Section S18 *ADVANCED QUANTUM MECHANICS: SOLUTIONS*

1. *A non-relativistic quantum particle of mass m is incident on the one-dimensional potential $U(x)$, where $U(x) \to 0$ for $|x| \to \infty$. Set up a scattering problem and write down the asymptotics of the wave function ψ_L (ψ_R) at $x \to \pm\infty$, assuming that the particle is incident on the potential from the left (right). Show that the reflection and transmission coefficients for a given potential depend only on the particle's energy and not on the direction from which it is incident on the potential. Hint: Multiply the Schrödinger equation obeyed by ψ_L by ψ_R and the equation obeyed by ψ_R by ψ_L, subtract and integrate over the real line. Compute the resulting quantity using the asymptotics of $\psi_{L,R}$.* *[4]*

The two sets of asymptotics can be written down as follows:

$$\psi_L(x) = \begin{cases} e^{ikx} + A_L e^{-ikx}, & x \to -\infty\,, \\ S_L e^{ikx}, & x \to +\infty\,, \end{cases}$$ [1]

and

$$\psi_R(x) = \begin{cases} e^{-ikx} + A_R e^{ikx}, & x \to +\infty\,, \\ S_R e^{-ikx}, & x \to -\infty\,, \end{cases}$$ [1]

where $\psi_{L,R}$ satisfy the equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_{L,R}}{dx^2} + U(x)\psi_{L,R} = E\psi_{L,R}\,.$$

Following the instructions given in the problem, we find

$$\int_{-\infty}^{\infty} dx\,(\psi_R\psi_L'' - \psi_L\psi_R'') = 0\,.$$ [1]

Integrating by parts, we obtain $\mathcal{F}(+\infty) = \mathcal{F}(-\infty)$, where $\mathcal{F} = \psi_R\psi_L' - \psi_L\psi_R'$. Substituting the expressions for asymptotics given above, we get $S_L = S_R$. Since $T = |S|^2$ and $R = 1 - T$, the required result follows. [1]

Using Fourier transform or any other method, show that the Green's function $G(x,x')$ obeying the equation $\hat{L}\,G(x,x') = \delta(x-x')$ and the boundary conditions $G(x,x') \to C\,\exp(ik|x|)$ for $|x| \to \infty$, where $\hat{L} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E$ is the one-dimensional Schrödinger operator for a free particle with $E > 0$, C is a constant and $k = \sqrt{2mE}/\hbar$, is given by $G(x,x') = \frac{im}{k\hbar^2}\exp(ik|x-x'|)$. *[5]*

The function $G(x,x')$ can be found by several methods:

1) Using Fourier transform.

2) From the solutions of the ODE with zero right-hand side.

3) By analytic continuation from the known result for $E < 0$.

These methods were discussed in lectures in detail. For example, solving the ODE directly with the b.c. $G(x,x') \to C\,\exp(ik|x|)$ for $|x| \to \infty$, we have

$$G(x,x') = \begin{cases} Ae^{-ik(x-x')}, & x < x'\,, \\ Be^{ik(x-x')}, & x > x'\,. \end{cases}$$ [2]

Matching conditions $G(x = x' + \epsilon, x') = G(x = x' - \epsilon, x')$ and $G(x = x' + \epsilon, x') - G(x = x' - \epsilon, x') = -2m/\hbar^2$ imply $A = B$ and $A = im/k\hbar^2$, so $G(x, x') = \frac{im}{k\hbar^2} \exp(ik|x - x'|)$. [3]

Alternatively, one can use Fourier transform to obtain the Green's functon for $E < 0$, $G(x - x') = \frac{m}{\kappa\hbar^2} e^{-\kappa|x-x'|}$ (see below) and then analytically continue to $E > 0$: this means setting $\kappa = \pm ik$, where $k = \sqrt{2mE}/\hbar$, $E > 0$. We get

$$G^{\pm}(x - x') = \pm \frac{im}{k\hbar^2} e^{\pm ik|x-x'|},$$

with $G^+(x, x')$ corresponding to the b.c. needed.

