Advanced Quantum Mechanics

S18 Exam 2017

Solutions

Problem 1

There are 3 ways to construct $G(x, x')$, all discussed in lectures:

- using Fourier transform
- using solutions of the free ODE
- analytic continuation from $E<0$.

(1.) Fourier transform

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E\right) G(x,x') = \delta(x-x'),$$

$E>0$.

$$G(x,x') = \frac{1}{2\pi\hbar}\int e^{ip(x-x')/\hbar}\, G(p)\, dp$$

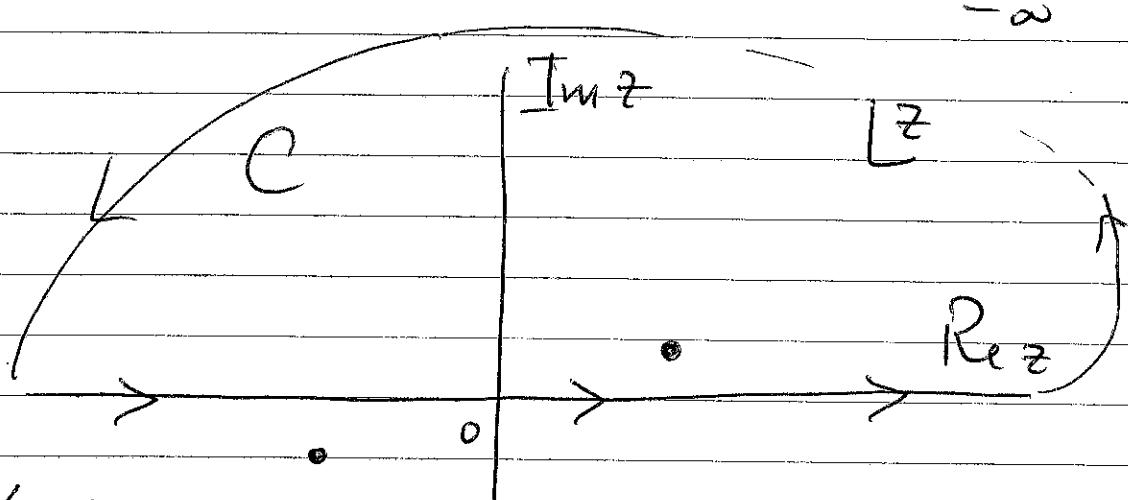
$$\Rightarrow \left(\frac{p^2}{2m} - E\right) G(p) = \mathbb{1}, \text{ since}$$

$$\delta(x-x') = \frac{1}{2\pi\hbar}\int e^{ip(x-x')/\hbar}\, dp$$

$$G(x-x') = \frac{2m}{\hbar^2}\frac{1}{2\pi}\int \frac{e^{iz(x-x')}}{z^2-k^2-i\varepsilon}\,dz,$$

$$z = p/\hbar, \quad k^2 = \frac{2mE}{\hbar^2} > 0, \quad \varepsilon > 0.$$

1) $x - x' > 0$: choose C in the upper half-plane of complex z ($\mathrm{Im}\, z > 0$, so $iz(x-x') < 0$ on the contour at ∞). Then the integral $\int_{-\infty}^{\infty} = \int_C$



choice $\varepsilon > 0$ guarantees $G \to e^{ik|x|}$ for $|x| \to \infty$.

$$z_* = \pm k\left(1 + \frac{i\varepsilon}{k^2}\right)^{1/2}$$

$$G(x-x') = \frac{2m}{\hbar^2}\frac{1}{2\pi}\,2\pi i\,\frac{e^{ik(x-x')}}{2k} = \frac{im}{k\hbar^2}\,e^{ik(x-x')}$$

2) Same procedure for $x - x' < 0$ but C is closed in the lower half-plane (note that the direction of C now

gives minus sign in the residue theorem:

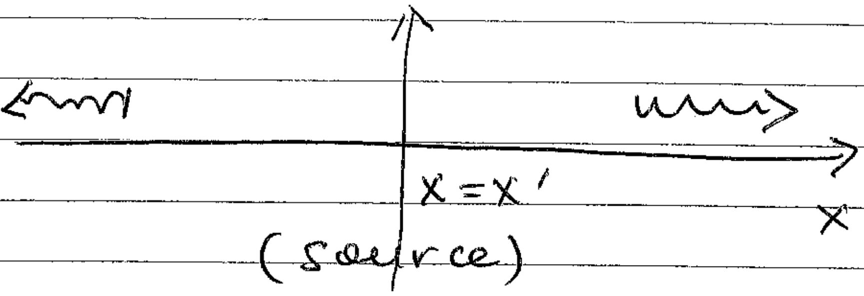
$$G(x-x') = \frac{2im}{\hbar^2}(-1)\frac{e^{-ik(x-x')}}{(-2k)} =$$

$$= \frac{im}{k\hbar^2} e^{-ik(x-x')}, \quad x-x'<0.$$

$$\Rightarrow G(x-x') = \frac{im}{k\hbar^2} e^{ik|x-x'|}, \text{ where}$$

$k = \sqrt{2mE}$, $E>0$.

(2.) Solving ODE directly with the b.c. $G \to e^{ik|x|}$, $|x|\to\infty$:



We have

$$G(x,x') = \begin{cases} A e^{-ik(x-x')}, & x<x' \\ B e^{ik(x-x')}, & x>x' \end{cases}$$

Conditions $\psi(x=x'-\varepsilon) = \psi(x=x'+\varepsilon)$

and $\psi'(x=x'+\varepsilon) - \psi'(x=x'-\varepsilon) = -\frac{2m}{\hbar^2}$

give $A = B$, $A = -\frac{m}{ik\hbar^2}$ $\Rightarrow$

$$G(x, x') = \frac{im}{k\hbar^2} e^{ik|x-x'|}$$

as before.

