Advanced Quantum Mechanics

S18 Exam 2015

Solutions

Problem 1

The Green's function $G(x,x')$ obeys

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}G(x,x') - E\,G(x,x') = \delta(x-x')$$

and $G(x,x') \to 0$ for $x - x' \to \infty$,

$G(0,x') = 0$.

Construct $G(x,x')$ as discussed in lectures:

a) $x < x'$: solution is

$$G(x,x') = A(x')\,e^{\varkappa(x-x')} + B(x')\,e^{-\varkappa(x-x')}$$

$G(0,x') = 0 \Rightarrow B = -A\,e^{-2\varkappa x'}$ 1

$$\Rightarrow G(x,x') = A(x')\left[e^{\varkappa(x-x')} - e^{-\varkappa(x+x')}\right]$$

Here $\varkappa = \sqrt{-2mE}/\hbar$, $E < 0$.

b) $x > x'$: b.c. at $x \to +\infty$ 1

leaves $G(x,x') = C(x')\, e^{-\varkappa(x-x')}$ 1-2

Conditions of continuity:

$G(x,x')$ is cont. at $x = x'$ but

$G'(x = x'+\varepsilon, x') - G'(x = x'-\varepsilon, x') = -\frac{2m}{\hbar^2}$ 1

(discussed in lectures in detail)

$\Rightarrow \quad C = A\left[1 - e^{-2\varkappa x'}\right]$

$A\left[1 + e^{-2\varkappa x'}\right] + C = \frac{2m}{\varkappa\hbar^2}$

$\Rightarrow \quad A = \frac{m}{\varkappa\hbar^2}$ and

$C = \frac{m}{\varkappa\hbar^2}\left[1 - e^{-2\varkappa x'}\right] \Rightarrow$ 1

$$\Rightarrow \quad G(x,x') = \begin{cases} \dfrac{m}{\varkappa\hbar^2}\left[e^{\varkappa(x-x')} - e^{-\varkappa(x+x')}\right] & x<x' \\[2ex] \dfrac{m}{\varkappa\hbar^2}\left(1 - e^{-2\varkappa x'}\right) e^{-\varkappa(x-x')}, & x>x' \end{cases}$$

as expected.

The Schrödinger eq.

$-\frac{\hbar^2}{2m}\psi'' - E\psi = f(x)$, 1 where $f = -U(x)\psi(x)$, can be 1-3

written as 1

$$\psi(x) = Ae^{-\varkappa x} + Be^{\varkappa x} + \int_{-\infty}^{\infty} G(x,x') f(x') dx'$$

as discussed in detail in lectures

Here $U(x) = -q\,\delta(x-a)$, $q > 0$ 1

B.c. at $x \to \infty$: $\Rightarrow B = 0$,

b.c. at $x = 0 \Rightarrow A = 0 \Rightarrow$ 1

$$\psi(x) = -\int_0^{\infty} G(x,x')\,U(x')\,\psi(x')\,dx'$$

In fact, $\psi(x) = q\,G(x,a)\,\psi(a)$. 1

This gives eq. for the bound states 2

energies: $\psi(a) = q\,G(a,a)\,\psi(a)$

$\Rightarrow G(a,a) = 1/q$ or 1

$$\boxed{\varkappa = \frac{mq}{\hbar^2}\left(1 - e^{-2\varkappa a}\right)} \quad (*)$$

Consider now stationary states with $E > 0$ in $U(x)$.

For $0 < x < a$:

$\psi(x) = A\left(e^{ikx} - e^{-ikx}\right)$ with 1

$\psi(0) = 0$

For $x > a$: $\psi = e^{-ikx} + Be^{ikx}$ 1

Conditions at $x = a$:

$$\begin{cases} \psi(a-\varepsilon) = \psi(a+\varepsilon) \\ \psi'(a+\varepsilon) - \psi'(a-\varepsilon) = -\dfrac{2mg}{\hbar^2}\psi(a) \end{cases}$$ 1

or

$$\begin{cases} e^{-ika} + Be^{ika} = A\left(e^{ika} - e^{-ika}\right) \\ -ike^{-ika} + ikBe^{ika} - ikAe^{ika} - ikAe^{-ika} = -\dfrac{2mg}{\hbar^2}A\left(e^{ika} - e^{-ika}\right) \end{cases}$$

$$\Rightarrow \quad A = -\frac{1}{1 - i\left(1 - e^{i2ka}\right)\dfrac{mg}{k\hbar^2}}$$ 1

$$B = -\frac{1 + i\left(1 - e^{-i2ka}\right)\dfrac{mg}{k\hbar^2}}{1 - i\left(1 - e^{i2ka}\right)\dfrac{mg}{k\hbar^2}}$$

This can be written as

$$A = -\frac{i\lambda \bar{k}}{1 - e^{i2\bar{k}} + i\lambda\bar{k}}$$

with $\lambda = \hbar^2/mga$, $\bar{k} = ka$.