The Fourier method to get $G(x, x')$ for $E < 0$ works as follows. Using the standard Fourier representation of $G(x, x')$,

$$G(x, x') = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} e^{ip(x-x')/\hbar} G(p) dp,$$

we find from the equation $\hat{L}\, G(x, x') = \delta(x - x')$

$$\left(\frac{p^2}{2m} + |E| \right) G(p) = 1,$$

since

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-x')} dk.$$

Therefore,

$$G(x - x') = \frac{m}{\pi\hbar^2} \int_{-\infty}^{\infty} \frac{e^{ik(x-x')}}{k^2 + \kappa^2} dk,$$

where $\kappa^2 = 2m|E|/\hbar^2$. The integrand has poles at $k = \pm i\kappa$. The integral can be evaluated using the residue theorem. For $x - x' > 0$, we close the contour in the upper half-plane, since the integrand then vanishes for $\mathrm{Im}k \to \infty$. By residue theorem,

$$\int_{-\infty}^{\infty} \frac{e^{ik(x-x')}}{k^2 + \kappa^2} dk = \frac{\pi}{\kappa} e^{-\kappa(x-x')}.$$

Therefore, $G(x - x') = \frac{m}{\kappa\hbar^2} e^{-\kappa(x-x')}$ for $x - x' > 0$. Similarly, by closing the contour in te lower half-plane, we find $G(x - x') = \frac{m}{\kappa\hbar^2} e^{\kappa(x-x')}$ for $x - x' < 0$. Thus, $G(x - x') = \frac{m}{\kappa\hbar^2} e^{-\kappa|x-x'|}$, and then we can use analytic continuation. Of course, yet another way is to obtain the expression $G(x, x') = \frac{im}{k\hbar^2} \exp(ik|x - x'|)$ directly using Fourier transform for $E > 0$. All this has been discussed in lectures.

A non-relativistic quantum particle of mass m is incident from the left on the one-dimensional potential $U(x) \leq 0$, where $U(x) \to 0$ for $|x| \to \infty$. Show that the wave function of a stationary scattering state of the particle satisfies the integral equation

$$\psi_s(x) = e^{ikx} - \frac{im}{k\hbar^2} \int_{-\infty}^{\infty} e^{ik|x-x'|} U(x') \psi_s(x') dx'.$$

[3]

This is the standard procedure discussed in lectures in detail. The Schrödinger equation $-\frac{\hbar^2}{2m}\frac{d^2\psi_s}{dx^2} - E\psi_s = f(x) \equiv -U(x)\psi_s$ For $E > 0$ can be written as an integral equation with the help of the Green's function $G(x, x') = \frac{im}{k\hbar^2}\exp(ik|x - x'|)$ for a free particle:

$$\psi_s(x) = Ae^{ikx} + Be^{-ikx} - \frac{im}{k\hbar^2}\int\limits_{-\infty}^{\infty} e^{ik|x-x'|}U(x')\psi_s(x')dx' .$$ [2]

For particles incident from the left, $A = 1$ and $B = 0$ (e.g. for $U = 0$ we must have unperturbed wave e^{ikx} propagating from left to right). Also, for $x \to \infty$ only waves $\sim e^{ikx}$ must be present, so the Green's function must really be $G^+(x, x') = \frac{im}{k\hbar^2}\exp(ik|x - x'|)$. [1]

Find the solution of the integral equation for the potential $U(x) = -\alpha\delta(x)$, $\alpha > 0$. Define and find the transmission and reflection coefficients T and R, and show that $R + T = 1$. [3]

For the potential $U(x) = -\alpha\delta(x)$, $\alpha > 0$, the integral equation gives

$$\psi_s(x) = e^{ikx} + \frac{im\alpha}{k\hbar^2}e^{ik|x|}\,\psi_s(0)\,,$$ [1]

which also determines $\psi_s(0) = 1/(1 - i\kappa)$, where $\kappa \equiv m\alpha/\hbar^2$. Thus,

$$\psi_s(x) = e^{ikx} - \frac{\kappa}{ik + \kappa}e^{ik|x|}\,.$$ [1]

Considering the asymptotics at $x \to \pm\infty$ and comparing with the standard scattering set up (see the beginning of the present problem), we find $S = ik/(ik + \kappa)$ and $A = -\kappa/(ik + \kappa)$.Thus, $T = |S|^2 = k^2/(k^2 + \kappa^2)$ and $R = |A|^2 = \kappa^2/(k^2 + \kappa^2)$. Clearly, $T + R = 1$. [1]