(3.) Finally, one can first obtain $G(x, x')$ for $E<0$ using the same approach as in (1) and (2):

$$G(x, x') = \frac{m}{\varkappa \hbar^2} e^{-\varkappa|x-x'|},$$

$\varkappa = \sqrt{-2mE}/\hbar$, $E<0$.

Continuing to $E>0$, $\varkappa = \pm ik$, where $k = \sqrt{2mE}/\hbar$, $E>0$. Thus,

$$G^{\pm}(x', x) = \pm \frac{im}{k\hbar^2} e^{\pm ik|x-x'|},$$

with $G^{+}(x, x')$ corresponding to the b.c. needed.

As discussed in lectures, writing the Schrödinger eq as

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - E\psi = f(x), \text{ where}$$

$f = -U(x)\,\psi(x)$, and using $G^+(x,x')$ for $E>0$ stationary states, we can write

$$\psi_s(x) = A e^{ikx} + B e^{-ikx} - \int_{-\infty}^{\infty} G(x,x')\,U(x')\,\psi_s(x')\,dx'$$

For particles incident from the left, $A=1$, $B=0$ (e.g. for $U=0$ we must have unperturbed wave e^{ikx} propag. from left to right), also for $x\to\infty$ only $\sim e^{ikx}$ must be present $\Rightarrow$ it must be $G^+(x,x')$. $\Rightarrow$

$$\psi_s(x) = e^{ikx} - \frac{im}{k\hbar^2}\int_{-\infty}^{\infty} e^{ik|x-x'|}\,U(x')\,\psi_s(x')\,dx'.$$

Considering the asymptotics at $x\to-\infty$, where $\psi_s = e^{ikx} + Ae^{-ikx}$, with $R = |A|^2$, one can read off A and

$$R = \frac{m^2}{k^2\hbar^4}\left|\int_{-\infty}^{\infty} e^{ikx}\, U(x)\, \psi_s(x)\, dx\right|^2$$

and $T = 1 - R$.

For $U(x) = -\alpha\delta(x)$, $\alpha > 0$, we have

$$\psi_s(x) = e^{ikx} + \frac{im\alpha}{k\hbar^2}\psi_s(0)\, e^{ik|x|},$$

$$\psi_s(0) = 1 + \frac{im\alpha}{k\hbar^2}\psi_s(0) \quad \Rightarrow$$

$$\psi_s(0) = \frac{1}{1 - \frac{im\alpha}{k\hbar^2}}.$$

$$R = \frac{m^2\alpha^2}{k^2\hbar^4}|\psi_s(0)|^2,$$ considering the coefficient of e^{-ikx} for $x \to -\infty$, and

$$T = \left|1 + \frac{im\alpha}{k\hbar^2}\psi_s(0)\right|^2,$$

considering the coeffi. of e^{ikx} for $x \to +\infty$

Simple algebra shows that

$$T = \frac{1}{1+\beta^2}, \quad R = \frac{\beta^2}{1+\beta^2},$$

where $\beta = m\alpha/k\hbar^2$. Clearly, $T+R=1$.

Consider now a pert. theory with $\psi_s = \psi_s^{(0)} + \psi_s^{(1)} + \dots$, where $\psi_s^{(0)} = e^{ikx}$.

To leading order

$$\psi_s = e^{ikx} + \psi_s^{(1)} + \dots = e^{ikx} - \frac{im}{k\hbar^2}\int_{-\infty}^{\infty} e^{ik|x-x'|} U(x') e^{ikx'} dx' + \dots$$

To find R, we look at asymptotics at $x \to -\infty$ (the coeffi in front of e^{-ikx}):

$$R = \frac{m^2}{k^2\hbar^4}\left|\int_{-\infty}^{\infty} e^{2ikx} U(x)\,dx\right|^2 .$$

For $U = -\alpha\,\delta(x)$, we get

$R = \dfrac{m^2\alpha^2}{k^2\hbar^4}$ which can be compared with the exact result:

$$R_{exact} = \frac{\beta^2}{1+\beta^2} = \beta^2 - \beta^4 + \ldots ,$$

$\beta = m\alpha / k\hbar^2$. Clearly, the leading order pert. theory approximates the exact result for $\beta \ll 1$ fairly well. The condition $\beta \ll 1$ is $k \gg \frac{m\alpha}{\hbar^2}$ or $E = \frac{\hbar^2 k^2}{2m} \gg \frac{m\alpha^2}{2\hbar^2} = \frac{m U_0^2}{2\hbar^2}$, i.e. the approx. works well for high-energy particles. Note that $E_0 = \frac{m U_0^2}{2\hbar^2} = \frac{m\alpha^2}{2\hbar^2}$ is the energy level (magnitude of it) in the well $U(x) = -\alpha\delta(x)$, so the condition can be written as

$$E \gg E_0 .$$

Note that in this approximation there is no difference between potentials $U = -\alpha\delta(x)$ and $U = \alpha\delta(x)$, $\alpha > 0$.

