SR Exam 2018 Solutions

1. (a) $D = (ct_d, \bar{x}_d)$ $\quad G = (ct_g, \bar{x}_g)$

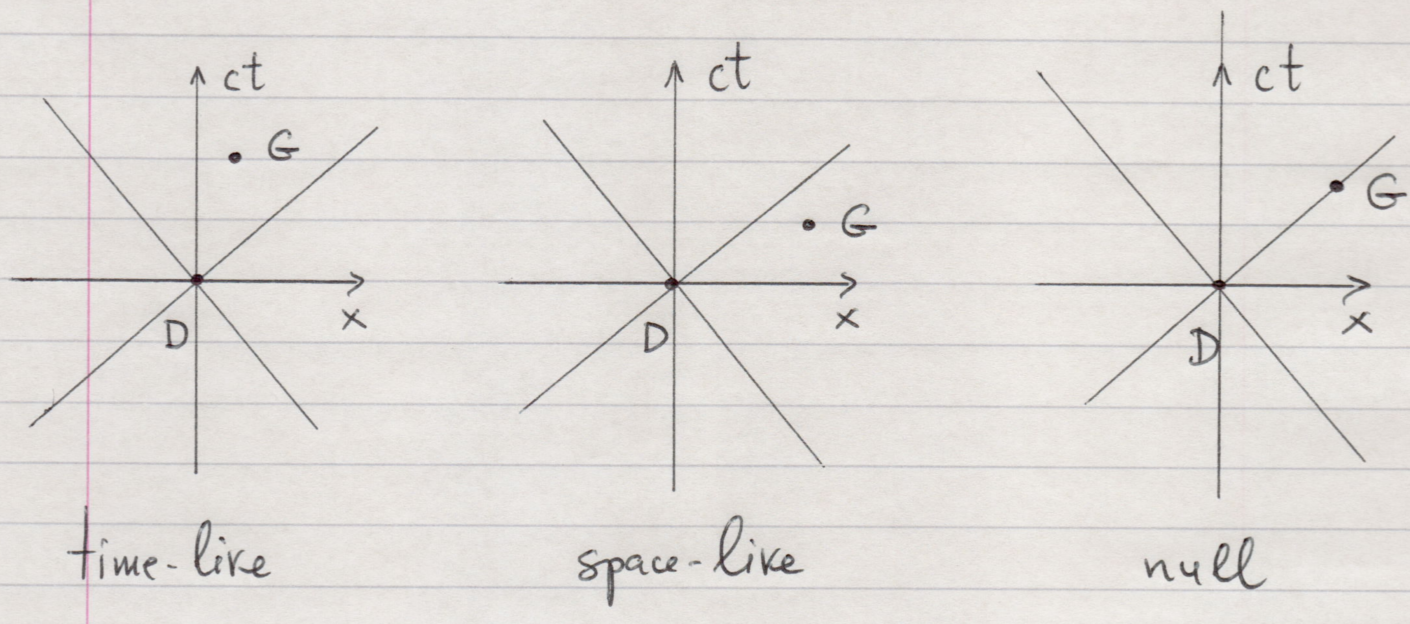
Interval: Δs, where

$$\Delta s^2 = -c^2(t_d - t_g)^2 + (\bar{x}_d - \bar{x}_g)^2$$

$\Delta s^2 < 0$ time-like interval

$\Delta s^2 > 0$ space-like

$\Delta s^2 = 0$ null



- $Y^\mu X_\mu = 0 \Rightarrow \eta_{\mu\nu} X^\mu Y^\nu = 0 \Rightarrow$

$$-X^0 Y^0 + \bar{X} \cdot \bar{Y} = 0 \quad \text{or}$$

$$-X^0 Y^0 + |\bar{X}||\bar{Y}| \cos\varphi = 0$$

This can be written as (assuming $|\bar{X}| \neq 0$, $|\bar{Y}| \neq 0$)

$$\frac{X^0}{|\bar{X}|} \frac{Y^0}{|\bar{Y}|} = \cos\varphi$$

$$\Rightarrow \frac{|X^0|}{|\bar{X}|} \frac{|Y^0|}{|\bar{Y}|} = |\cos\varphi| \leq 1$$

- if one vector is time-like (e.g. $|X^0| > |\bar{X}|$), the other should be space-like
- or they both can be space-like

- if one is null (e.g. $|X^0| = |\bar{X}|$), the other should be space-like or null. In the latter case, $|\cos\varphi| = 1$, i.e. $\bar{X}$ and $\bar{Y}$ are collinear.

