Elements of relativistic hydrodynamics

Hydrodynamics describes small deviations from thermal equilibrium. The equilibrium state is characterized by the density operator

$$\hat{\rho} = \frac{1}{Z} e^{\beta_\mu \hat{P}^\mu + \gamma_A \hat{Q}_A},$$

where $\hat{P}^\mu$ is the 4-momentum, $\hat{Q}_A$ are conserved charges (baryon etc), and β_μ, γ_A parametrize the manifold of equilibrium states: $\beta_\mu = \beta u_\mu$, $\gamma_A = \beta \mu_A$, $u_\mu u^\mu = -c^2$, $\beta = 1/T$, $Z = \mathrm{tr}\,\hat{\rho}$.

Deviations from equilibrium are described by $u^\mu(x)$, $\mu_A(x)$, $T(x)$

Conservation laws: $\partial_\mu T^{\mu\nu} = 0$

$$\partial_\mu J^\mu = 0$$

$T^{\mu\nu}$: $\frac{(d+1)(d+2)}{2}$ components in $d+1$ dim, whereas $\partial_\mu T^{\mu\nu} = 0$ gives $(d+1)$ eqs

With $\partial_\mu J^\mu = 0$, we have $d+2$ eqs and

$$\frac{(d+1)(d+2)}{2} + (d+1) \quad \text{components.}$$

Hydro regime = simplifying assumption that sufficiently close to equilibrium states are described by $d+2$ variables $T(x)$, $\vec{v}(x)$, $\mu(x)$. In kinetic theory, this can be explained more explicitly (see N.N. Bogolubov's papers).

Constitutive relations:

$$\partial_\mu J^\mu = \partial_0 J^0 + \partial_i J^i = 0$$

Need to express $J^i = J^i(J^0)$

$$J^i = -D\partial^i J^0 + \ldots \qquad \text{Fick's Law.}$$

$$\Rightarrow \partial_0 J^0 = D\partial_i^2 J^0$$

$$J^0(x) \sim e^{-i\omega t + i\bar{q}\bar{x}} \Rightarrow \omega = -iDq^2 + \ldots$$

Similarly: $\partial_\mu T^{\mu\nu} = 0$, fund. variables in the hydro regime are T^{00}, T^{0i} (i.e. $T(x)$, $\vec{v}(x)$).

$$T_{\mu\nu} = \mathrm{diag}(\varepsilon, P, P, P), \quad \text{where } P = P(\varepsilon),$$

$$\varepsilon(x) = T^{00}(x).$$

$$T^{\mu\nu} = T^{\mu\nu}_{eq} + \tilde{T}^{\mu\nu}(x)$$

$\partial_\mu T^{\mu\nu}(x) = 0 \Rightarrow$

$$\begin{cases} \partial_0 \tilde{T}^{00} + \partial_i \tilde{T}^{0i} = 0, \\ \partial_0 \tilde{T}^{0i} + \partial_j \tilde{T}^{ij} = 0. \end{cases}$$

Zeroth order: $T^{00} = \varepsilon$

$$T^{ij} = P(\varepsilon)\delta^{ij}$$

First order:

$$T^{00} = \varepsilon + \tilde{T}^{00}(x)$$

$$T^{ij} = \delta^{ij}\left(P(\varepsilon) + \frac{\partial P}{\partial \varepsilon}\tilde{T}^{00}(x)\right) -$$

$$-\frac{1}{\varepsilon+P}\left[\eta\left(\partial_i \tilde{T}^{0j}+\partial_j \tilde{T}^{0i}-\frac{2}{d}\delta^{ij}\partial_k \tilde{T}^{0k}\right)\right.$$

$$\left.+\zeta\,\delta^{ij}\,\partial_k \tilde{T}^{0k}\right]$$

Remark: an equivalent form is

$$T^{\mu\nu}=\varepsilon u^\mu u^\nu + P(\varepsilon)\,\Delta^{\mu\nu}(u) -$$

$$-\eta(\varepsilon)\,\sigma^{\mu\nu}-\zeta(\varepsilon)\,\Delta^{\mu\nu}(\nabla\cdot u),$$