For stationary states with $E<0$ the free particle Green's function satisfies (+ b.c. $G \to 0$ for $|x| \to \infty$):

$$\left(-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} + |E| \right) G = \delta(x-x')$$

$$\Rightarrow \quad G(x,x') = \begin{cases} A\, e^{\varkappa(x-x')}, & x < x' \\ C\, e^{\varkappa(x'-x)}, & x > x' \end{cases}$$

$G(x,x')$ is contin. at $x=x'$ but

$$G'(x=x'+\varepsilon) - G'(x=x'-\varepsilon) = -\frac{2m}{\hbar^2}$$

$$\Rightarrow A = C = m/\varkappa\hbar^2, \quad \varkappa = \sqrt{2m|E|}/\hbar.$$

$$\Rightarrow \quad G(x,x') = \frac{m}{\varkappa\hbar^2}\, e^{-\varkappa|x-x'|}.$$

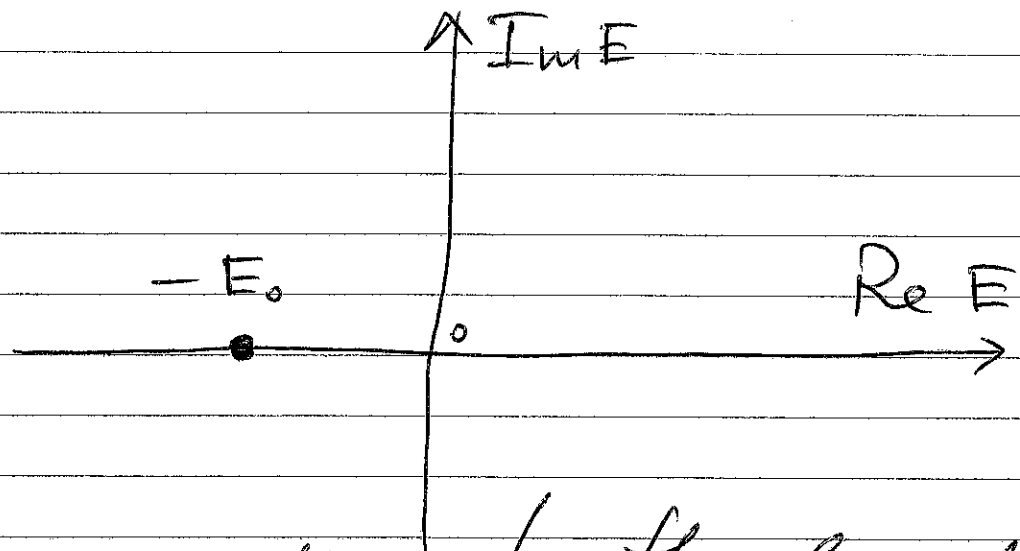
This result can also be obtained via Fourier transf. or analytic continuation from $E>0$ case, as discussed in lectures.

The transmission coefficient was found earlier: $T = \dfrac{1}{1+\beta^2} \Rightarrow$

$\beta^2 = -1$ is a singularity for

$$\frac{m^2 \alpha^2}{k^2 \hbar^4} = -1 \quad \text{or} \quad E = \frac{\hbar^2 k^2}{2m} = -\frac{m\alpha^2}{2\hbar^2}.$$

In other words, T has a singularity in the complex E-plane



corresponding to the bound state energy in the same potential $U(x)$. This also corresponds to the pole of $S(k)$, $T = |S|^2$, where $S(k) = \dfrac{ik}{ik+\varkappa}$ is the transmission amplitude.

Problem 2

$$\hat{L}\, G(x,x') = \delta(x-x')$$

$$\hat{L} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} - E, \quad \text{with } E<0.$$

$G(x,x')$ can be found by various methods.

1) Fourier transf.

$$G(x,x') = \frac{1}{2\pi\hbar}\int e^{ip(x-x')/\hbar}\, G(p)\, dp$$

Then the eq. gives

$$\left(\frac{p^2}{2m} + |E|\right) G(p) = 1,$$

since $\delta(x-y) = \frac{1}{2\pi}\int e^{ik(x-y)}\, dk$.

So,

$$G(x-x') = \frac{2m}{\hbar^2}\frac{1}{2\pi}\int \frac{e^{ik(x-x')}}{k^2+\varkappa^2}\, dk, \quad \text{where}$$

$\varkappa^2 = 2m|E|/\hbar^2$. Integrand has poles at $k = \pm i\varkappa$.

a) $x-x'>0$: close contour in the upper half-plane, since the integrand vanishes for $\mathrm{Im}\, k \to \infty$, and use residue theorem:

$$\int \frac{e^{ik(x-x')}}{k^2+\varkappa^2}\, dk = 2\pi i \frac{e^{-\varkappa(x-x')}}{2i\varkappa} = \frac{\pi}{\varkappa} e^{-\varkappa(x-x')}$$

$$\Rightarrow \quad G(x-x') = \frac{m}{\varkappa \hbar^2} e^{-\varkappa(x-x')} \quad \text{for } x-x'>0.$$

Similarly, β) $x-x'<0$:

$$G(x-x') = \frac{m}{\varkappa \hbar^2} e^{\varkappa(x-x')}, \qquad x-x'<0$$

(contour closed in the lower half-plane)

$$\Rightarrow \quad G(x-x') = \frac{m}{\varkappa \hbar^2} e^{-\varkappa|x-x'|}.$$

2) From ODE:

For $x \neq x'$: free eq.

α) $x<x'$

$$G(x,x') = A(x')\, e^{\varkappa(x-x')} + B(x')\, e^{-\varkappa(x-x')}$$

$x \to -\infty$ should have $G \to 0$

$$\Rightarrow B(x') = 0 \Rightarrow G = A(x')\, e^{\varkappa(x-x')}, \quad x<x'$$

β) Similarly, $x > x'$:

$$G(x, x') = C(x')\, e^{-\varkappa(x-x')}$$

γ) at $x = x'$: G is contin. but G' isn't and (integrating the eq) we have

$$G'(x = x'+\varepsilon, x') - G'(x = x'-\varepsilon, x') = -\frac{2m}{\hbar^2}$$

These 2 conditions give $A = C = \frac{m}{\varkappa\hbar^2}$.

$$\Rightarrow G(x-x') = \frac{m}{\varkappa\hbar^2}\, e^{-\varkappa|x-x'|}.$$

Now we can write Schröd. eq. as integral eq, since for

$$-\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi - E\psi = f(x),$$

where $f = -U(x)\,\psi(x)$, we can write

$$\psi(x) = A e^{-\varkappa x} + B e^{\varkappa x} + \int_{-\infty}^{\infty} G(x, x') f(x')\, dx'.$$