- cases with $|\bar{X}| = 0$ or $|\bar{Y}| = 0$ fall into the same category: $(0, \bar{x})$ and $(Y^0, \bar{0})$ and $(0, \bar{X})$, $(0, \bar{Y})$ with $\varphi = \pi/2$ or $X = (0, \bar{0})$ (or $Y = (0, \bar{0})$).

We now consider special 4-vectors: A, K, J.

Recall that $A^\mu = \dfrac{dU^\mu}{d\tau}$, where

$U^\mu = dx^\mu/d\tau = (\gamma c, \gamma \bar{v})$

$\Rightarrow A^\mu = (c\gamma\dot{\gamma}, \gamma\dot{\gamma}\bar{v} + \gamma^2 \dot{\bar{v}}) =$

$= \left(\gamma^4 \dfrac{\bar{v}\cdot\bar{a}}{c}, \gamma^4 \dfrac{\bar{v}\cdot\bar{a}}{c^2}\bar{v} + \gamma^2 \bar{a}\right)$, since

$\dot{\gamma} = \dfrac{\bar{v}\cdot\bar{a}}{c^2}\gamma^3$.

In particle's rest frame, $A^\mu = (0, \bar{a}_0)$.

Also, $K^\mu = (K^0, \bar{K})$, $K^2 = 0$, i.e.

$|K^0| = |\bar{K}|$.

$J^\mu = (\rho c, \rho \bar{v})$, where ρ is the charge density in the lab frame. Or $J^\mu = \rho_0 U^\mu$, where ρ_0 is the charge density in particle's own frame.

$A_\mu J^\mu$, $A_\mu K^\mu$, $J_\mu K^\mu$ are invariants and can be computed in any frame.

In particle's rest frame (inertial frame comoving with particle at each instant of time): $A^\mu = (0, \bar{a}_0)$, $J^\mu = (\rho_0 c, \bar{0})$,

so $A_\mu J^\mu = 0$ (in any frame).

$A_\mu K^\mu \neq 0$ and $J_\mu K^\mu \neq 0$, generically.

(6) $X^\mu = X^\mu(\tau) \Rightarrow u^\mu = dX^\mu/d\tau$

$A^\mu = du^\mu/d\tau$. The 4-momentum is

$$p^\mu = m u^\mu = (\gamma m c, \gamma m \bar{v}) = \left(\frac{\mathcal{E}}{c}, \bar{p}\right)$$

$$p_\mu p^\mu = -\gamma^2 m^2 c^2 + \gamma^2 m^2 \bar{v}^2 = -\gamma^2 m^2 c^2 \left(1 - \frac{\bar{v}^2}{c^2}\right)$$

$$= -m^2 c^2.$$

We had above:

$$A^\mu = \left(\gamma^4 \frac{\bar{v}\cdot\bar{a}}{c}, \gamma^4 \frac{\bar{v}\cdot\bar{a}}{c^2}\bar{v} + \gamma^2 \bar{a}\right)$$

When $\bar{v} \parallel \bar{a}$, we have

$$A^\mu = \left(\gamma^4 \frac{v \cdot a}{c}, \gamma^4 \frac{v\cdot a}{c^2}\bar{v} + \gamma^2 \bar{a}\right)$$

So, $A_\mu A^\mu = -\gamma^8 \frac{v^2a^2}{c^2} + \gamma^4 \left[\bar{a} + \gamma^2 \frac{va}{c^2}\bar{v}\right]^2 =$ 5

$= -\gamma^8 \frac{v^2a^2}{c^2}\left(1 - \frac{v^2}{c^2}\right) + \gamma^4 a^2 + 2\gamma^6 \frac{v^2a^2}{c^2}$

$= \gamma^6 \frac{v^2a^2}{c^2} + \gamma^4 a^2 = a^2\gamma^4\left(1 + \gamma^2\frac{v^2}{c^2}\right) = \gamma^6 a^2$

Note: $\gamma^6 a^2 = a_0^2$.