Introducing $\bar{k} = \bar{k}_R + i\bar{k}_I$, we 1
have for the zeros of denomin. of A:

$$1 - e^{i2(\bar{k}_R + i\bar{k}_I)} + i\lambda(\bar{k}_R + i\bar{k}_I) = 0$$ 1

On the Im axis ($\bar{k}_R = 0$):

$$\bar{k}_I = \frac{1}{\lambda}\left(1 - e^{-2\bar{k}_I}\right)$$ 2

This is eq (*) for the bound state energy. Approximately,

$$\bar{k}_I \approx \frac{1}{\lambda}\left(1 - e^{-2/\lambda}\right)$$

for large λ.

As discussed in detail in lectures, we also expect singularities corresp.

to resonances to appear.

For $\lambda \ll 1$, to leading order they are given by the solutions of the eq. $e^{i2\bar{k}} = 1 \Rightarrow$

$ka = n\pi$, $n = \pm 1, \pm 2, \ldots$ (Note that $n = 0$ is <u>NOT</u> a singularity of A). We can solve the eq.

$$1 - e^{i2ka} + i\lambda ka = 0$$

perturbatively in λ by writing

$$ka = n\pi + \lambda X_1 + \lambda^2 X_2 + \ldots$$

and subst. into the eq. This gives

$$X_1 = n\pi/2 \,, \quad X_2 = \frac{n\pi}{4} - i\frac{n^2\pi^2}{4}$$

$$\Rightarrow ka = n\pi\left(1 + \frac{\lambda}{2} + \frac{\lambda^2}{4} + \ldots\right) - i\left(\frac{\lambda n\pi}{2}\right)^2 + \ldots$$

for $\lambda \ll 1$, $n = \pm 1, \pm 2, \ldots$

Problem 2

$R = r^{-1/2} y$ 1

$R' = -\frac{1}{2} r^{-3/2} y + r^{-1/2} y'$ 1

$R'' = \frac{3}{4} r^{-5/2} y - r^{-3/2} y' + r^{-1/2} y''$

This can be subst. into

$$-R'' - \frac{2}{r} R' + \frac{l(l+1)}{r^2} R + \frac{2m\alpha}{r^2\hbar^2} R = k^2 R$$ 1

to find $y'' + \frac{1}{x} y' + \frac{1}{x^2}\left(-\frac{1}{4} - l(l+1) - 2m\alpha/\hbar^2\right) y + y = 0$ 1

Comparing to Bessel eq, we find 1

$\nu^2 = (l + 1/2)^2 + 2m\alpha/\hbar^2$.

For $x \to 0$, solutions of the eq. $x^2 y'' + x y' + (x^2 - \nu^2) y = 0$

behave as $y \sim x^{\pm\nu}$ (this can 1

be seen by subst. $y = x^{\lambda}$ in the

equation in the limit $x \to 0$) 1

The radial wave-function

$R \sim r^{-1/2} y^{\pm\nu} \sim r^{-1/2 \pm \nu}$ is

non-singular at $r=0$ for $y \sim x^{\nu}$) 1

(situation is more subtle for $\alpha<0$)

$\Rightarrow$ must choose $y(x) = A\, J_{\nu}(x)$ as 1

a solution, so $B=0$.

At $r \to \infty$, we have

$J_{\nu}(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi\nu}{2} + \pi/4\right)$, 1

and for free particle ($\alpha = 0$,

$\nu = l + 1/2$): 1

$y(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi l}{2}\right)$.

The phase difference (phase shift) 2

is thus

$$\delta_l = x - \frac{\pi\nu}{2} + \frac{\pi}{4} - x + \frac{\pi l}{2} =$$

$$= -\frac{\pi}{2}\left[\nu - l - 1/2\right] = \qquad 1$$

$$= -\frac{\pi}{2}\left[\sqrt{\left(l+\frac{1}{2}\right)^2 + \frac{2m\alpha}{\hbar^2}} - \left(l+\frac{1}{2}\right)\right].$$

This can be expanded for $\frac{m\alpha}{\hbar^2} \ll 1$ | 1

$$\delta_\ell \approx -\frac{\pi m \alpha}{(2\ell+1)\hbar^2}, \quad \text{and } |\delta_\ell| \ll 1.$$

So, $e^{2i\delta_\ell} \approx 1 + 2i\delta_\ell$ and the | 1

scattering amplitude

$$f(\theta, k) = \frac{1}{2ik} \sum_{\ell=0}^{\infty} (2\ell+1)(e^{i2\delta_\ell} - 1) P_\ell$$

$$\approx -\frac{\pi m \alpha}{k \hbar^2} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta).$$ | 1

Using the formula for Legendre polynomials given in the hint, | 1

we get $\sum_{\ell=0}^{\infty} P_\ell(z) = (2-2z)^{-1/2} \Rightarrow$

$$2^{-1/2}(1-\cos\theta)^{-1/2} = \frac{1}{2\sin(\theta/2)} \Rightarrow$$ | 1

$$\Rightarrow f \approx -\frac{\pi m \alpha}{2\hbar^2 k \sin\theta/2}.$$ The diff.

cross-section $\frac{d\sigma}{d\Omega} = |f|^2 = \frac{\pi^2 m^2 \alpha^2}{4\hbar^4 k^2 \sin^2\frac{\theta}{2}} \sim$ | 1

$\sim 1/E$.

