n \,(x - x_0)^n \right] + B\,(x - x_0)^{\alpha_+} \left[1 + \sum_{n\geq 1} b_n \,(x - x_0)^n \right]$$

with

$$a_n = \frac{c_n(\alpha_-)}{c_0} \ , \qquad\qquad b_n = \frac{c_n(\alpha_+)}{c_0} \ .$$

For completeness let us state the form of the general local solution in the case $\alpha_+ - \alpha_- \in \mathbb{N}_0$ (see e.g. [2]). For $\alpha \equiv \alpha_+ = \alpha_-$ one obtains

$$y = A\,(x - x_0)^{\alpha} \log(x - x_0) \left[1 + \sum_{n\geq 1} a_n \,(x - x_0) \right] + B\,(x - x_0)^{\alpha} \left[1 + \sum_{n\geq 1} b_n \,(x - x_0) \right]$$

where the $a_{n\geq 1}$ are completely determined by A, and the $b_{n\geq 1}$ are completely determined by A and B. For $\alpha_+ - \alpha_- \in \mathbb{N}$ one obtains

$$y = A \left\{ (x - x_0)^{\alpha_-} \left[1 + \sum_{n\geq 1} a_n \,(x - x_0)^n \right] + (x - x_0)^{\alpha_+} \log(x - x_0) \sum_{n\geq 0} \tilde{a}_n \,(x - x_0)^n \right\}$$
$$+ B\,(x - x_0)^{\alpha_+} \left[1 + \sum_{n\geq 1} b_n \,(x - x_0)^n \right]$$

where the $a_{n\geq 1}$ and $\tilde{a}_{n\geq 0}$ are completely determined by A, and the $b_{n\geq 1}$ are completely determined by A and B. In summary, as the local solution is completely determined by A and B it is often written in the reduced form

$$y = A\,(x-x_0)^{\alpha_-} + \cdots + B\,(x-x_0)^{\alpha_+} + \ldots\ ,\ \alpha_+ \neq \alpha_-$$
$$y = A\,(x-x_0)^{\alpha}\log(x-x_0) + B\,(x-x_0)^{\alpha} + \ldots\ ,\ \alpha \equiv \alpha_+ = \alpha_-\ ,$$

only indicating the two independent *modes* with coefficients A and B respectively. The radius of convergence of such a local solution is given by the distance to the closest singularity of $p(x)$ or $q(x)$ in the complex plane.

Question 6 *Show that the local solution of* (2) *near the boundary is*

$$\varphi = A z^{d-\Delta_+} + \cdots + B z^{\Delta_+} + \ldots\ ,$$

where Δ_+ is the larger root of

$$\Delta\,(\Delta - d) = m^2$$

and where $d = 2$. $A\,z^{d-\Delta_+}$ is called the leading mode *of φ and is interpreted as source of the dual operator $\mathcal{O}$.*
Further show that the local solution of (2) *near the horizon is*

$$\varphi = A_{\text{in}}\,(1-z)^{-i\omega/2} + \cdots + A_{\text{out}}\,(1-z)^{i\omega/2} + \ldots$$

By switching to a new radial coordinate $\rho = -\frac{1}{2}\log(1-z)$, show that $A_{\text{in/out}}\,(1-z)^{\mp i\omega/2}$ describe incoming and outgoing waves at the horizon respectively, that is, waves that propagate towards the inside and the outside of the horizon respectively.

Causal Correlators from AdS/CFT

We have sketched how to construct a local solution to a second-order ODE near a RSP, but this does not tell us how to obtain the global solution which is valid everywhere in the complex plane. Yet we can make the following observation: If the two circles of convergence of two local solutions around two RSPs x_0 and x_0' overlap, then the very existence of a global solution implies that it must be possible to transform these local solutions into one another. In particular, the solution to this so-called *connection problem* must provide a unique relation between the parameter pairs (A, B) and (A', B') of the two local solutions around x_0 and x_0' that descend from the same global solution.