(c) $\bar{f} = \alpha \bar{r}/r^3$

We have $\frac{d\bar{p}}{dt} = \bar{f}$ and $F^\mu = \frac{dp^\mu}{d\tau}$,

with $p^\mu = \left(\frac{\mathcal{E}}{c}, \bar{p}\right) \Rightarrow$

$F^\mu = \left(\gamma\frac{\dot{\mathcal{E}}}{c}, \gamma\vec{f}\right)$.

Since $U^\mu = (\gamma c, \gamma\bar{v})$, the invariant

$F_\mu U^\mu$ is (one can compute it in the particle's rest frame, where $U_0^\mu = (c, \bar{0})$ and $F_0^\mu = (\dot{m}c, \bar{f}_0)$) equal to $-\dot{m}c^2$.

Since the force is „pure", $\dot{m} = 0$.

$\Rightarrow F_\mu U^\mu = 0 \Rightarrow -\gamma^2\dot{\mathcal{E}} + \gamma^2\bar{f}\cdot\bar{v} = 0$

$\Rightarrow \dot{\mathcal{E}} = \bar{f}\cdot\bar{v}$ for the „pure force".

Then, since for potential force

$\bar{f} = -\nabla\varphi, \qquad \dot{\mathcal{E}} + \bar{\nabla}\varphi \cdot \bar{v} = 0.$

The last eq is $\frac{d}{dt}\left(\mathcal{E} + \varphi\right) = 0$, since

$$\frac{d}{dt}\varphi(x) = \frac{\partial\varphi}{\partial x^i}\dot{x}^i = \bar{\nabla}\varphi \cdot \bar{v}.$$

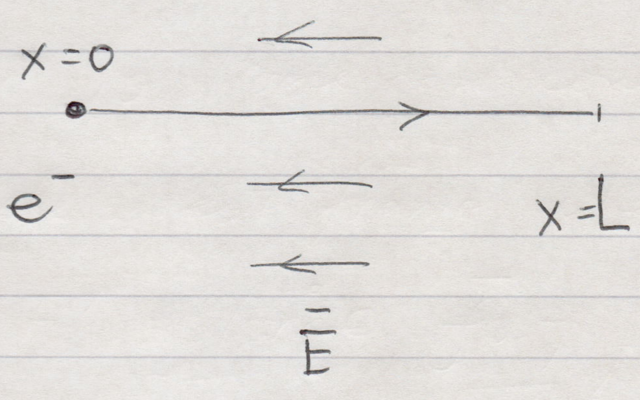
For $\bar{f} = \alpha\bar{r}/r^3$ we have $\varphi = \alpha/r$

$\Rightarrow \quad \mathcal{E} + \alpha/r = \gamma m c^2 + \alpha/r = \text{const}.$

For grav. or Coulomb force, $\alpha < 0$, $\alpha = -|\alpha|$. Then $\bar{f} = -|\alpha|\bar{r}/r^3$

and $\gamma m c^2 - |\alpha|/r = \text{const}.$

(d)



$L = 5\,m \qquad |\bar{E}| = 10\ MV/m$

$\varphi(x) = e|\bar{E}|\cdot x, \quad e < 0.$

We have $\mathcal{E} + \varphi = \text{const} \Rightarrow$

$\gamma mc^2 + e|\bar{E}|x = \text{const}, \quad e < 0.$

At $x = 0$: $\gamma = 1 \Rightarrow mc^2 = \text{const}$

At $x = L$: $\gamma mc^2 + e|\bar{E}|L = \text{const} = mc^2$

$\Rightarrow \gamma mc^2 - |e||\bar{E}|L = mc^2$

$$\Rightarrow \gamma = 1 + \frac{|e||\bar{E}|L}{mc^2}$$

Since $|\bar{E}|L = 50\ MV$, $|e||\bar{E}|L = 50\ MeV$

$mc^2 \simeq 0.5\ MeV$

$\Rightarrow \gamma = 1 + 100 = 101.$

- Equation of motion for the electron:

$$\frac{d\bar{p}_x}{dt} = \bar{f}_x = |e||\bar{E}| = \text{const}$$

With $\bar{p}_{0x} = 0$, we get $\bar{p} = (p_x, 0, 0)$,

$p_x = \gamma m v_x = |e||\bar{E}|t$ or, with

$\beta = v_x/c$, $\gamma\beta m = \frac{|e|}{c}|\bar{E}|t$.

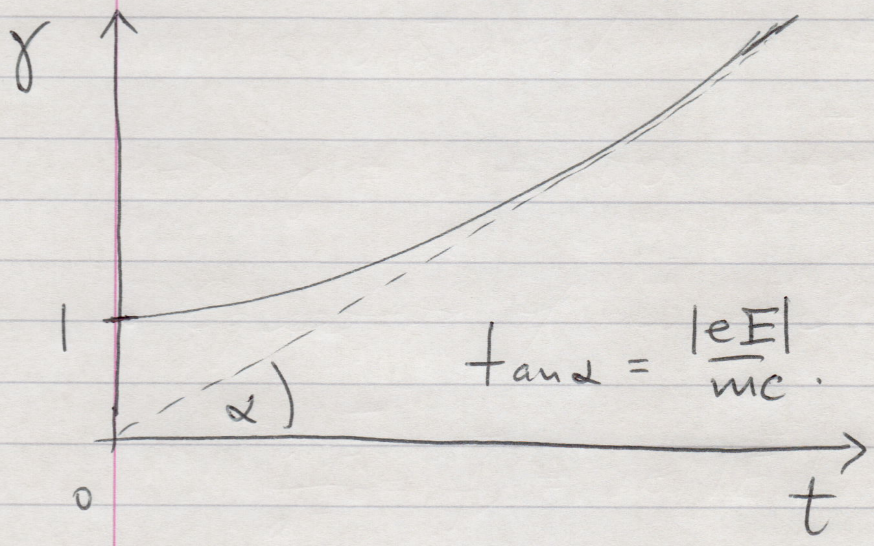
<u>Note</u> : $\gamma\beta = \sqrt{\gamma^2 - 1}$

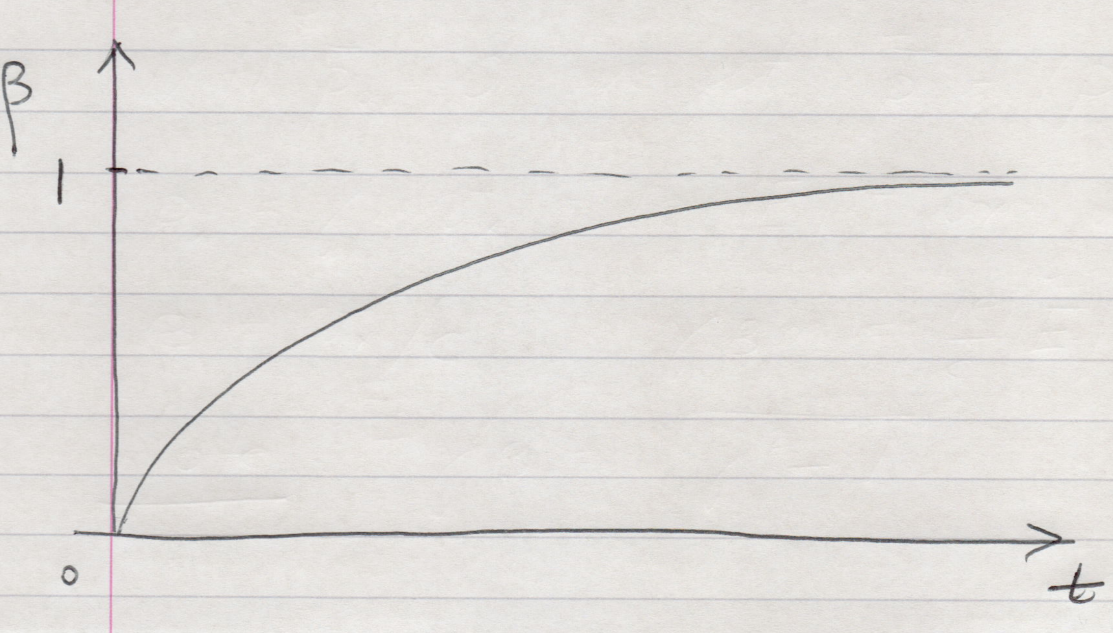
Then $\gamma\beta = \frac{|e|}{mc}|\bar{E}|t$