where $\Delta^{\mu\nu}=g^{\mu\nu}+u^\mu u^\nu$ (symmetric and transverse, $u_\mu\Delta^{\mu\nu}=0$), $\sigma^{\mu\nu}$ is symmetric, transverse, traceless:

$$\sigma^{\mu\nu}=2\langle\nabla^\mu u^\nu\rangle,$$

$$\langle A^{\mu\nu}\rangle \equiv \frac{1}{2}\Delta^{\mu\alpha}\Delta^{\nu\beta}(A_{\alpha\beta}+A_{\beta\alpha})-\frac{1}{d}\Delta^{\mu\nu}\Delta^{\alpha\beta}A_{\alpha\beta}$$

Remark: we consider field theories without anomalies here. See Son, Surowka 0906.5044 and Kovtun 1205.5040 for more details

For theories with e.g. axial currents there are more terms in the constitutive relations.

Solving the system of fluct. eq. with $\tilde{T}^{00}, \tilde{T}^{0i} \sim e^{-i\omega t + iqz}$ we find the eigenmodes:

$$\boxed{\omega = -\frac{i\eta}{\varepsilon+P} q^2 + \ldots = -i\frac{\eta}{sT} q^2 + \ldots}$$

$$\omega^2 + i\Gamma\omega q^2 - v_s^2 q^2 = 0,$$

$$v_s = \sqrt{\partial P/\partial \varepsilon}, \quad \Gamma = \frac{1}{\varepsilon+P}\left(\zeta + \frac{2(d-1)}{d}\eta\right)$$

If $v_s \neq 0$,

$$\boxed{\omega = \pm v_s q - \frac{i\Gamma}{2} q^2 + \ldots}$$

Now consider the hydro limit of AdS/CFT. At finite temperature, the AdS - Schwarzschild metric is dual to $\mathcal{N}=4$ SYM at finite T :

$$ds_5^2 = \frac{(\pi T L)^2}{u}\left(-f\,dt^2 + d\vec{x}^2\right) + \frac{L^2}{4fu^2}\,du^2$$

(recall $u = r_0^2/r^2$, $T = r_0/\pi L^2$).

The e.o.m. :

$$R_{\mu\nu} = \frac{2\Lambda}{3}\,g_{\mu\nu}$$

$\Lambda = -6/L^2$

With $g_{\mu\nu} = \overset{\circ}{g}_{\mu\nu} + h_{\mu\nu}(t, z, r)$:

$$R_{\mu\nu} = R^{(0)}_{\mu\nu} + R^{(1)}_{\mu\nu} + \ldots = \frac{2\Lambda}{3}\left(\overset{\circ}{g}_{\mu\nu} + h_{\mu\nu} + \ldots\right)$$

$$\Rightarrow \quad R^{(1)}_{\mu\nu} = \frac{2\Lambda}{3}\,h_{\mu\nu}$$

E.o.m. split into 3 classes: gauge $h_{rM} = 0$

$h_{xy} \neq 0$

* $h_{xt}, h_{xz} \neq 0$ (or $h_{yt}, h_{yz} \neq 0$)

h_{tz} and diag. terms $\neq 0$

Consider * now: $H_t \equiv u\, h_{tx} / (\pi T L)^2$, $H_z \equiv \dfrac{u h_{zx}}{(\pi T L)^2}$

$$
\begin{cases}
H_t' + \dfrac{\bar{q} f}{\bar{\omega}} H_z' = 0 & (a) \\[2ex]
H_t'' - \dfrac{1}{u} H_t' - \dfrac{\bar{\omega}\bar{q}}{uf} H_z - \dfrac{\bar{q}^2}{uf} H_t = 0 & (b) \\[2ex]
H_z'' - \dfrac{1+u^2}{uf} H_z' + \dfrac{\bar{\omega}^2}{uf^2} H_z + \dfrac{\bar{\omega}\bar{q}}{uf^2} H_t = 0 & (c)
\end{cases}
$$

not indep.

