Advanced Quantum Mechanics
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Problem Sheet III
Relativistic wave equations

Note: some problems are optional, problems with one or more stars are more difficult and can be treated as optional as well, although an ambitious student should attempt all problems.

1. Consider the Klein-Gordon equation for a spinless particle with mass m and electric charge e in an external electromagnetic field. Show that the function $\psi_c = \hat{C}\psi$, where C is the charge conjugation operator, obeys the Klein-Gordon equation for a spinless particle with mass m and electric charge $-e$.

2. Show that a *scalar* external field $U(\mathbf{r}, t)$ acts identically on relativistic spinless particle and antiparticle. *Hint: Write down the appropriate Klein-Gordon equation.*

3*. Show that the energy spectrum of a relativistic spinless particle in an external constant and uniform magnetic field $\mathbf{B} = (0, 0, B)$ is given by

$$\varepsilon^2_{p_z,n} = m^2c^4 + p_z^2c^2 + 2mc^2\hbar\omega(n + 1/2)\,,$$

where $n = 0, 1, ...,$ $\omega = |e|B/mc$. *Hint: Reduce the problem to the one for a non-relativistic particle in an external constant and uniform magnetic field and use the solution of the non-relativistic problem (available in most courses on Quantum Mechanics, e.g. Landau & Lifshitz, vol. III).*

4. A spinless relativistic particle of mass m in an external scalar field $U(\mathbf{r}, t)$ obeys the equation

$$\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta + \frac{m^2c^2}{\hbar^2} + \frac{2m}{\hbar^2}U(\mathbf{r}, t)\right]\psi = 0\,.$$

a) Show that in the non-relativistic limit $U(\mathbf{r}, t)$ has the meaning of the usual potential energy.

b) Show that the wave function $R(r) = r\psi(r)$ of the s-wave spinless particle in the external field

$$\Phi(r) = \begin{cases} -U_0, & r \leq a\,, \\ 0, & r > a \end{cases}$$

satisfying the boundary condition $R(0) = 0$ is

$$R(r) = \begin{cases} A\sin\left(r\sqrt{\frac{2mU_0}{\hbar^2} - \kappa^2}\right), & r \leq a\,, \\ B\,e^{-\kappa r}, & r > a\,, \end{cases}$$

where $\kappa = \sqrt{m^2c^4 - \epsilon^2}/\hbar c > 0$.

c) Show that the discrete energy spectrum ϵ_n is determined by the equation

$$\tan\sqrt{\frac{2mU_0a^2}{\hbar^2} - \kappa_n^2a^2} = -\frac{1}{\kappa_n a}\sqrt{\frac{2mU_0a^2}{\hbar^2} - \kappa_n^2a^2}\,,$$

where $\kappa_n = \sqrt{m^2c^4 - \epsilon_n^2}/\hbar c$. What is the spectrum of an antiparticle in this field?

d) Find the algebraic equation determining the critical value $U_{0,crit}$ of the external field corresponding to $\epsilon_{n=0} = 0$. What physical processes one may expect to occur for external fields exceeding the critical value? Is the one-particle equation adequate in this case?

5. Find the leading relativistic correction to the Schrödinger equation. *Hint: Expand the expression for relativistic energy to the appropriate order and make the usual quantum-mechanical substitutions.*

6. The Dirac equation in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ is

$$\left[\gamma^\mu\left(p_\mu - \frac{e}{c}A_\mu\right) - mc\right]\psi = 0\,,$$

where $\psi = \begin{pmatrix} \varphi \\ \chi \end{pmatrix}$ is the four-component Dirac spinor. The Minkowski metric is given by $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$, $p_\mu = i\hbar\partial_\mu$, and the Dirac matrices are

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix},$$

where I is the identity matrix, and σ^k are the Pauli matrices obeying $\sigma_i \sigma_k = \delta_{ik} + i\epsilon_{ikl}\sigma_l$.

Assuming the external field is time-independent, consider stationary solutions of the Dirac equation with the time dependence of the form $\psi \sim \exp(-i\epsilon t/\hbar)$.

a) Write down the system of coupled equations for the two-component spinors φ and χ.

b) Consider the positive energy solution with $\epsilon = mc^2 + E$. Show that in the non-relativistic limit, where $|E| \ll mc^2$, $|e\Phi| \ll mc^2$, the spinor φ obeys the Pauli equation

$$i\hbar \frac{\partial \varphi}{\partial t} = \left[\frac{\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2}{2m} + e\Phi - \mu_0\, \boldsymbol{\sigma} \cdot \mathbf{B} \right] \varphi$$

and find the value of the magnetic moment μ_0.

c) Is the value of μ_0 universal for all charged particles with spin 1/2?

d) Do you expect the theoretical prediction for μ_0 following from the Dirac equation to be exact?

e) Is the value of the magnetic moment fixed uniquely by the Dirac equation? *Hint: Consider a non-minimal coupling to an electromagnetic field.*

f) Consider further the case of $\mathbf{A} = 0$, and let $e\Phi = U(r)$. By expanding the Dirac equation to the next order in $|E|/mc^2 \ll 1$, $|U|/mc^2 \ll 1$, show that the spinor φ obeys the equation

$$i\hbar \frac{\partial \varphi}{\partial t} = \left(\frac{\mathbf{p}^2}{2m} + U(r) + H_1 \right) \varphi \,,$$

where the perturbation operator is given by

$$H_1 = -\frac{\mathbf{p}^4}{8m^3c^2} + \frac{1}{2m^2c^2}\frac{1}{r}\frac{dU(r)}{dr}\mathbf{L}\cdot\mathbf{S} + \frac{\hbar^2}{4m^2c^2}\frac{dU}{dr}\frac{d}{dr} \,.$$

What is the physical meaning of terms in H_1?

Hint: First prove the following identity: $(\boldsymbol{\sigma}\mathbf{p})f(\boldsymbol{\sigma}\mathbf{p}) = f\mathbf{p}^2 - \hbar^2(\partial_i f)\partial_i - i\hbar^2\sigma^i\epsilon_{ijk}(\partial_j f)\partial_k$.