$$\Rightarrow \gamma^2 - 1 = \left(\frac{e}{mc}\right)^2 E^2 t^2$$

$$\Rightarrow \gamma(t) = \sqrt{1 + \left(\frac{eE}{mc}\right)^2 t^2}$$

$$\beta(t) = \frac{|e||E|}{mc}\frac{t}{\gamma} = \frac{|eE|}{mc}\frac{t}{\sqrt{1+\left(\frac{eE}{mc}\right)^2 t^2}}.$$





The time to reach the other side (9)
of the gap can be found from

$$t = \frac{mc}{|eE|}\left(\gamma^2-1\right)^{1/2}$$

$$\Rightarrow \frac{ct}{L} = \frac{mc^2}{|eE|L}\left(\gamma^2-1\right)^{1/2}$$

$$t \simeq \frac{L}{c}\cdot 1.01 \simeq 1.7\cdot 10^{-8}\,\mathrm{s}.$$

- If $|\bar{E}_{new}| = 0.8|\bar{E}|$, then

$$\gamma_{new} = 1 + \frac{|e||\bar{E}_{new}|L}{mc^2} = 1 + 0.8\cdot 100$$
$$= 81.$$

$$\beta_{new} = \sqrt{1 - 1/\gamma_{new}^2} \approx 0.999924,$$

whereas $\beta = \left(1 - 1/101^2\right)^{1/2} \approx 0.999951$

- For proton: $m_e/m_p \approx 1/1836$

$$\gamma_p = 1 + \frac{|e||E|L}{m_e c^2}\frac{m_e}{m_p} \approx 1 + \frac{100}{2000} = 1.05$$

$$\Rightarrow \beta_p \approx 0.3.$$

2. 1(a) Photon : $K^\mu = (K^0; \bar{K})$

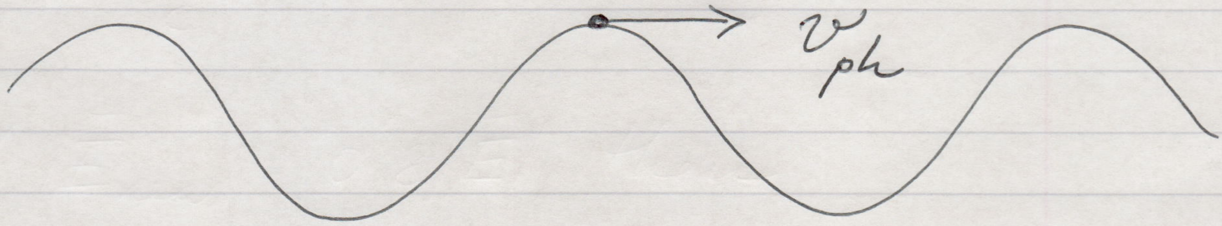
$K^2 = K_\mu K^\mu = 0$ (photon is massless)

$\Rightarrow \quad K^0 = |\bar{K}| = \omega/c$ (with $\hbar = 1$).

Photon propagating in vac: $\sim e^{-i\omega t + i\bar{k}\bar{x}}$

as solution of Maxwell's eqs.