In the Born approximation,

$$f = -\frac{2m}{\hbar^2}\int_0^\infty \frac{r\sin qr}{q}\frac{\alpha}{r^2}\,dr = \quad 1$$

$$= -\frac{2m\alpha}{\hbar^2 q}\int_0^\infty \frac{\sin x}{x}\,dx = -\frac{\pi m\alpha}{\hbar^2 q} = \quad 1$$

$$= -\frac{\pi m\alpha}{\hbar^2\, 2k\sin\theta/2}. \quad 1$$

Validity: $\bar{K} \gg \bar{U}$: $\frac{\hbar^2}{m\bar{a}^2} \gg \frac{\alpha}{\bar{a}^2}$ 2

$\Rightarrow$ $m\alpha/\hbar^2 \ll 1$ as discussed in lectures for a general case.

Problem 3

The equations satisfied by ψ and ψ^* are, resp. (let $c=1$, $\hbar=1$): 1

$$\left(\Box + ie\left(\partial_\mu A^\mu + 2A^\mu \partial_\mu + ieA_\mu A^\mu\right) + m^2\right)\psi = 0 \quad (a)$$

$$\left(\Box - ie\left(\partial_\mu A^\mu + 2A^\mu \partial_\mu - ieA_\mu A^\mu\right) + m^2\right)\psi^* = 0 \quad (b)$$

On the other hand,

$$\partial_\mu j^\mu = -\frac{i}{2}\left(\partial_\mu \psi \partial^\mu \psi^* + \psi \Box \psi^* - \partial_\mu \psi \partial^\mu \psi^* - \psi^* \Box \psi\right) - e\left(\psi A^\mu \partial_\mu \psi^* + \psi \psi^* \partial_\mu A^\mu + \psi^* A^\mu \partial_\mu \psi\right) \quad (c)$$ 1

Multiplying (a) by ψ^* and (b) by ψ and subtracting we get 1

$$\psi \Box \psi^* - \psi^* \Box \psi - 2ie\left(\psi\psi^* \partial_\mu A^\mu + A^\mu \psi^* \partial_\mu \psi + A^\mu \psi \partial_\mu \psi^*\right) = 0$$ 1

Subst. in (c) results in $\partial_\mu j^\mu = 0$.

With $\psi(t, \bar{r}) = e^{-i\varepsilon t/\hbar}\varphi(r)$ a direct substitution gives the 2 stationary KG-eq

$$\left[c^2\left(\hat{p}-\frac{e}{c}\bar{A}\right)^2+m^2c^4\right]\varphi=(\varepsilon-e\Phi)^2\varphi$$ 2

Dividing by $2mc^2$ and introducing $E=\varepsilon-mc^2$ gives:

$$\left(\frac{1}{2m}\left(\hat{p}-\frac{e}{c}\bar{A}\right)^2+e\Phi-E\right)\varphi=\frac{(E-e\Phi)^2}{2mc^2}\varphi$$ 1

Expanding for $|E|\ll mc^2$, $|e\Phi|\ll mc^2$, 2
to leading order we get

$$\left(\frac{1}{2m}\left(\hat{p}-\frac{e}{c}A\right)^2+e\Phi\right)\varphi_0=E\varphi_0,$$ 1

i.e. a Schrödinger eq.

Then: $\frac{(E-e\Phi)^2}{2mc^2}(\varphi_0+\varphi_1)\approx$ 2

$$\approx\frac{(E-e\Phi)^2}{2mc^2}\varphi_0=\frac{1}{2mc^2}\left(\frac{\left(\hat{p}-\frac{e}{c}\bar{A}\right)^2}{2m}\right)^2\varphi_0\approx$$ 2

$$\approx\frac{\left(\hat{p}-\frac{e}{c}\bar{A}\right)^4}{8m^3c^2}(\varphi_0+\varphi_1).$$ Therefore, 2

the first rel. correction is

$$\left[\frac{1}{2m}\left(\hat{p}-\frac{e}{c}\bar{A}\right)^2-\frac{1}{8m^3c^2}\left(\hat{p}-\frac{e}{c}\bar{A}\right)^4+e\phi\right]\varphi= \quad 2$$

$$=E\varphi.$$

One can expand the classical rel.
Hamiltonian

$$H=\left[c^2\left(p-\frac{e}{c}A\right)^2+m^2c^4\right]^{1/2}+e\phi-mc^2\approx \quad 2$$

$$\approx\frac{\left(p-\frac{e}{c}A\right)^2}{2m}-\frac{\left(\bar{p}-\frac{e}{c}\bar{A}\right)^4}{8m^3c^2}+e\phi+\ldots$$

and make the usual subst. $\bar{p}\to -i\hbar\bar{\nabla}$ 2
to obtain this result but this will
not work for generic corrections due
to non-commutativity of operators
$\hat{P}^2=\left(\hat{p}-\frac{e}{c}\bar{A}\right)^2$ and ϕ, as discussed 1
in lectures.