Since $E < 0$ and $\psi \to 0$ for $|x| \to \infty$ (note $U(x) \to 0$ for $|x| \to \infty$), we must have $A = 0$ and $B = 0$. Thus,

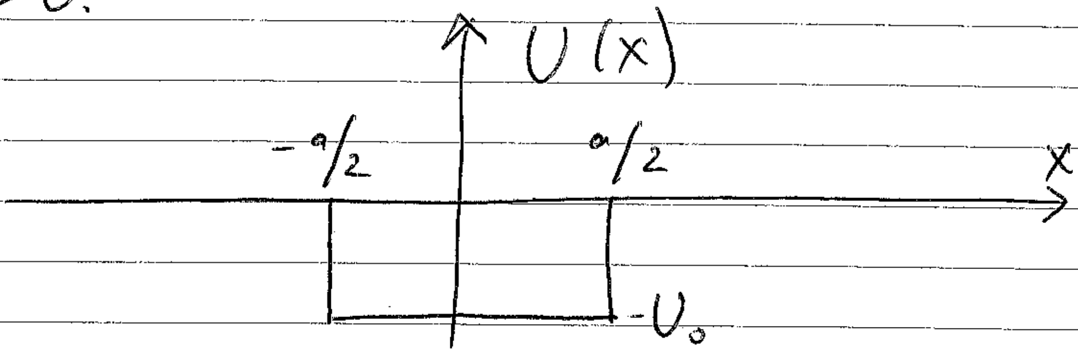
$$\psi(x) = -\frac{m}{\varkappa \hbar^2} \int_{-\infty}^{\infty} e^{-\varkappa|x-x'|} U(x')\psi(x')\,dx',$$

where $\varkappa = \sqrt{-2mE}/\hbar$, $E < 0$.

Consider a potential

$$U(x) = \begin{cases} -U_0, & |x| < a/2, \\ 0, & |x| > a/2, \end{cases}$$

with $U_0 > 0$.



Even wave-functions:

$$\psi(x) = A\cos\left[\frac{1}{\hbar}\sqrt{2m(|U_0|-|E|)}\,x\right]$$ are

sol. of

$$\left(\partial^2_{xx} + (U_0 - |E|)\frac{2m}{\hbar^2}\right)\psi = 0$$

with $\psi(-x) = \psi(x)$ for $|x| < a/2$

and $\psi(x) = B e^{-\sqrt{2m|E|}\,|x|/\hbar}$

for $|x| > a/2$.

Odd wave-functions:

$$\psi(x) = C \sin\left[\frac{1}{\hbar}\sqrt{2m(U_0 - |E|)}\, x\right]$$

for $|x| < a/2$

$$\psi(x) = D \,\mathrm{sgn}(x)\, e^{-\sqrt{2m|E|}\,|x|/\hbar}$$

for $|x| > a/2$, $\mathrm{sgn}(-x) = -\mathrm{sgn}(x)$,

$\mathrm{sgn}(x) = 2\Theta(x) - 1$. Note: $\mathrm{sgn}'(x) = 2\delta(x)$.

Note $\psi(x)$ and $\psi'(x)$ are contin. at $x = \pm a/2$. E.g. for odd functions at $x = a/2$:

$$C \sin\left[\frac{a}{2\hbar}\sqrt{2m(U_0 - |E|)}\right] = D e^{-\frac{\sqrt{2m|E|}\,a}{2\hbar}}$$

$$C \frac{\sqrt{\quad}}{\hbar} \cos\left[\frac{a}{2\hbar}\sqrt{\quad}\right] = -D \frac{\sqrt{2m|E|}}{\hbar} e^{-\frac{\sqrt{2m|E|}\,a}{2\hbar}}$$

Note that at $x = \frac{a}{2}$ all contributions from discont. functions can be ignored. We get

$$\cot\left[\frac{\sqrt{2m}\,a}{2\hbar}\sqrt{U_0-|E|}\right] = -\frac{\sqrt{|E|}}{\sqrt{U_0-|E|}}$$

or $$\cot\left[\frac{\varkappa a}{2}\sqrt{\frac{U_0}{|E|}-1}\right]\cdot\sqrt{\frac{U_0}{|E|}-1} = -1$$

and similarly for the even states

$$\sqrt{\frac{U_0}{|E|}-1}\,\tan\left[\frac{\varkappa a}{2}\sqrt{\frac{U_0}{|E|}-1}\right] = 1$$

Now consider scattering states with $E>0$
We have

$$\psi = e^{ikx} + Ae^{-ikx} \quad \text{for } x < -a/2$$

$$\psi = Be^{i\bar{\varkappa}x} + Ce^{-i\bar{\varkappa}x} \quad \text{for } |x| < a/2$$

$$\psi = Se^{ik(x-a)}, \quad x > a/2.$$

Here $k^2 = 2mE/\hbar^2 > 0$ and

$$\bar{\varkappa}^2 = \frac{2m}{\hbar^2}\left(E+U_0\right).$$

Continuity conditions for ψ and ψ' at $x = \pm a/2$ lead to:

$$e^{-ika/2} + A e^{ika/2} = B e^{-i\bar{\varkappa} a/2} + C e^{i\bar{\varkappa}\frac{a}{2}}$$

$$B e^{i\bar{\varkappa} a/2} + C e^{-i\bar{\varkappa} a/2} = S e^{-ika/2}$$

$$ik e^{-ika/2} - ik A e^{ika/2} = i\bar{\varkappa} B e^{-i\bar{\varkappa} a/2} - i\bar{\varkappa} C e^{i\bar{\varkappa} a/2}$$

$$i\bar{\varkappa} B e^{i\bar{\varkappa} a/2} - i\bar{\varkappa} C e^{-i\bar{\varkappa}\frac{a}{2}} = S e^{-ika/2} \cdot ik$$