Snapshot of const phase $-\omega t + \bar{K}\bar{x} = \text{const}$



$-\omega dt + \bar{K} d\bar{x} = 0$

$-\omega + \bar{K}\cdot\bar{V}_{ph} = 0 \quad \Rightarrow \bar{V}_{ph} = \frac{\omega}{K}\frac{\bar{K}}{K}$

Here $K \equiv |\bar{K}|$.

Group vel. : can have wave packet with dispersion $\omega = \omega(K)$. Then

$v_{gr} = \frac{\partial \omega}{\partial K}$.

The phase can be written in Lor-covar. way as $\varphi = K_\mu X^\mu$ - this is a product of 4-vectors and thus is Lor-invar. The phase vel., being a 3-vector, is not Lor-invar.

(b) Suppose we have $K = K' + P$

Then: $(K - K')^2 = P^2 = -m^2c^2$

$\Rightarrow \; -2KK' = -m^2c^2$

Since $|\bar{K}| = \mathcal{E}/c = \omega/c \quad |\bar{K}'| = \omega'/c$,

we have $\; -\dfrac{\omega\omega'}{c^2} + \dfrac{\omega\omega'}{c^2}\cos\varphi = \dfrac{m^2c^2}{2}$

$\Rightarrow \; \cos\varphi - 1 = \dfrac{m^2c^2}{2} \cdot \dfrac{c^2}{\omega\omega'} > 0 \quad$ if $m \neq 0$

$\Rightarrow \; \cos\varphi > 1$: impossible.

- We can have 2 (or more) photons producing e^+e^- pair (charge is conserved)

$$K_1 + K_2 = P_+ + P_-$$

$(K_1 + K_2)^2 = (P_+ + P_-)^2$ - can be computed in any frame. E.g. in CMF of e^+e^-

pair, $(p_+ + p_-)^2 = -\left(\frac{\mathcal{E}_+ + \mathcal{E}_-}{c}\right)^2$

Now, $(k_1 + k_2)^2 = \underbrace{k_1^2}_{=0} + 2k_1k_2 + \underbrace{k_2^2}_{=0} =$

$$= 2\left(-\frac{\omega^2}{c^2} + \frac{\omega^2}{c^2}\cos\varphi\right) = \frac{2\omega^2}{c^2}(\cos\varphi - 1),$$

since $\omega_1 = \omega_2 = \omega$ by assumption.

$$\Rightarrow \omega^2 = \frac{(\mathcal{E}_+ + \mathcal{E}_-)^2}{2(1-\cos\varphi)}$$

To find min ω: min $\mathcal{E}_+ + \mathcal{E}_- = 2m_e c^2$

Also, $\max(1-\cos\varphi) = 2$ at $\varphi = \pi$.

$\Rightarrow \omega_{min} = m_e c^2$ in head-on collision

$(\varphi = \pi)$ in the lab frame: $\underset{\gamma}{\rightsquigarrow}\ \underset{\gamma}{\leftsquigarrow}$

One can initially assume N photons and then repeat above calculation showing that only $N > 1$ can work.

(c) $\bar{v}_0 = (v_0, 0, 0)$

$$p^\mu = \left(\frac{\mathcal{E}}{c}, \bar{p}\right)$$

i) Eq. of motion: $\frac{d\bar{p}}{dt} = \bar{f} \quad \Rightarrow$

$$\frac{dp_x}{dt} = eE_x \qquad \frac{dp_y}{dt} = 0 \qquad \frac{dp_z}{dt} = 0$$

$$\Rightarrow \quad p_x(t) = eE_x t + p_{0x} \;, \quad p_{0x} = \gamma(v_0) m v_0$$

$$\dot{\mathcal{E}} = \bar{f} \cdot \bar{v} = eE_x v_x(t) \neq 0$$

$\Rightarrow$ only $p_y = p_{0y}$ and $p_z = p_{z0}$ components of 4-momentum are conserved in this case.