Consider scattering of "slow" particles with energies obeying $ka \ll 1$ by a generic potential $U(x)$ characterised by the typical strength U_0 and width a. Assume that the potential is "weak", i.e. that $U_0 \ll \hbar^2/ma^2$.

a) Set up a scattering problem and show, by analysing the Schrödinger equation and matching its solution to the asymptotics at $x \to \pm\infty$, that to leading order in $ka \ll 1$ and $U_0 \ll \hbar^2/ma^2$, the solution in the region $|x| \le a$ is well approximated by $\psi \approx C = const.$ [3]

We can write the Schrödinger equation in the form

$$a^2\psi'' = \frac{2mUa^2}{\hbar^2}\psi - k^2a^2\psi\,.$$ [1]

The conditions $ka \ll 1$ and $U_0 \ll \hbar^2/ma^2$ imply that $\psi'' \approx 0$ for slow particles and weak potentials. Thus, $\psi \approx C + \tilde{C}x$. Matching to the asymptotics at $x \to \pm\infty$, to leading order in $ka \ll 1$ we find $\tilde{C} = 0$ and $C = 1 + A = S$. [2]

b) By considering the integral equation for ψ in the limit $ka \ll 1$, show that $C = (1 + \frac{im\gamma}{k\hbar^2})^{-1}$. Relate γ and $U(x)$. Compare with the delta-function potential above. [3]

The integral equation

$$\psi(x) = e^{ikx} - \frac{im}{k\hbar^2} \int_{-\infty}^{\infty} e^{ik|x-x'|} U(x')\psi(x')dx'$$

with all the aproximations taken into account becomes

$$\psi(x) = e^{ikx} - \frac{imC}{k\hbar^2} \int_{-\infty}^{\infty} U(x')dx' .$$

[1]

In the region $x < |a|$, we have $\psi \approx C$. Thus, $C = (1 + \frac{im\gamma}{k\hbar^2})^{-1}$, where $\gamma = \int_{-\infty}^{\infty} U(x)dx$. The delta-function potential above mimics this approximation, provided $\alpha = -\gamma$. [2]

c) Show the location of singularities of the transmission coefficient in the complex plane of k and E. *[2]*

The situation is exactly the same as in the problem with delta-function potential, so the scattering amplitude $S = ik/(ik + \kappa)$ with $\kappa \equiv -m\gamma/\hbar^2$. For $\gamma < 0$, it has a pole in the upper half-plane of complex k or, in terms of E, a pole on the negative real axis of the "physical sheet" of E at $E = -\hbar^2\kappa^2/2m$. [2]

d) Using the scattering data found above, identify the bound states in the potential $U(x) < 0$ and compute their energies to leading order in $U_0 \ll \hbar^2/ma^2$. What states do we expect to appear if $U(x) > 0$? *[2]*

For $U(x) < 0$ we have $\gamma < 0$ and there is a pole of S at $E = -\hbar^2\kappa^2/2m$ corresponding to the single bound state in the "weak" potential $U(x)$ at low energies. For $\gamma > 0$ we have a virtual state instead, as discussed in lectures. [2]

2. *Show that the Green's function* $G(x,x')$ *obeying* $\hat{L}\,G(x,x') = \delta(x-x')$, *where* $\hat{L} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x) - E$ *is the Schrödinger operator, for a particle in an infinite potential well of width* a,

$$U(x) = \begin{cases} 0, & |x| < \frac{a}{2}\,, \\ \infty, & |x| > \frac{a}{2}\,, \end{cases}$$

is given by

$$G(x,x') = -\frac{2m}{k\hbar^2 \sin ka}\,\sin\left[\frac{k}{2}\left(x + x' - |x'-x| + a\right)\right]\cdot \sin\left[\frac{k}{2}\left(x + x' + |x'-x| - a\right)\right], \quad (*)$$

where $k^2 = 2mE/\hbar^2$. *You may use the identity* $\sin\alpha\cos\beta - \cos\alpha\sin\beta = \sin(\alpha-\beta)$. *[5]*

