University of Oxford
Department of Physics

Oxford Master Course in Mathematical and Theoretical Physics

Introduction to Gauge-String Duality

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Problem Set II

Elements of General Relativity (continued)

- **Anti - de Sitter space**

  In Poincaré coordinates, the AdS metric is given by the line element

$$ds^2 = \frac{dz^2 + dx_\mu dx^\mu}{z^2} \qquad , \qquad x^\mu = \left(x^0, \dots, x^{d-2}\right) . \tag{1}$$

In this exercise, for x^μ we consider the signature $x \cdot x = -x_0^2 + x_1^2 + \dots + x_{d-2}^2$.

Let us now consider something (naively) completely different: flat $R^{2,d-1}$ spacetime with signature $(-, -, +, ..., +)$ and coordinates $Y_{-1}, Y_0, \dots, Y_{d-1}$. Let us embed a hyperboloid in this space-time as the set of points $\vec{Y}$ such that

$$\vec{Y} \cdot \vec{Y} \equiv -Y_{-1}^2 - Y_0^2 + Y_1^2 + Y_2^2 + \dots + Y_{d-1}^2 = -1 . \tag{2}$$

We shall see that this hyperboloid is also a description of AdS.

1. Draw a two-dimensional sphere S^2 in the three dimensional Euclidean space R^3.
2. Draw the two-dimensional AdS_2 in the three dimensional $R^{2,1}$.
3. Verify that the following parametrization, known as the Poincaré coordinates, solves the Hyperboloid constraint (2)

$$\begin{aligned} Y_{-1} &= \frac{1 + z^2 + x_\mu x^\mu}{2z} \\ Y_\mu &= \frac{x_\mu}{z}, \qquad \mu = 0, \dots, d-2 \\ Y_{d-1} &= \frac{1 - z^2 - x_\mu x^\mu}{2z} . \end{aligned} \tag{3}$$

4. Compute the line element $ds^2 = d\vec{Y} \cdot d\vec{Y}$ in this parametrization. You should find (1).

5. Note that $Y_{-1} + Y_{d-1} = \frac{1}{z} > 0$ in this parametrization. Therefore, the Poincaré coordinates do not cover the full hyperboloid (2). Instead they cover only a patch of it denoted as the *Poincaré patch*. Represent the Poincaré patch in the two-dimensional hyperboloid drawn above. You can check your result by running

```
ContourPlot3D[x^2+y^2-z^2==1,{x,-4,4},{y,-4,4},{z,-4,4},ContourStyle->Blue
Mesh->False]~Show~ContourPlot3D[x^2+y^2-z^2==1.1,{x,-4,4},{y,-4,4},{z,-4,4},
RegionFunction->Function[{x,y,z},x+z<0],ContourStyle->Orange,Mesh->False]
```

6. Another very useful parametrization of the embedding coordinates Y_a is

$$\begin{aligned} Y_{-1} &= \cosh\rho\cos t \\ Y_0 &= \cosh\rho\sin t \\ Y_i &= \sinh\rho\,\Omega_i\,, \qquad i = 1,\ldots,d-1 \end{aligned} \tag{4}$$

where $\Omega^2 = 1$, that is Ω parametrizes a S^{d-2} sphere. Verify that (4) does solve the constraint (2). Note that this parametrization does cover the full hyperboloid.

7. Show that the line element in this parametrization reads

$$ds^2 = -\cosh^2\rho\,dt^2 + d\rho^2 + \sinh^2\rho\,d\Omega^2_{S^{d-2}}\,. \tag{5}$$

These are the so called global coordinates. We also encountered them in the lectures. The boundary is $R \times S^{d-2}$.

8. Given the parametrization (4) we have $t \sim t + 2\pi$. From the point of view of (5) this is physically quite unpleasant. Why?

9. There is a simple solution to the problem found in the previous point. We simply do *not* identify t and $t + 2\pi$. That is, the space (4) will cover the hyperboloid infinitely many times. Represent this many wrapping structure in the two dimensional hyperboloid drawn above. This is called the universal covering. To check your result you can run

```
ParametricPlot3D[{Cosh[r]Cos[t]Exp[t/100],Sinh[r],Cosh[r]Sin[t]Exp[t/100]},
{t,0,6\[Pi]},{r,-2,2},PlotPoints->100,BoundaryStyle->Directive[Black,Thick]]
```

10. From now on we consider $d > 2$. Consider the change of variables $\tan\frac{r}{2} = \tanh\frac{\rho}{2}$. Note that $r \in [0, \pi/2]$ for $\rho \in [0, \infty]$. Verify that

$$ds^2 = \sec^2 r\left[-dt^2 + dr^2 + \sin^2 r\,d\Omega^2_{S^{d-2}}\right] \tag{6}$$

11. We can drop the $\sec^2 r$ prefactor in the previous metric to study its causal structure, that is its Penrose diagram. Consider $d = 3$, that is AdS_3 space time.

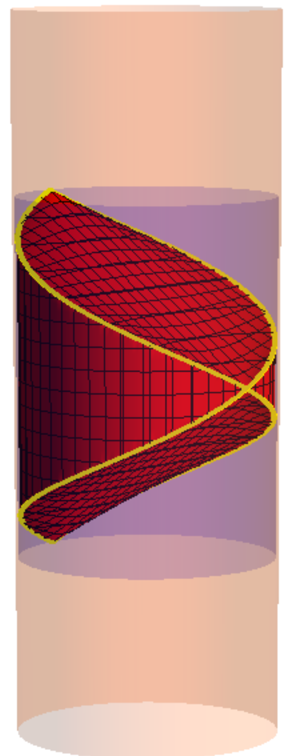

Figure 1: Penrose diagram representation for AdS in global coordinates.