ii) $\bar{B} = (0, B_y, 0)$

Eq. of motion: $\dfrac{d\bar{p}}{dt} = \bar{f} = e\bar{v} \times \bar{B}$

$$\begin{vmatrix} i & j & k \\ v_x & v_y & v_z \\ 0 & B_y & 0 \end{vmatrix} \Rightarrow \begin{cases} \dfrac{dp_x}{dt} = -ev_z B_y \\[2ex] \dfrac{dp_y}{dt} = 0 \Rightarrow p_y = p_{y0} = \text{const} \\[2ex] \dfrac{dp_z}{dt} = e v_x B_y \end{cases}$$

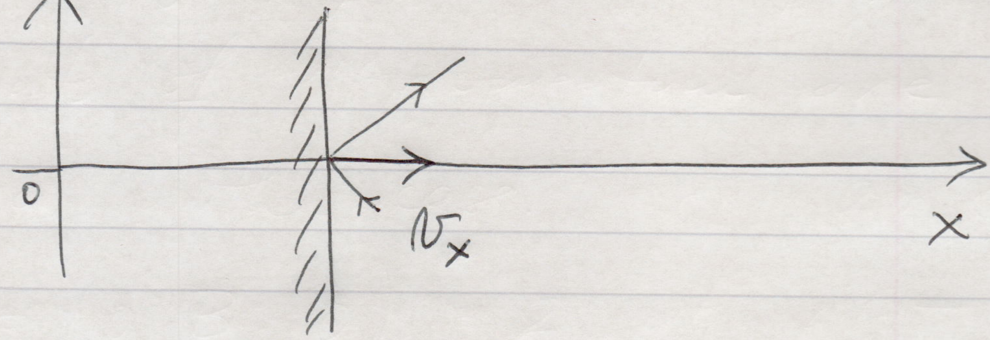
Also, $\dot{\mathcal{E}} = \bar{f} \cdot \bar{v} = e(\bar{v} \times \bar{B}) \cdot \bar{v} = 0$

$\Rightarrow$ $\mathcal{E} = const$ and, since $\mathcal{E} = \gamma mc^2$,

$\gamma = const$, i.e. $|\bar{v}| = const$

$\Rightarrow$ in 4-momentum, $\mathcal{E}$ is conserved, p_y is conserved, and also $|\bar{p}|$ is conserved.

(d)



$\beta_x = v_x / c = 0.99$ (Lab frame S)

$\lambda_1 = 1\ \mu m$ (Lab frame S)

Comoving frame S':

$$K'^{\mu}_{i} = \left(\frac{\omega'}{c}, -\frac{\omega'}{c}\cos\theta'_i, \frac{\omega'}{c}\sin\theta'_i, 0 \right)$$

$$K'^{\mu}_{r} = \left(\frac{\omega'}{c}, \frac{\omega'}{c}\cos\theta'_r, \frac{\omega'}{c}\sin\theta'_r, 0 \right)$$

- $\theta'_i = \theta'_r$ and $\omega'_i = \omega'_r$ in S'.

In S:

$$K_i^\mu = \left(\frac{\omega_i}{c}, -\frac{\omega_i}{c}\cos\theta_i, \frac{\omega_i}{c}\sin\theta_i, 0\right)$$

$$K_r^\mu = \left(\frac{\omega_r}{c}, \frac{\omega_r}{c}\cos\theta_r, \frac{\omega_r}{c}\sin\theta_r, 0\right)$$

We have Lor. transf. for K^μ:

$$\begin{cases} K^0 = \gamma\left(K'^0 + \beta K'_x\right) & (*) \\ K_x = \gamma\left(K'_x + \beta K'^0\right) & (**) \\ K_y = K'_y \\ K_z = K'_z \end{cases}$$

So, $K_y = K'_y$ gives $\begin{cases} \omega_i \sin\theta_i = \omega' \sin\theta'_i \\ \omega_r \sin\theta_r = \omega' \sin\theta'_r \end{cases}$

but $\theta'_r = \theta'_i \Rightarrow \omega_i \sin\theta_i = \omega_r \sin\theta_r$.

$$\Rightarrow \sin\theta_r = \frac{\omega_i}{\omega_r}\sin\theta_i = \frac{\lambda_r}{\lambda_i}\sqrt{1-\cos^2\theta_i}.$$