The Green's function can be constructed from the solutions of the equation $\hat{L}\,\phi = 0$, taking into account the appropriate boundary conditions: $G(\pm\frac{a}{2}, x') = 0$. We have $G(x,x') = 0$ outside of the interval $[-\frac{a}{2}, \frac{a}{2}]$. For $-\frac{a}{2} < x' < \frac{a}{2}$ and $x > x'$, one can write the solution as

$$G(x,x') = B(x')\sin k\left(x - \frac{a}{2}\right) + \tilde{B}(x')\cos k\left(x - \frac{a}{2}\right)$$ [1]

and the boundary condition at $x = \frac{a}{2}$ implies $\tilde{B}(x') = 0$. Similarly, for $x < x'$, we have

$$G(x,x') = A(x')\sin k\left(x + \frac{a}{2}\right) + \tilde{A}(x')\cos k\left(x + \frac{a}{2}\right)$$ [1]

and the boundary condition at $x = -\frac{a}{2}$ gives $\tilde{A}(x') = 0$. Thus,

$$G(x,x') = \begin{cases} A(x')\sin k\,(x + \frac{a}{2}), & x < x'\,, \\ B(x')\sin k\,(x - \frac{a}{2}), & x > x'\,. \end{cases}$$

[1]

The coefficients A and B are determined by the matching conditions at $x = x'$ (these conditions are obtained by integrating the equation, as discussed in lectures in detail):

$$G'(x = x' + \epsilon, x') - G'(x = x' - \epsilon, x') = -\frac{2m}{\hbar^2}\,,$$
$$G(x = x' + \epsilon, x') = G(x = x' - \epsilon, x')\,.$$

[1]

We find

$$A(x') = -\frac{2m}{\hbar^2 k \sin ka}\sin k\left(x' - \frac{a}{2}\right).$$
$$B(x') = -\frac{2m}{\hbar^2 k \sin ka}\sin k\left(x' + \frac{a}{2}\right).$$ [1]

The expression (*) for $G(x,x')$ follows immediately.

Identify the singularities of the Green's function and show that they correspond to the energy levels E_n *of the particle in the potential well.* *[2]*

The only singularities are the zeros of the denominator, located at $k_n a = n\pi$, $n = 1, 2, \ldots$ (note that $k = 0$ is not a singularity), i.e. at $E_n = \hbar^2 n^2 \pi^2 / 2ma^2$ which are the energy levels in an infinite potential well. [2]

Let $\{\varphi_n(x)\}$ be an orthonormal set of eigenfunctions of the Schrödinger operator $\hat{L}$ with eigenvalues $\lambda_n = E_n - E$. Argue that the Green's function can be expanded as $G(x, x') = \sum_n c_n \varphi_n(x)$ and show that the coefficients c_n are given by $c_n = \varphi_n^*(x')/\lambda_n$. Thus show that the Green's function can be written as

$$G(x, x') = \sum_n \frac{\varphi_n^*(x')\varphi_n(x)}{E_n - E}. \qquad (**)$$

[3]

We have $\hat{L}\, G(x, x') = \delta(x - x')$, and $\hat{L}\, \varphi_n = \lambda_n \varphi_n = (e_n - E)\varphi_n$, where $\hat{L} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + U(x) - E$. Since $\hat{L}$ is Hermitian, the eigenfuctions form a complete orthonormal set. Thus, we can expand

$$G(x, x') = \sum_n c_n \varphi_n(x).$$

[1]

Acting by $\hat{L}$ on the left, we find

$$\hat{L}\, G(x, x') = \sum_n c_n (E_n - E)\varphi_n = \delta(x - x').$$

[1]

Since the eigenfuctions are orthonormal, one immediately finds c_n:

$$c_n = \frac{\varphi^*(x')}{E_n - E}.$$

[1]

Therefore,

$$G(x, x') = \sum_n \frac{\varphi_n^*(x')\varphi_n(x)}{E_n - E}. \qquad (**)$$

as expected.