Then $d\Omega^2_{S^1} = d\phi^2$. We see that this spacetime can be nicely represented as a cylinder. Draw this cylinder. Draw also the region corresponding to a single cover of the hyperboloid.

12. The red region in figure 1 represents a Poincaré patch. Find the shape of the yellow curves. How would you go about finding the precise form of the red region? (An explanation is enough)

13. Consider a light ray sent from the middle of global AdS, that is for the center of the cylinder at $\rho = 0$. Suppose it bounces back when it reaches the boundary. How long does it take to get back to the center of AdS, $\Delta t =$?

14. Geodesics of a massive particle can be found by minimizing $L = \int d\tau g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$. Write down the resulting equations of motion in global coordinates. Consider a geodesic going through the origin with some velocity. Plot its trajectory inside the cylinder.

- **(Extremal) Reissner-Nordström black holes**

  To give a black hole a charge we need to couple gravity with a gauge field (here we shall consider Abelian gauge fields but generalization is possible). This coupling, in four dimensions, is given by the *Einstein-Maxwell action*[1]

$$S_{EM} = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} \left(R - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \,. \tag{7}$$

a) Write the Einstein's equations of motion $\delta S_{EM}/\delta g^{\mu\nu} = 0$ resulting from this action and the Maxwell's equations of motion $\delta S_{EM}/\delta A^\mu = 0$.

b) From the equations of motion compute the scalar curvature R.

c) Now consider the following ansatz for the metric and gauge field:

$$\begin{aligned} ds^2 &= -\frac{1}{H(r)^2} dt^2 + H(r)^2 \left(dr^2 + r^2 d\Omega_2^2 \right) , \\ F &= -\alpha \, dt \wedge d \left(\frac{1}{H(r)} \right) . \end{aligned}$$

Here α is some constant to be determined. If you are not familiar with the form notation for F, convince yourself that this notation means that the field strength $F = \frac{1}{2} F_{\mu\nu} \, dx^\mu \wedge dx^\nu$ has only two non-vanishing components:

$$F_{tr} = -F_{rt} = \alpha \frac{1}{H(r)^2} \frac{\partial H(r)}{\partial r} \,. \tag{8}$$

In particular, this means that the gauge field we are considering is an electric field. Substitute the ansatz above in the equations of motion (you may use `Mathematica`) and find the correct value of the constant α and the equation that $H(r)$ must satisfy in order for the ansatz to be a solution of the equations of motion.

d) What is the general solution of the equation for $H(r)$ that you have found?

e) In the most general solution there are two integration constants (because the equation is second order). Fix one of them by requiring asymptotic flatness (that is, at $r \to \infty$ the metric should be Minkowski). The second integration constant should be fixed by requiring that the black hole has charge Q. This is done by computing the flux threading the S^2 of the dual field strength:

$$Q \equiv \int_{S^2} \star F \,, \tag{9}$$

where $\star F$ is the Hodge dual of F. This is oriented along the S^2 (unlike F which in our ansatz has legs along the t and r directions, but not along the S^2 directions). The explicit definition of the Hodge dual of a form $F_{\mu\nu}$ is

$$\star F_{\rho\sigma} = \frac{1}{2} \sqrt{-g} \, \epsilon_{\rho\sigma}{}^{\mu\nu} F_{\mu\nu} \,, \tag{10}$$

[1] Here we are rescaling the gauge field so that the $1/(16\pi G_N)$ factor becomes an overall factor.

where $\epsilon_{\rho\sigma\mu\nu}$ is the totally antisymmetric tensor ($\epsilon_{0123} = +1$, with even permutation of the indices being equal to $+1$, odd permutations being equal to -1, and repeated indices being equal to 0).

f) Can you superimpose different solutions? What does this represent?

g) Take the so-called *near-horizon limit*, $r \to 0$. Show that the resulting metric is $AdS_2 \times S^2$. Write down the relation between the radius of curvature of the space L (both AdS_2 and S^2 have the same radius L) and the charge Q.

h) Do you know how to add a magnetic field? (No need to repeat the whole computation, it is enough to write down how you would modify the ansatz for F).

- **Generic RN black holes**

  What we have seen in the previous exercise is in fact a very special case of RN black hole, the case in which the mass M and the charge Q of the black hole are the same (this is why it was called "extremal").

  Consider the more general RN black hole

$$
\begin{aligned}
ds^2 &= -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_2^2\,, \\
F &= \alpha \frac{Q}{r^2} dt \wedge dr\,.
\end{aligned}
$$

  a) Using `Mathematica` check that this is a solution to the equations of motion that you have obtained in the previous problem. In particular determine what α is.

  b) Find the position of the horizons $r_\pm$. What do we need to assume about M and Q? Compute the temperature of the black hole (using the outer horizon $r = r_+$). What happens when $M = Q$?

  c) Compute the appropriate curvature invariant to understand the nature of the singularities at $r = r_\pm$ and at $r = 0$.