This can be solved for S

$$S = \frac{k\bar{\varkappa}}{k\bar{\varkappa}\cos\bar{\varkappa}a - \frac{i}{2}(k^2+\bar{\varkappa}^2)\sin\bar{\varkappa}a}$$

and thus for $T = S^* S$:

$$T = \left[1 + \frac{(k^2-\bar{\varkappa}^2)^2}{4k^2\bar{\varkappa}^2}\sin^2\bar{\varkappa}a\right]^{-1}.$$

(Notation $\bar{\zeta} \equiv \bar{\varkappa}$ used in the problem)

One can also write T as

$$T = \left[\cos^2 \zeta a + \frac{(k^2+\zeta^2)^2}{4k^2\zeta^2}\sin^2 \zeta a\right]^{-1}$$

The singularities on the Im axis of k can be analyzed e.g. by setting $k = iy$, $y \in \mathbb{R}$, then

$$\zeta^2 = k^2 + \frac{2mU_0}{\hbar^2} = -y^2 + \frac{2mU_0}{\hbar^2}$$

Zeros of denom. of T are at

$$\cot \zeta a = \pm \frac{\frac{2mU_0}{\hbar^2} - 2y^2}{2y\zeta}, \quad (*)$$

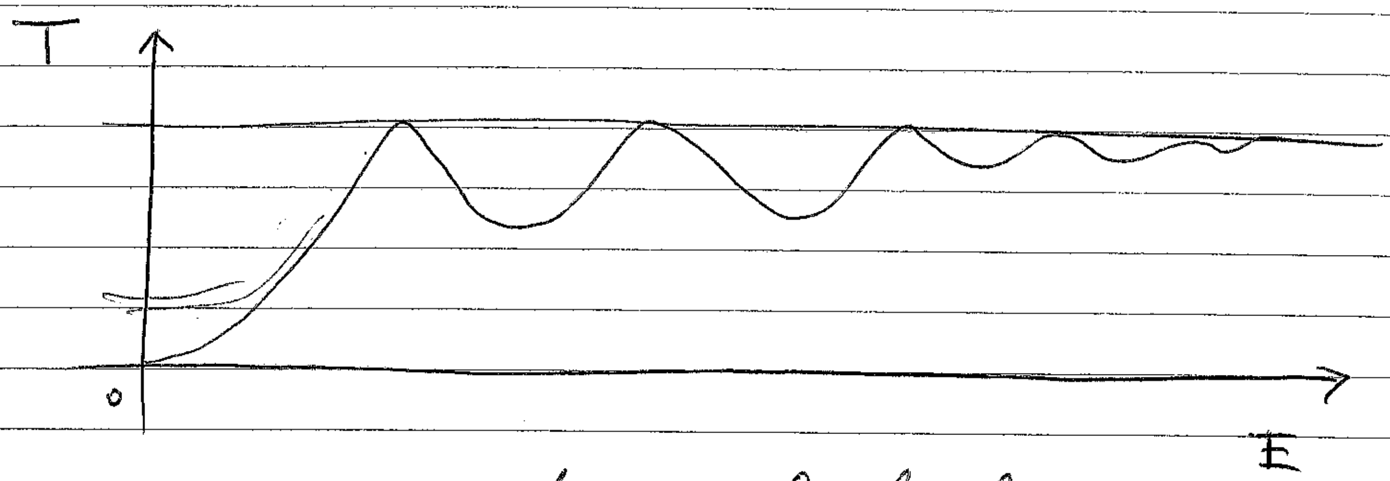
whereas the bound states found earlier are given by eqs

$$\tan \frac{\zeta a}{2} = \pm \frac{y}{\zeta} \quad (**)$$

Use e.g. $\tan 2X = \dfrac{2\tan X}{1-\tan^2 X}$ to see that $(*)$ and $(**)$ are the same.

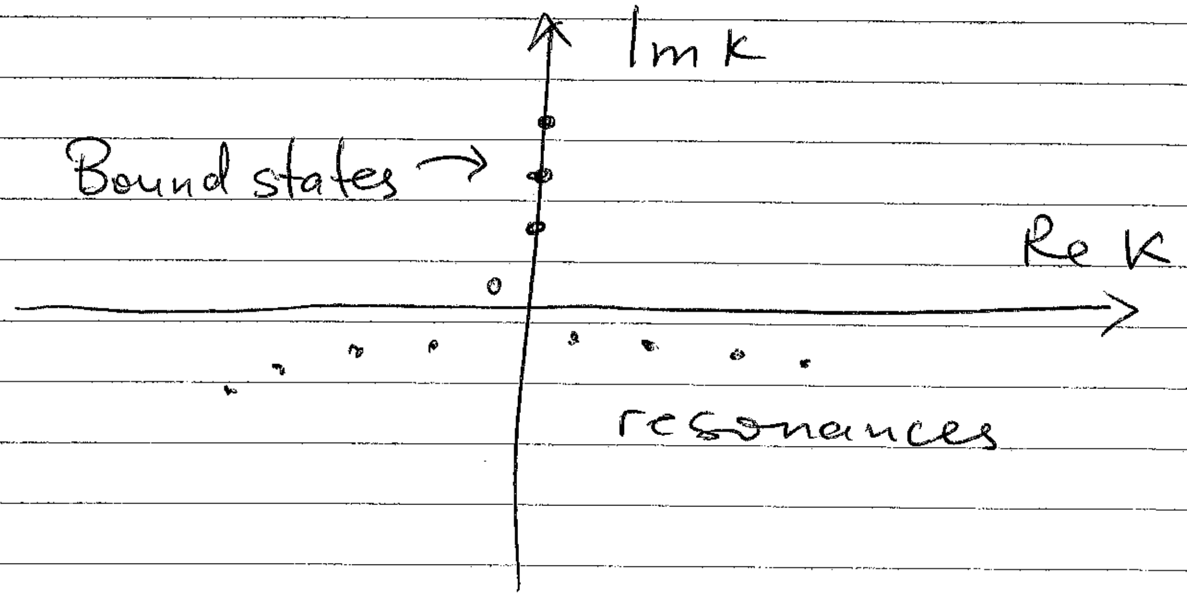
The max value of T is 1, and it is achieved for $\sin \zeta a = 0$, i.e. for $E_n = \frac{n^2 \pi^2 \hbar^2}{2ma^2} - U_0 > 0$,

$n = 1, 2, \ldots$



For $E = E_n$ the potential is completely transparent.