Fix $x > x'$ (or $x < x'$), compute the limits $\lim_{E \to E_n} (E_n - E) G(x, x')$, where $G(x, x')$ is given by Eq. (*) and Eq. (**), correspondingly, and find the expression for $\varphi_n^*(x')\varphi_n(x)$ by comparing the two results. *[4]*

Taking the limit in the expansion (**) gives simply $\varphi_n^*(x')\varphi_n(x)$ for fixed n. To take the limit in (*), note that

$$E_n - E = \frac{\hbar^2 \pi^2 n^2}{2ma^2} - \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2 \pi^2}{2ma^2}\left(\pi^2 n^2 - k^2 a^2\right).$$

[1]

Also, expanding around $k = k_n$ gives $\sin ka = \cos(k_n a)(ka - k_n a) + \ldots$. [1]

Fixing e.g. $x < x'$, we get

$$\begin{aligned}
\lim_{E \to E_n} (E_n - E) G(x, x') &= (-1)^n \frac{2}{a} \sin\left(\frac{n\pi x'}{a} - \frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{a} + \frac{n\pi}{2}\right) \\
&= (-1)^n \frac{2}{a} \sin\left(\frac{n\pi x'}{a} - \frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{a} - \frac{n\pi}{2} + n\pi\right) \\
&= \frac{2}{a} \sin\left(\frac{n\pi x'}{a} - \frac{n\pi}{2}\right) \sin\left(\frac{n\pi x}{a} - \frac{n\pi}{2}\right).
\end{aligned}$$

[2]

Find the explicit form of $\varphi_n(x)$. Compare the result with the normalised wave-functions of the particle in an infinite potential well. [2]

Comparing the two results above, we get

$$\varphi_n^*(x')\varphi_n(x) = \frac{2}{a}\sin\left(\frac{n\pi x'}{a} - \frac{n\pi}{2}\right)\sin\left(\frac{n\pi x}{a} - \frac{n\pi}{2}\right),$$

therefore

$$\varphi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a} - \frac{n\pi}{2}\right),$$

which is the properly normalised wavefunction in an infinite potential well of width a. [2]

Now consider the Schrödinger operator with a non-local potential, whose action on the wave-functon ψ is given by $U(x)\psi(x) = \int\limits_{-\infty}^{\infty} dx' V(x,x')\psi(x')$. Using the Green's function $G(x,x') = \frac{m}{\kappa\hbar^2}\exp(-\kappa|x-x'|)$ of a free particle with $E < 0$, where $\kappa = \sqrt{-2mE}/\hbar$, show that the Schrödinger equation $\hat{L}\psi = 0$ with such a potential can be written as an integral equation

$$\psi(x) = -\frac{m}{\kappa\hbar^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} e^{-\kappa|x-x'|}V(x',x'')\psi(x'')dx'dx''.$$

[2]

The Schrödinger equation $-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} - E\psi = -U(x)\psi$ For $E < 0$ can be written as an integral equation with the help of the Green's function $G(x,x')$ (as discussed in lectures):

$$\psi = \int G(x,x')U(x')\psi(x')dx'.$$

Substituting the expressions for U and $G(x,x')$, we get the required result. [2]

Assume that the potential $V(x,x')$ is separable, i.e. $V(x,x') = -\alpha F(x)F^*(x')$, where F is a smooth function. Show that the solution of the Schrödinger equation has the form

$$\psi(x) = \frac{\alpha mC}{\kappa\hbar^2}\int\limits_{-\infty}^{\infty} e^{-\kappa|x-x'|}F(x')dx', \qquad (A)$$

where C is a constant. Write an explicit expression for C. [3]

Substituting $V(x,x') = -\alpha F(x)F^*(x')$ into the integral equation above we get the required result with

$$C = \int\limits_{-\infty}^{\infty} F^*(x'')\psi(x'')dx''. \qquad (B)$$

[3]

Show that the energy levels are determined by the equation

$$\kappa = \frac{\alpha m}{\hbar^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x) F^*(x')\, e^{-\kappa|x-x'|} dx dx' \,.$$

[4]

Substituting (A) into (B) and cancelling C on both sides results in the equation

$$\kappa = \frac{\alpha m}{\hbar^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F(x) F^*(x')\, e^{-\kappa|x-x'|} dx dx'$$

which can be solved for κ to determine the energy levels. [4]