~~Here~~ $\bar{\omega} = \omega / 2\pi T$, $\bar{q} = q/2\pi T$.

Combining the first 2 eqs, we get

$$H_t''' - \frac{2u}{f} H_t'' + \frac{2uf - \bar{q}^2 f + \bar{\omega}^2}{u f^2} H_t' = 0$$

Let $H_t' = (1-u)^{-i\bar{\omega}/2}\, G(u)$:

$$G'' - \left(\frac{2u}{f} - \frac{i\bar{\omega}}{1-u}\right) G' +$$

$$+ \frac{1}{f}\left(2 + \frac{i\bar{\omega}}{2} - \frac{\bar{q}^2}{u} + \frac{\bar{\omega}^2\left[4 - u(1+u)^2\right]}{4uf}\right) G$$

$$= 0$$

Recall: $u = 1$ - horizon

$u = 0$ - boundary

Find G regular at $u = 1$ perturbatively in $\bar{\omega}, \bar{q} \ll 1$ (hydro regime):

$$G(u) = C\left[u - i\bar{\omega}\left(1 - u - \frac{u}{2}\ln\frac{1+u}{2}\right) + \frac{\bar{q}^2(1-u)}{2}\right] +$$

$$+ \mathcal{O}\left(\bar{\omega}^2, \bar{q}^4, \bar{\omega}\bar{q}^2\right)$$

Now the limit of eq (6) $u \to 0$ gives

$$C = \frac{\bar{q}^2 H_t^{(0)} + \bar{q}\bar{\omega} H_z^{(0)}}{i\bar{\omega} - \bar{q}^2/2}$$

The action on shell is

$$S = -\frac{\pi^2 N_c^2 T^4}{8} \int du\, d^4x \, \frac{1}{u} \left(-H_t'^2 + f H_z'^2 \right)$$

Computing the correlators, we find:

$$\langle T_{tx}(\omega, q)\, T_{tx}(-\omega, -q) \rangle_{ret} = \frac{N_c^2 \pi T^3 q^2}{8(i\omega - Dq^2)} + \dots$$

$$\langle T_{tx}\, T_{xz} \rangle_{ret} = -\frac{N_c^2 \pi T^3 \omega q}{8(i\omega - Dq^2)},$$

$$\langle T_{xz}, T_{xz} \rangle_{ret} = \frac{N_c^2 \pi T^3 \omega^2}{8(i\omega - Dq^2)},$$

where $D = \frac{1}{4\pi T}$.

But $D = \eta/(\varepsilon + P)$,

and $\varepsilon + P = sT$, $s = \frac{\pi^2}{2} N_c^2 T^3$ $\begin{pmatrix} \lambda \to \infty \\ N_c \to \infty \end{pmatrix}$