In the complex k - plane



Problem 3

$$\left[\gamma^0 \hat{p}_0 + \gamma^1\left(\hat{p}_x - \frac{e}{c}A_x\right) + \gamma^2 \hat{p}_y + \gamma^3 \hat{p}_z - mc\right]\psi = 0$$

$$(\gamma^0)^2 = \mathbb{1}, \qquad \hat{p}_0 = \frac{i\hbar}{c}\partial_t$$

$$\frac{i\hbar}{c}\frac{\partial\psi}{\partial t} = \Big(-\gamma^0\gamma^1\left(\hat{p}_x - \frac{e}{c}A_x\right) - \gamma^0\gamma^2\hat{p}_y -$$

$$- \gamma^0\gamma^3\hat{p}_z + \gamma^0 mc\Big)\psi$$

$$\vec{A} = (-By, 0, 0) \Rightarrow \vec{B} = (0, 0, B).$$

$$\begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ -By & 0 & 0 \end{vmatrix} \quad \text{OK.} \qquad A^x = -By$$

$$A_x = By$$

$$\hat{p}_x = i\hbar\partial_x$$

$$\hat{p}_x - \frac{e}{c}A_x = i\hbar\partial_x - \frac{eB}{c}y.$$

$$\gamma^0\gamma^k = \begin{pmatrix} 0 & \sigma^k \\ \sigma^k & 0 \end{pmatrix}$$

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}_D\psi, \text{ where}$$

$$\hat{H}_D = \begin{bmatrix} mc^2 & -\sigma^1 c\left(i\hbar\partial_x - \frac{eBy}{c}\right) - \sigma^2 c i\hbar\partial_y - \sigma^3 c i\hbar\partial_z \\ '''''' & -mc^2 \end{bmatrix}$$

$$[\hat{H}_D, \hat{p}_x] = 0 \quad \text{and} \quad [\hat{H}_D, \hat{p}_z] = 0$$

$\Rightarrow$ can use common basis with eigenvalues E, p_x, p_z $\left(\hat{p}_x \psi = p_x \psi \quad \text{etc}\right)$

$$\psi = e^{-\frac{iE}{\hbar} + i\frac{p_x}{\hbar}x + i\frac{p_z}{\hbar}z} \begin{pmatrix} \varphi(y) \\ \chi(y) \end{pmatrix}$$

$\Rightarrow$ eq. becomes

$$E\varphi = mc^2\varphi + \left[\sigma^1\left(p_x c + \frac{eBy}{\cancel{c}}\right) + \sigma^3 c p_z - \sigma^2 c i\hbar\partial_y\right]\chi$$

$$E\chi = \left[\sigma^1 c\left(p_x + \frac{eBy}{c}\right) + \sigma^3 c p_z - \sigma^2 c i\hbar\partial_y\right]\varphi - mc^2\chi$$

$$\Rightarrow \chi = \frac{1}{E + mc^2}\left[\sigma^1(p_x c + eBy) + \sigma^3 p_z c - i\hbar c\sigma^2\partial_y\right]\varphi$$

$$\Rightarrow \left(E^2 - m^2c^4\right)\varphi =$$

$$\left[\sigma^1(p_x c + eBy) + \sigma^3 p_z c - i\hbar c\sigma^2 \partial_y\right] \times$$

$$\left[\sigma^1(p_x c + eBy) + \sigma^3 p_z c - i\hbar c\sigma^2 \partial_y\right] \varphi(y)$$

Use $\sigma_i \sigma_k = \delta_{ik} + i\epsilon_{ikl}\sigma_l$ to find

$$\Big[\underline{(p_x c + eBy)^2} - \cancel{i\sigma^2 (p_x c + eBy) p_z c} -$$

$$\cancel{-i\sigma^1\sigma^2 (p_x c + eBy)\hbar c\, \partial_y} +$$

$$+ \cancel{i\sigma^2 (p_x c + eBy) p_z c} + \underline{p_z^2 c^2} - i\hbar c\, \cancel{p_z c\sigma^3\sigma^2 \partial_y}$$

$$- i\hbar c\, \cancel{\sigma^2\sigma^1 (p_x c + eBy)\partial_y}\ \underline{- i\hbar c eB\sigma^2\sigma^1} -$$

$$- i\hbar c\, \cancel{\sigma^2\sigma^3 p_z c\, \partial_y} - \underline{(\hbar c)^2 \partial^2_{yy}}\ \Big] \varphi(y) =$$

$$= \left[(p_x c + eBy)^2 + p_z^2 c^2 - \hbar c eB\sigma^3 - (\hbar c)^2 \partial^2_{yy}\right]\varphi$$

$$= \left(E^2 - m^2c^4\right)\varphi.$$

$$\left[\frac{d^2}{dy^2} - \frac{(p_x c + eBy)^2}{\hbar^2 c^2} - \frac{p_z^2 \cancel{c^2}}{\hbar^2 \cancel{c^2}} + \frac{eB}{\hbar c}\sigma^3 + \right.$$

