University of Oxford
Department of Physics

Oxford Master Course in Mathematical and Theoretical Physics

Introduction to Gauge-String Duality

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Problem Set I

Elements of General Relativity

- **Einstein's field equations**

  a) Show that Einstein's field equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \frac{8\pi G_D}{c^4} T_{\mu\nu}$ can be written for $D = 4$ in the form
$$R_{\mu\nu} = \frac{8\pi G_4}{c^4} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) ,$$
  where T is the trace of the energy-momentum tensor. Generalize to arbitrary D. Repeat the exercise for the case of a non-vanishing cosmological constant, when the equations of motion read $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G_D}{c^4} T_{\mu\nu}$.

  b) Determine the dimensions of G_D for arbitrary D.

  c) Consider $D = 4$. Use dimensional analysis to build combinations of $G_4 = 6.674 \times 10^{-11} \mathrm{m}^3/\mathrm{kg} \cdot \mathrm{s}^2$, $\hbar = 1.055 \times 10^{-34} \mathrm{kg} \cdot \mathrm{m}^2/\mathrm{s}$ and $c = 2.998 \times 10^8 \mathrm{m/s}$ which have dimensions of length, time, and mass. These quantities are called Planck length, Planck time, and Planck mass. Discuss their physical meaning.

  d) Higher derivative terms: in generalizations of Einsten's theory (e.g. in string theory), one adds higher derivative terms such as $\alpha R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}$ to the Einstein-Hilbert action in a systematic way. What is the dimension of α and similar coefficients? Can one detect physical effects due to the presence of such terms on the scales currently used to test GR?

  e) Verify that the Schwarzschild metric[1]
$$ds^2 = - \left(1 - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{2M}{r}} + r^2 d\Omega_2^2$$
  is a solution of the vacuum Einstein's equations. Compute the Kretschmann invariant $K = R_{\mu\nu\lambda\sigma} R^{\mu\nu\lambda\sigma}$ for this metric.

[1] Here and in other formulas below we have set $G_4 = 1$ and $c = 1$.

- **Einstein's field equations (continued)**

  a) Derive the Einstein's field equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 8\pi G_D T_{\mu\nu}$$

  from the action (classical Einstein-Hilbert gravity in D dimensions coupled to matter)

$$S_{EH} = \frac{1}{16\pi G_D} \int d^D x \sqrt{-g}\,(R - 2\Lambda) + \int d^D x\, \mathcal{L}_{matter}$$

  by computing the variation $\delta S_{EH}/\delta g^{\mu\nu} = 0$.

  Note: The variation of the matter action gives, by definition, the matter energy-momentum tensor

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}} \frac{\delta \mathcal{L}_{matter}}{\delta g^{\mu\nu}} \,.$$

  Note: There are two tricky parts in this computation: the first one is to realize that you should write $R = g^{\mu\nu} R_{\mu\nu}$ and that the variation $\delta R_{\mu\nu}$ actually gives a boundary term that you can neglect (you may assume this part of the computation without proving it). The second tricky part is to find out what $\frac{\delta}{\delta g^{\mu\nu}}\sqrt{-g}$ is. Hint: Use the identity $\ln(\det M) = Tr(\ln M)$ valid for any matrix M (you may want to think how to prove this useful identity).

- **Electromagnetism in curved spacetime**

  a) In curved space-time, the electromagnetic field strength tensor $F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu$ satisfies Maxwell's equations: $\nabla_\mu F^{\mu\nu} = -J^\nu$. Check that the covariant derivatives in the definition of the field strength can actually be replaced by usual partial derivatives.

  b) The energy-momentum tensor of the electromagnetic field in $D = 4$ is

$$T_{\mu\nu} = F_{\mu\sigma} F_\nu{}^\sigma - \frac{1}{4} g_{\mu\nu} F_{\sigma\tau} F^{\sigma\tau} \,.$$

  Show that in the absence of charged matter

  i) $\nabla_\mu F_{\nu\rho} + \nabla_\rho F_{\mu\nu} + \nabla_\nu F_{\rho\mu} = 0$ (Bianchi identities).

  ii) $\nabla_\mu T^{\mu\nu} = 0\,.$

  iii) $T^\mu_\mu = 0\,.$

  How would the above change in arbitrary dimension?

  c) Prove that Maxwell's equations can also be written as

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} F^{\mu\nu}) = -J^\nu \,.$$

- **AdS black holes in 5 dimensions**

  a) Verify that the following 5-dimensional metric

$$ds^2 = -\left(1 + \frac{r^2}{L^2} - \frac{r_0^4}{L^2 r^2}\right) dt^2 + \left(1 + \frac{r^2}{L^2} - \frac{r_0^4}{L^2 r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2$$

  is a solution of the Einstein's equations with cosmological constant $\Lambda = -6/L^2$. This solution is called an *AdS-Schwarzschild black hole*.

  b) Find the position r_H of the horizon of this black hole and compute its Hawking temperature. Plot $1/T$, the inverse temperature, on the vertical axis versus the position of the horizon r_H on the horizontal axis. Discuss this plot. In particular, can you understand the terminology "large" and "small" AdS black holes?

  Useful (optional) reference:

  S. W. Hawking and D.N. Page, "Thermodynamics of Black Holes in anti-De Sitter Space," Commun. Math. Phys. **87**, *577 (1983).*

- **Optional problems**

  a) Gibbons-Hawking term: discuss applying the variational principle to the Einstein-Hilbert action on a manifold with boundary. Argue that a boundary term should be added to the action to make the variational principle well defined (you may consider a simple classical mechanics model which captures the relevant features).

  You may want to look at the discussion in hep-th/0406264.

  b) Ostrogradsky instability: generically, higher derivative terms in the gravitational action can be treated only as small perturbations. Investigate (using e.g. a simple classical mechanics model) what happens if this requirement is relaxed.

  You may want to consult articles such as R.P. Woodard, "Ostrogradsky's theorem on Hamiltonian instability," arXiv:1506.02210 [hep-th].