3. *The Dirac equation for a free relativistic particle of mass m and spin $s = 1/2$ in $D = 3+1$ dimensions can be written in the form*

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\left(\alpha_1\frac{\partial\psi}{\partial x^1} + \alpha_2\frac{\partial\psi}{\partial x^2} + \alpha_3\frac{\partial\psi}{\partial x^3}\right) + \beta mc^2\psi \equiv H_D\psi\,,$$

where $\alpha_i^2 = I$, $\beta^2 = I$, $\alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}$, $\alpha_i\beta + \beta\alpha_i = 0$ and I is the identity matrix. Show that each component ψ_α, $\alpha = 1, ...N$, of the Dirac spinor ψ satisfies the Klein-Gordon equation. *[2]*

The Klein-Gordon equation contains second derivatives. By acting with $i\hbar\partial/\partial t$ on the left hand side of the Dirac equation, we find

$$-\hbar^2\frac{\partial^2\psi}{\partial t^2} = H_D(H_D\psi)\,. \qquad [1]$$

The right hand side contains products of matrices α_i and β:

$$-\hbar^2\frac{\partial^2\psi_\alpha}{\partial t^2} = -\frac{\hbar^2c^2}{2}\sum_{i,j=1}^{3}(\alpha_j\alpha_i + \alpha_i\alpha_j)\frac{\partial^2\psi}{\partial x^i\partial x^j} - i\hbar mc^3\sum_{i=1}^{3}(\alpha_i\beta + \beta\alpha_i)\frac{\partial\psi}{\partial x^i} + \beta^2m^2c^4\psi\,.$$

Simplifying the products with the help of identities $\alpha_i^2 = I$, $\beta^2 = I$, $\alpha_i\alpha_k + \alpha_k\alpha_i = 2\delta_{ik}$, $\alpha_i\beta + \beta\alpha_i = 0$, we find that each component ψ_α, $\alpha = 1, 2, ...N$ of the spinor satisfies the equation

$$-\hbar^2\frac{\partial^2\psi_\alpha}{\partial t^2} = \left(-\hbar^2c^2\nabla^2 + m^2c^4\right)\psi_\alpha$$

which is the Klein-Gordon equation. [1]

Show that the eigenvalues of α_i and β are equal to ± 1. *[2]*

Using the relations $\alpha_i^2 = I$, $\beta^2 = I$, we find for $\alpha_i\phi = \lambda\phi$

$$\alpha_i^2\phi = \lambda\alpha_i\phi = \lambda^2\phi\,. \qquad [1]$$

Thus, $\lambda^2 = 1$. Since α_1 are Hermitian, all λ are real. Therefore, $\lambda = \pm 1$. Same for β. [1]

Show that $Tr\,\alpha_i = 0$ and $Tr\,\beta = 0$. *[2]*

Using the properties $\alpha_i\beta + \beta\alpha_i = 0$ and $\alpha_i^2 = I$, $\beta^2 = I$, we find $\alpha_i = -\beta\alpha_i\beta$. Thus, $\mathrm{tr}\,\alpha_i = -\mathrm{tr}\,\beta\alpha_i\beta = -\mathrm{tr}\,\alpha_i$, so $\mathrm{tr}\,\alpha_i = 0$. Same for β. Note that all the matrices here are finite-dimensional, and the property $\mathrm{tr}\,AB = \mathrm{tr}\,BA$ holds. [2]

Show that N must be an even number greater than 2. *[2]*

Since $\mathrm{tr}\,\alpha_i = \sum_{p=1}^{N}\lambda_p = +1+1+1+...-1-1-1 = 0$, N must be even, $N \geq 2$. For $N = 2$, we can use 3 Paili matrices. But for the Dirac equation in $D = 3+1$ dimensions, we need 4 traceless matrices, not 3. So $N > 2$ and even. [1]

The case $N = 4$ gives irreducible representation, all representations with $N > 4$ are reducible. Note: all of this has been discussed in the lectures. [1]