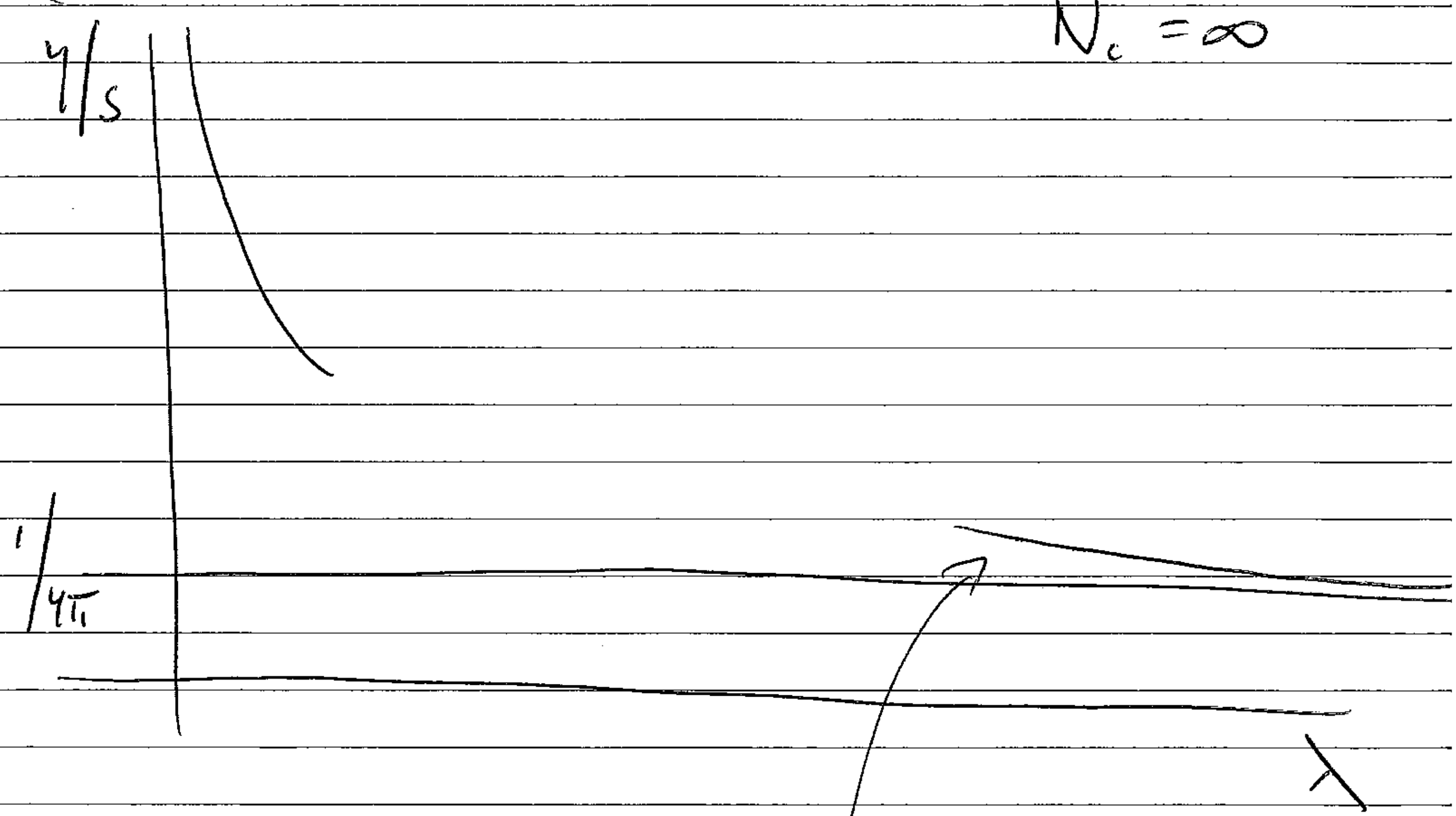
$\Rightarrow \eta/s = 1/4\pi$ for $\mathcal{N}=4$ SYM at $\lambda \to \infty$, $N_c \to \infty$.

Viscosity: $N_c \to \infty$ limit

$$\eta = f(\lambda)\, N_c^2\, T^3$$

$$f \simeq \begin{cases} \lambda^{-2} \ln^{-1}\left(\frac{1}{\lambda}\right) & \lambda \ll 1 \\ \frac{\pi}{8} & \lambda \gg 1 \end{cases}$$

(See 0205052 hep-th for details)



corrections known

$$\eta/s = \frac{\hbar}{4\pi k_B} + O\left(\lambda^{-3/2}\right) = \frac{\hbar}{4\pi k_B}\left(1 + \frac{15\zeta(3)}{\lambda^{3/2}} + \dots\right).$$