Let us now put this into the context of gauge-/gravity duality. In the appropriate limits of AdS/CFT, the generating function W for a dimension-Δ_+ operator $\mathcal{O}$ with source A in the flat CFT_d is given by the on-shell action S of the dual gravity theory in asymptotically AdS_{d+1} containing a scalar $\Phi = Az^{d-\Delta_+} + \ldots$:

$$W = \left\langle \int \mathrm{d}^d x \mathcal{O}(x) A(x) \right\rangle = \left\langle \int \frac{\mathrm{d}^d k}{(2\pi)^d} \mathcal{O}(-k) A(k) \right\rangle = S\ .$$

It turns out that the 1-point function (expectation value) of $\mathcal{O}$ in the presence of the source A is given by Φ's *sub-leading mode* B (see the optional question 9):

$$\langle \mathcal{O}(k) \rangle_A = (2\pi)^d \frac{\delta W}{\delta A(-k)} = -2\eta \, (2\Delta_+ - d) \, B(A) \ . \tag{3}$$

The sub-leading mode B is a function of the source A as causality forces one to impose a certain boundary condition on Φ at the horizon: in order to obtain the retarded response of $\langle \mathcal{O} \rangle$ to turning on the source A in the field theory we must demand that the bulk field Φ represent an incoming wave at the horizon, meaning that no information comes out of the horizon.

The *retarded correlator* in the source-free vacuum $A = 0$ is then finally given by

$$\langle \mathcal{O}(k_1)\mathcal{O}(k_2) \rangle_{\text{retarded}} = (2\pi)^{2d} \left. \frac{\delta^2 W}{\delta A(k_1) \delta A(k_2)} \right|_{A=0}$$

or, inferring translational invariance,

$$\begin{aligned} \langle \mathcal{O}(-k)\mathcal{O}(k) \rangle_{\text{retarded}} &= (2\pi)^{d} \left. \frac{\delta^2 W}{\delta A(-k) \delta A(k)} \right|_{A=0} \\ &= -2\eta \, (2\Delta_+ - d) \left. \frac{\delta B(A)}{\delta A(k)} \right|_{A=0} \end{aligned}$$

To compute correlators in AdS/CFT we must therefore solve the connection problem relating the local solution near the boundary, $\Phi = A z^{d-\Delta_+} + \dots + B z^{\Delta_+} + \dots$, to the local solution near the horizon, $\Phi = A_{\text{in}} \, (1-z)^{-i\omega/d} + \dots + A_{\text{out}} \, (1-z)^{i\omega/d} + \dots$, in order to find A_{out} in terms of A and B. Setting $A_{\text{out}} = 0$ to zero then yields the desired relation $B(A)$. Usually, the solution to the connection problem can only be obtained numerically, but the BTZ geometry is one of the very few cases in which it can be obtained analytically.

To that end note that changing the independent and dependent variables in (2) to u and w defined by

$$u = z^2 \ , \qquad \varphi = u^{(2-\Delta_+)/2} \, (1-u)^{-i\omega/2} \, w(u) \ ,$$

transforms (2) into the *hypergeometric equation*

$$u \, (1-u) \, \frac{\mathrm{d}^2 w}{\mathrm{d}u^2} + [c - (a+b+1) \, u] \, \frac{\mathrm{d}w}{\mathrm{d}u} - a \, b \, w = 0 \tag{4}$$

with

$$a = \frac{2-\Delta_+}{2} - \frac{i}{2} \, (\omega + q) \ , \qquad b = \frac{2-\Delta_+}{2} - \frac{i}{2} \, (\omega - q) \ , \qquad c = 2 - \Delta_+ \ .$$

Assuming that $c \notin \mathbb{N}_0$, the most general solution to (4) is given by

$$w(u) = A \, {}_2F_1(a, b; c; u) + B \, u^{1-c} \, {}_2F_1(a - c + 1, b - c + 1; 2 - c; u) \ ,$$

where ${}_2F_1(a, b; c; u)$ is the Gaussian hypergeometric function. Near $u = 0$ it is represented by the hypergeometric series whose first two terms are

$${}_2F_1(a, b; c; u) = 1 + \frac{a \, b}{c} u + \mathcal{O}(u^2)$$

and whose radius of convergence is $|u| = 1$.

Question 7 *Assuming that* $\sqrt{1+m^2} \notin \mathbb{N}_0$*, confirm that* A *and* B *are the coefficients of* φ*'s leading and sub-leading mode respectively.*

Question 8 *Use the transformation rule*

$$
\begin{aligned}
{}_2F_1(a,b;c;u) = &\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\,{}_2F_1(a,b;a+b-c+1;1-u) \\
&+ (1-u)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\,{}_2F_1(c-a,c-b;c-a-b+1;1-u)\ ,
\end{aligned}
$$

valid if $c-a-b \notin \mathbb{N}_0$*, to obtain the series expansion of* φ *near the horizon. In particular, find* $A_{\text{in/out}}$ *in terms of* A*,* B*.*
Impose the incoming-wave boundary condition $A_{\text{out}} = 0$ *to obtain* $B(A)$ *and to determine the retarded correlator* $\langle \mathcal{O}(-k)\mathcal{O}(k)\rangle_{\text{retarded}}$*.*