It is often useful to consider the Dirac equation in dimensions other than four. Following the steps above, demonstrate that the Dirac equation in $D = 2+1$ dimensions is given by

$$i\hbar\frac{\partial\psi}{\partial t} = \frac{\hbar c}{i}\left(\sigma_x\frac{\partial\psi}{\partial x} + \sigma_y\frac{\partial\psi}{\partial y}\right) + \sigma_z mc^2\psi\,,$$ *[2]*

where σ_x, σ_y, σ_z are Pauli matrices $\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *obeying $\sigma_i\sigma_k = \delta_{ik} + i\epsilon_{ikl}\sigma_l$.*

In $D = 2+1$ dimensions, repeating the steps above we see that we need only 3 Dirac matrices. Thus $N = 2$ is an option now, and one can take Pauli matrices as $\alpha_{1,2}$ and β. The equation follows immediately. [2]

In $D = 3+1$ dimensions, consider the spin and angular momentum operators, whose components, correspondingly, are given by $S_i = -i\varepsilon_{ijk}\alpha_j\alpha_k/4$ and $L_i = i\varepsilon_{ijk}x_jp_k$, and introduce $J_i = L_i + S_i$.

Compute the commutators $[S_i, S_j]$, $[L_i, L_j]$ and $[J_i, J_j]$. *[3]*

This is a direct computation using the properties of Dirac matrices. We have $[\alpha_i\alpha_k, \alpha_j] = -2\delta_{ij}\alpha_k + 2\delta_{kj}\alpha_i$. Then $[S_i, \alpha_j] = i\varepsilon_{ijk}S_k$ and, finally, $[S_i, S_j] = i\varepsilon_{ijk}S_k$. [2]

The commutator $[L_i, L_j]$ is the standard quantum-mechanical result, $[L_i, L_j] = i\varepsilon_{ijk}L_k$. Thus $[J_i, J_j] = i\varepsilon_{ijk}J_k$. [1]

Show that $[H_D, J_i] = 0$. What are the quantum numbers associated with the Dirac wavefunction? *[2]*

To compute $[H_D, J_i]$, the Dirac Hamiltonian operator can be written as $H_D = c\alpha_i p_i + mc^2\beta$. Then one finds $[L_i, H_D] = ic\varepsilon_{ijk}\,\alpha_j p_k$ and $[S_i, H_D] = ic\varepsilon_{ijk}\,\alpha_k p_j$. Thus, $[H_D, J_i] = 0$ and, since $[H_D, J_z] = 0$ and $[H_D, \mathbf{J}^2] = 0$, the common set of eigenfunctions of the commuting operators H_D, J_z and $\mathbf{J}^2$ can be labeled by E, M, J, where $H_D\psi_{E,J,M} = E\psi_{E,J,M}$, $J_z\psi_{E,J,M} = M\psi_{E,J,M}$, $\mathbf{J}^2\psi_{E,J,M} = J(J+1)\psi_{E,J,M}$. [2]

The Dirac equation in an external electromagnetic field $A^\mu = (\Phi, \mathbf{A})$ is

$$\left[\gamma^\mu\left(\hat{p}_\mu - \frac{e}{c}A_\mu\right) - mc\right]\psi = 0\,, \qquad (*)$$

where $\psi = e^{-imc^2t/\hbar}\begin{pmatrix} \tilde{\varphi} \\ \tilde{\chi} \end{pmatrix}$ *is the four-component Dirac spinor. The Minkowski metric is given by $\eta_{\mu\nu} = diag(+1, -1, -1, -1)$, $\hat{p}_\mu = i\hbar\partial_\mu$, and the Dirac matrices are* $\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$, $\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ -\sigma^k & 0 \end{pmatrix}$.

An alternative equation is proposed,

$$\left[\gamma^\mu\hat{p}_\mu - mc - \frac{i\kappa}{2c}F_{\mu\nu}\gamma^\mu\gamma^\nu\right]\psi = 0\,, \qquad (**)$$

where $F_{\mu\nu}$ is the electromagnetic field strength tensor related to the electric and magnetic fields via $E_i = cF_{0i}$ and $B_i = -\frac{1}{2}\varepsilon_{ijk}F_{jk}$, respectively. For equations (*) and (**), write down the system of coupled equations obeyed by the spinors $\tilde{\varphi}$ and $\tilde{\chi}$. *[4]*