$$\left. + \frac{E^2 - m^2c^4}{(\hbar c)^2}\right]\varphi(y) = 0$$

Introduce $\xi = (p_x c + eBy)/\hbar c$

$$\Rightarrow \frac{d^2}{dy^2} = \frac{d^2}{d\xi^2}\cdot\left(\frac{eB}{\hbar c}\right)^2 \Rightarrow$$

$$\left[\frac{d^2}{d\xi^2} - \left(\frac{\hbar c}{eB}\right)^2 \xi^2 - \left(\frac{\hbar c}{eB}\right)^2 \frac{p_z^2}{\hbar^2} + \left(\frac{\hbar c}{eB}\right)^2 \frac{eB}{\hbar c}\sigma_3 + \right.$$

$$\left. + \frac{E^2 - m^2c^4}{(eB)^2}\right]\varphi = 0$$

Compare with

$$\left(-\frac{\hbar^2}{2M}\frac{d^2}{d\xi^2} + \frac{M\omega^2\xi^2}{2}\right)\psi = \varepsilon\psi$$

$$\Rightarrow \frac{2M}{\hbar^2}\frac{M\omega^2}{2} = \left(\frac{\hbar c}{eB}\right)^2 \Rightarrow \left(\frac{M\omega}{\hbar}\right)^2 = \left(\frac{\hbar c}{eB}\right)^2$$

and

$$\frac{2M\mathcal{E}}{\hbar^2} = \frac{E^2 - m^2c^4}{(eB)^2} - \left(\frac{\hbar c}{eB}\right)^2 \frac{p_z^2}{\hbar^2} + \frac{\hbar c \cancel{eB}}{(eB)^{\cancel{2}}}\sigma_3$$

But $\varepsilon = \hbar\omega(n + 1/2)$, $n = 0, 1, 2, \ldots$

$$\frac{2M\mathcal{E}}{\hbar^2} = \frac{M\omega(2n+1)}{\hbar} = \frac{\hbar c}{|eB|}(2n+1) =$$

$$= \frac{E^2 - m^2c^4}{(eB)^2} - \frac{p_z^2c^2}{(eB)^2} + \frac{\hbar c}{eB}\sigma_3$$

$$\Rightarrow E^2 - m^2c^4 = p_z^2c^2 + \hbar c\, eB\, \sigma_3 + \hbar c|eB|(2n+1)$$

$$\Rightarrow E^2 = m^2c^4 + p_z^2c^2 + \hbar c|eB|(2n+1) \pm \hbar c eB$$

$$(\sigma_3 \varphi = \pm \varphi)$$

or

$$E^2 = m^2c^4 + 2mc^2\left[\frac{p_z^2}{2m} + \mu B(2n+1) \pm \mu B\right],$$

$$\mu = \frac{|e|\cancel{\hbar}}{2mc}.$$

Spin 0 particle:

$$\left[c^2\left(-i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}\right)^2 + m^2c^4\right]\phi = E^2\phi$$

$$c^2\left(-i\hbar\frac{\partial}{\partial x} + \frac{eBy}{c}\right)\left(-i\hbar\partial_x + \frac{eBy}{c}\right)\phi -$$

$$-(\hbar c)^2\frac{\partial^2\phi}{\partial y^2} - (\hbar c)^2\frac{\partial^2\phi}{\partial z^2} + m^2c^4\phi = E^2\phi$$

Or

$$\left[-(\hbar c)^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) - i\,2\hbar c eBy\,\partial_x + \right.$$

$$\left. + m^2c^4 + e^2B^2y^2 - E^2\right]\phi = 0$$

i.e. (with ansatz):

$$\left[c^2p_x^2 + c^2p_z^2 - (\hbar c)^2\frac{\partial^2}{\partial y^2} + 2ceByp_x + \right.$$

$$\left. + m^2c^4 + e^2B^2y^2 - E^2\right]\phi = 0$$

Or

$$\left[\frac{d^2}{dy^2} - \frac{(p_xc + eBy)^2}{\hbar^2c^2} - \frac{p_z^2}{\hbar^2} + \frac{E^2 - m^2c^4}{\hbar^2c^2}\right]\phi = 0 \quad [B]$$

where $S_z = 0$ for $s = 0$,
$S_z = \pm\hbar/2$ for $s = 1$, $S_z = 0, \pm\hbar$ for $s = 1$ etc, with $2s+1$ values in general.

For $s = 1$, in particular, we have an instability ($\varepsilon^2 < 0$) in the spectrum (for $p_z = 0$, $n = 0$, $eB > \frac{m^2c^4}{\hbar c}$)

=> the solution is no longer stationary
=> one-particle wave eq. not applicable. This is a known instab. of a gauge boson in an external magnetic field.

Solutions : problem 3

3. The first step is to write the Dirac equation in the form (this can be done by multiplying the equation by γ^0 from the left and remembering that $(\gamma^0)^2 = 1_{4x4}$)

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}_D\psi\,,$$

where the Dirac Hamiltonian is

$$\hat{H}_D = -c\gamma^0\gamma^1\hat{p}_x + \gamma^0\gamma^1 eBy - \gamma^0\gamma^2 c\hat{p}_y - \gamma^0\gamma^3 c\hat{p}_z + mc^2\gamma^0\,.$$

One should not forget that with the Minkowski metric given by $\eta_{\mu\nu} = \mathrm{diag}(+1,-1,-1,-1)$, we have $A^\mu = (\Phi, \mathbf{A})$ and $A_\mu = (\Phi, -\mathbf{A})$. Also,

$$\hat{p}^\mu = i\hbar\partial^\mu = \left(\frac{i\hbar}{c}\frac{\partial}{\partial t}, -i\hbar\frac{\partial}{\partial x}, -i\hbar\frac{\partial}{\partial y}, -i\hbar\frac{\partial}{\partial z}\right)$$

and

$$\hat{p}_\mu = i\hbar\partial_\mu = \left(\frac{i\hbar}{c}\frac{\partial}{\partial t}, i\hbar\frac{\partial}{\partial x}, i\hbar\frac{\partial}{\partial y}, i\hbar\frac{\partial}{\partial z}\right)\,.$$