[Optional] Holographic 1-Point Function and Renormalisation

We shall now show how the formula (3) for the holographic 1-point function of a scalar operator is obtained in the simplest example. Recall that the AdS/CFT correspondence states that the on-shell action of a gravitational theory in asymptotically AdS_{d+1} with a scalar field $\varphi(z,k) = A(k)z^{d-\Delta_+} + \dots$ equals the generating functional W of a dimension-Δ_+ scalar operator $\mathcal{O}$ with source A in the dual CFT_d "living on the boundary" $z \to 0$. For simplicity, let us restrict ourselves to a CFT_d in flat space. The expectation value of $\mathcal{O}$ in the presence of the source A is given by

$$
\langle \mathcal{O}(-k)\rangle_A = (2\pi)^d \frac{\delta W}{\delta A(k)} = (2\pi)^d \lim_{z_0\to 0}\left(\frac{\delta\left(S^{\text{on-shell}}(z_0)+S_{\text{ct}}\right)}{z_0^{\Delta_+-d}\,\delta\varphi(z_0,k)}\right) .
$$

$S^{\text{on-shell}}(z)$ denotes the on-shell action, S_{ct} contains counterterms that remove the UV-divergencies in the QFT correlators. S_{ct} is a functional of the field $\varphi(z_0,k)$ on the near-boundary cut-off $z = z_0 \to 0$ and is essentially determined by the requirement that it remove all divergencies and be covariant. In the simplest case of a scalar satisfying

$$
\nu := \Delta_+ - \frac{d}{2} = \sqrt{\left(\frac{d}{2}\right)^2 + m^2} < 1\ ,
$$

the counterterms are simply

$$
\begin{aligned}
S_{\text{ct}} &= \eta\,(d-\Delta_+)\int \mathrm{d}^d x\, z_0^{-d}\phi(z_0,x)^2 \\
&= \eta\,(d-\Delta_+)\int \frac{\mathrm{d}^d k}{(2\pi)^d} z_0^{-d}\varphi(z_0,-k)\varphi(z_0,k)\ .
\end{aligned}
$$

The variation of $S^{\text{on-shell}}(z)$ with respect to $\varphi(z_0,k)$ is most easily evaluated by making use of the following observation. Consider an action

$$
S(z_0) = \int_{z_0}^{z_H} L(\varphi,\varphi')
$$

with Lagrangian L. The variation of S is

$$\delta S(z_0) = \int\limits_{z_0}^{z_H} \left(\frac{\partial L}{\partial \varphi} \delta\varphi + \frac{\partial L}{\partial \varphi'} \delta\varphi' \right)$$
$$= \int\limits_{z_0}^{z_H} \delta\varphi \left(\frac{\partial L}{\partial \varphi} - \frac{\mathrm{d}}{\mathrm{d}z} \left(\frac{\partial L}{\partial \varphi'} \right) \right) + \delta\varphi \frac{\partial L}{\partial \varphi'} \bigg|_{z_0}^{z_H} \; .$$

The variation of the on-shell action with respect the field $\varphi(z_0, k)$ on the boundary cut-off z_0 is therefore simply given by the on-shell canonical momentum at z_0:

$$\frac{\delta S^{\mathrm{on-shell}}(z_0)}{\delta \varphi(z_0)} = -\frac{\partial L}{\partial \varphi'}(z_0) \; .$$

In our case, the action is simply

$$S = \eta \int\limits_{z_0}^{z_H} \mathrm{d}z \int \frac{\mathrm{d}^2 k}{(2\pi)^d} \sqrt{-g} \left(g^{zz} \partial_z \varphi(-k) \partial_z \varphi(k) + \left[g^{\mu\nu} k_\mu k_\nu + m^2 \right] \varphi(-k) \varphi(k) \right) \; ,$$

giving

$$(2\pi)^d \frac{\delta S^{\mathrm{on-shell}}}{\delta \varphi(z_0, k)} = -\,(2\pi)^d \frac{\partial L}{\partial \varphi'} = -2\eta \sqrt{-g} g^{zz} \varphi'(z_0, -k) \; .$$

[Optional] Question 9 *Show that for $\nu < 1$ and $d = 2$ one indeed obtains*

$$\langle \mathcal{O}(-k) \rangle_A = -2\eta \, (2\Delta_+ - 2) \, B(A) \; .$$

References

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