Find the equations obeyed by the spinor $\tilde{\varphi}$ to leading order in the non-relativistic limit of the equations (*) and (**). Compare the two results. What is the physical meaning of the parameter κ? *[6]*

The Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$, we have $A^\mu = (\Phi, \mathbf{A})$ and $A_\mu = (\Phi, -\mathbf{A})$. Also,

$$\hat{p}^\mu = i\hbar\partial^\mu = \left(\frac{i\hbar}{c}\frac{\partial}{\partial t}, -i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z}\right)$$

and

$$\hat{p}_\mu = i\hbar\partial_\mu = \left(\frac{i\hbar}{c}\frac{\partial}{\partial t}, i\hbar\frac{\partial}{\partial x}, i\hbar\frac{\partial}{\partial y}, i\hbar\frac{\partial}{\partial z}\right).$$

Equation (*) is the Dirac equation with the minimal coupling to electromagnetic field. Using the explicit form of the Dirac matrices, we find

$$i\hbar\frac{\partial\tilde{\varphi}}{\partial t} = c\sigma_i\left(p_i - \frac{e}{c}A_i\right)\tilde{\chi} + e\Phi\tilde{\varphi},$$

$$i\hbar\frac{\partial\tilde{\chi}}{\partial t} = c\sigma_i\left(p_i - \frac{e}{c}A_i\right)\tilde{\varphi} + e\Phi\tilde{\chi} - 2mc^2\tilde{\chi}.$$ [2]

In the non-relativistic limit $|\frac{\partial\tilde{\chi}}{\partial t}| \ll mc^2$, $|e\Phi| \ll mc^2$, and the second equation gives

$$\tilde{\chi} \approx \frac{\sigma_i\left(p_i - \frac{e}{c}A_i\right)}{2mc}\tilde{\varphi}.$$ [1]

Substituting this into the first equation and using the identity $(\sigma_i A_i)(\sigma_j B_j) = A_k B_k + i\sigma_i\varepsilon_{ijk}A_j B_k$ (it follows from the given identity $\sigma_i\sigma_k = \delta_{ik} + i\epsilon_{ikl}\sigma_l$ and was discussed in the lectures), we find that $\tilde{\varphi}$ satisfies the non-relativistic Pauli equation

$$i\hbar\frac{\partial\tilde{\varphi}}{\partial t} = \left[\frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\Phi - \mu_0\sigma_i B_i\right]\tilde{\varphi},$$

where $\mathbf{B} = \mathrm{curl}\,\mathbf{A}$ and $\mu_0 = e\hbar/2mc$. [1]

Now the same procedure is applied to equation (**). We find

$$i\hbar\frac{\partial\tilde{\varphi}}{\partial t} = \left(c\sigma_i p_i + \frac{i\kappa}{c}\sigma_i E_i\right)\tilde{\chi} - \kappa\sigma_i B_i\tilde{\varphi},$$

$$i\hbar\frac{\partial\tilde{\chi}}{\partial t} = \left(c\sigma_i p_i - \frac{i\kappa}{c}\sigma_i E_i\right)\tilde{\varphi} - \left(2mc^2 - \kappa\sigma_i B_i\right)\tilde{\chi}.$$ [2]

In the non-relativistic limit the second equation gives

$$\tilde{\chi} \approx \frac{\sigma_i p_i}{2mc}\tilde{\varphi}.$$ [1]

Substituting this into the first equation, we obtain

$$i\hbar\frac{\partial\tilde{\varphi}}{\partial t} = \left[\frac{\mathbf{p}^2}{2m} - \kappa\sigma_i B_i\right]\tilde{\varphi},$$ [1]

which is a non-relativistic Pauli equation for a *neutral* particle (e.g. a neutron) with a non-vanishing magnetic moment κ in an external electromagnetic field. [2]

One can combine the minimal and non-minimal coupling to obtain the equation

$$\left[\gamma^\mu\left(\hat{p}_\mu - \frac{e}{c}A_\mu\right) - mc - \frac{i\kappa}{2c}F_{\mu\nu}\gamma^\mu\gamma^\nu\right]\psi = 0.$$

This would correspond to a relativistic particle of mass m, charge e, spin $s = 1/2$ and the magnetic moment $\mu = e\hbar/2mc + \kappa$.