Clearly, $\hat{H}_D$ commutes with $\hat{p}_x$ and $\hat{p}_z$, but not with $\hat{p}_y$. Thus, one can choose common eigenstates of $\hat{H}_D$, $\hat{p}_x$ and $\hat{p}_z$ in the form

$$\psi = e^{-i\frac{E}{\hbar}t + i\frac{p_x}{\hbar}x + i\frac{p_z}{\hbar}z}\begin{pmatrix}\varphi(y)\\ \chi(y)\end{pmatrix},$$

where E, p_x, p_z are the corresponding eigenvalues, and the spinors $\varphi(y)$ and $\chi(y)$ satisfy the system of coupled equations

$$\begin{cases} E\varphi = mc^2\varphi + (cp_x\sigma^1 + eBy\sigma^1 + cp_z\sigma^3 - i\hbar c\sigma^2\partial_y)\,\chi\,,\\ E\chi = -mc^2\chi + (cp_x\sigma^1 + eBy\sigma^1 + cp_z\sigma^3 - i\hbar c\sigma^2\partial_y)\,\varphi\,.\end{cases}$$

Combining the two equations (and using the identity $\sigma_i\sigma_k = \delta_{ik} + i\epsilon_{ikl}\sigma_l$), we find the equation for φ:

$$\left(\frac{d^2}{dy^2} - \frac{(p_xc + eBy)^2}{\hbar^2c^2} + \frac{E^2 - m^2c^4}{\hbar^2c^2} - \frac{p_z^2}{\hbar^2} + \frac{eB}{\hbar c}\sigma_3\right)\varphi(y) = 0\,.$$

Now we introduce a new variable $\xi = (p_xc + eBy)/\hbar c$. The equation above can be rewritten in the form of the Schrödinger equation for a harmonic oscillator,

$$\left(-\frac{\hbar^2}{2M}\frac{d^2}{d\xi^2} + \frac{M\omega^2\xi^2}{2}\right)\Psi(\xi) = \varepsilon\Psi(\xi)\,,$$

with identifications

$$\frac{M^2\omega^2}{\hbar^2} = \left(\frac{\hbar c}{eB}\right)^2$$

and

$$\frac{2M}{\hbar^2}\varepsilon = -\left(\frac{cp_z}{eB}\right)^2 + \left(\frac{\hbar c}{eB}\right)\sigma_3 + \frac{E^2 - m^2c^4}{e^2B^2}\,.$$

This, and the fact that $\sigma_3\varphi = \pm\varphi$ lead to the spectrum E_{n,p_z} determined by

$$E_{n,p_z}^2 = m^2c^4 + 2mc^2\left[\frac{p_z^2}{2m} + \mu B(2n+1) \pm \mu B\right], \qquad n = 0, 1, 2, ...,$$

where $\mu = |e|\hbar/2mc$.

For a spinless relativistic particle of mass m and charge e in an external constant magnetic field $\mathbf{B} = (0, 0, B)$ obeying the stationary Klein-Gordon equation,

$$\left[c^2\left(-i\hbar\boldsymbol{\nabla} - \frac{e}{c}\mathbf{A}\right)^2 + m^2c^4\right]\phi(x,y,z) = E^2\phi(x,y,z)\,,$$

the procedure is almost identical. Using the same ansatz as before (for the same reason), the equation can be rewritten as

$$\left[p_x^2c^2 + p_z^2c^2 - \hbar^2c^2\frac{\partial^2}{\partial y^2} + 2ceByp_x + m^2c^4 + e^2B^2y^2 - E^2\right]\phi = 0$$

or

$$\left(\frac{d^2}{dy^2} - \frac{(p_xc + eBy)^2}{\hbar^2c^2} + \frac{E^2 - m^2c^4}{\hbar^2c^2} - \frac{p_z^2}{\hbar^2}\right)\phi(y) = 0\,.$$

Changing variable to $\xi = (p_xc + eBy)/\hbar c$, we again obtain the equation for the harmonic oscillator with identifications

$$\frac{M^2\omega^2}{\hbar^2} = \left(\frac{\hbar c}{eB}\right)^2$$

and

$$\frac{2M}{\hbar^2}\varepsilon = -\left(\frac{cp_z}{eB}\right)^2 + \frac{E^2 - m^2c^4}{e^2B^2}\,.$$

This gives the energy levels:

$$E^2_{n,p_z} = m^2c^4 + 2mc^2\left[\frac{p_z^2}{2m} + \mu B(2n+1)\right]\,, \qquad n = 0, 1, 2, ...,$$

where $\mu = |e|\hbar/2mc$.

Based on these results, we can guess a formula for the energy levels of a relativistic particle of spin s in a constant magnetic field

$$E^2_{n,p_z} = m^2c^4 + 2mc^2\left[\frac{p_z^2}{2m} + \mu B(2n+1) + \frac{2S_z}{\hbar}\mu B\right]\,, \qquad n = 0, 1, 2, ...,$$

where $S_z = 0$ for $s = 0$, $S_z = \pm\hbar/2$ for $s = 1/2$, $S_z = 0, \pm 1$ for $s = 1$ etc, with $2s + 1$ values in general. For $s = 1$, there exists an instability ($E^2 < 0$) in the spectrum for $p_z = 0$, $n = 0$ and $eB > m^2c^4/\hbar c$. The solution is no longer stationary, one may expect particle production by the field, and the one-particle wave equation description breaks down. This is in fact a known instability of a gauge boson in an external magnetic field.