Advanced Quantum Mechanics

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Problem Sheet I
One-dimensional scattering

Note: some problems are optional, problems with one or more stars are more difficult and can be treated as optional as well, although an ambitious student should attempt all problems.

1. Show that for a generic real potential $U(x)$ the sum of the transmission coefficient T and the reflection coefficient R is equal to one, $T + R = 1$.

2. Show that T and R do not depend on the direction from which particles are incident on the (real) potential $U(x)$.

3. [*Optional*] Find the Green's function for the Schrödinger equation of the free particle with $E < 0$ by using the Fourier transform.

4. Use the integral equation equivalent to the Schrödinger equation to find the wave function and the energy of the bound state in the potential $U(x) = -q\delta(x)$.

5. Find the equation determining the energy values for which particles are *not* reflected by the potential $U(x) = \alpha\,[\delta(x) + \delta(x-a)]$, where $\alpha > 0$.

6. Consider the rectangular one-dimensional potential well

$$U(x) = \begin{cases} -U_0, & |x| \leq a \, , \\ 0, & |x| > a \, . \end{cases}$$

a) Find the transmission amplitude $S(E)$ and the transmission probability.

b) Show that the trasmission amplitude has poles (singularities) in the complex k plane determined by the equations

$$\tan \kappa a = -\frac{ik}{\kappa} \, , \qquad \cot \kappa a = \frac{ik}{\kappa} \, .$$

where $\kappa = \sqrt{2m(E+|U_0|)}/\hbar$. *Hint: You may find the identity* $\tan 2x = 2/(\cot x - \tan x)$ *useful.*

c) Find the even and odd parity wave functions corresponding to the stationary states with $E < 0$ (bound states) in this potential well.

d) Find equations determining the bound state energies in the potential well and show that these energies coincide with the poles of the transmission amplitude.

e) Find the energies E for which the transmission probability $T(E)$ reaches its maximum value.

f) Treating E formally as a complex variable, sketch (qualitatively) the location of the poles of the transmission amplitude in the complex k plane and the complex E plane.

7*. Show that the discrete energy spectrum eigenvalues $\{E_n\}$ in the generic (real) $U(x) \leq 0$ with $U(x) \to 0$ for $x \to \pm\infty$ obey

$$|E_n| \leq \frac{m}{2\hbar^2}\left(\int\limits_{-\infty}^{\infty} U(x)dx\right)^2 .$$

Hint: For the ground state $E = E_0 < 0$ *(*$|E_n| \leq |E_0|$*) the wave function* $\psi_0(x)$ *can be chosen to be real and positive. Show first that* $\psi_0(x_0) \leq \frac{m}{\kappa_0\hbar^2}\psi_0(x_0)\int\limits_{-\infty}^{\infty}|U(x)|dx$*, where* x_0 *is the point where* $\psi(x)$ *has a maximum.*

8**. Find $T(E)$ for the potential $U(x) = \frac{U_0}{\cosh^2(\frac{x}{a})}$. *Hint: See Landau & Lifshitz, volume III, Section 25.*