Now, from $(*)$ we have $\frac{\omega_i}{c} = \gamma\left(\frac{\omega'}{c} - \frac{\beta\omega'}{c}\cos\theta'_i\right)$

$$\Rightarrow \omega_i = \gamma\omega'(1-\beta\cos\theta_i')$$

$$\omega_r = \gamma\omega'(1+\beta\cos\theta_r')$$

Since $\theta_i' = \theta_r'$, we get (let $\theta_i' = \theta_r' = \theta'$):

$$\begin{cases} \omega_i = \gamma\omega'(1-\beta\cos\theta') \\ \omega_r = \gamma\omega'(1+\beta\cos\theta') \end{cases}$$

$$\frac{\omega_i}{\omega_r} = \frac{1-\beta\cos\theta'}{1+\beta\cos\theta'} = \frac{\lambda_r}{\lambda_i}.$$

From (* *): $-\frac{\omega_i}{c}\cos\theta_i = \gamma\left(-\frac{\omega'}{c}\cos\theta' + \beta\frac{\omega'}{c}\right)$

$$\Rightarrow -\omega_i\cos\theta_i = \gamma\omega'(\beta-\cos\theta');$$

$$\omega_r\cos\theta_r = \gamma(\omega'\cos\theta' + \beta\omega')$$

$$\Rightarrow \omega_i\cos\theta_i = \gamma\omega'(\cos\theta'-\beta)$$

$$\omega_r\cos\theta_r = \gamma\omega'(\cos\theta'+\beta)$$

$$\Rightarrow \cos\theta_i = \frac{\cos\theta' - \beta}{1 - \beta\cos\theta'}$$

$$\cos\theta_r = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

$$\frac{\lambda_r}{\lambda_i} = \frac{1 - \beta\cos\theta'}{1 + \beta\cos\theta'}$$

3. (a) $x^\mu = x^\mu(\tau)$

$u^\mu = u^\mu(\tau) = (\gamma c, \gamma \bar{v})$, $\quad d\tau = dt/\gamma$,

- $A^\mu = \dfrac{du^\mu}{d\tau} = (c\gamma\dot{\gamma},\ \gamma\dot{\gamma}\bar{v} + \gamma^2\dot{\bar{v}}) =$

$$= \left(\gamma^4 \frac{\bar{v}\cdot\bar{a}}{c},\ \gamma^4 \frac{\bar{v}\cdot\bar{a}}{c^2}\bar{v} + \gamma^2\bar{a}\right),$$

where $\bar{a} = d\bar{v}/dt$.

Since $\gamma = (1 - \bar{v}^2/c^2)^{-1/2} \in [1, \infty)$, there is no bound on components of A^μ.

- Forces: $\dfrac{d\bar{p}}{dt} = \bar{f}$, $\quad \bar{p} = \gamma m \bar{v}$

4-vector of force $F^\mu = dp^\mu/d\tau$, where

$p^\mu = m u^\mu = (\gamma m c, \gamma m \bar{v}) = \left(\dfrac{\mathcal{E}}{c}, \bar{p}\right)$

Explicitly, $F^\mu = \left(\gamma \dfrac{\dot{\mathcal{E}}}{c}, \gamma \bar{f}\right)$.

Since $\gamma \in [1, \infty)$, there is no bound on components of F^μ.

- Phase velocity: wave propagation

$\sim e^{-i\omega t + i\bar{k}\bar{x}}$: constant phase

$\Rightarrow\ -\omega t + \bar{k}\bar{x} = \text{const} \Rightarrow$

$-\omega\, dt + \bar{k}\, d\bar{x} = 0 \Rightarrow \bar{v}_{ph} = \frac{d\bar{x}}{dt} =$

$= \frac{\omega}{k}\frac{\bar{k}}{k}$. No apriori bound on components of $\bar{v}_{ph}$. In vac., $\frac{\omega}{k} = c$, but in a medium $|\bar{v}_{ph}|$ can be less or greater than c, depending on refract. index.

• Compton scattering: $\gamma + e^- \to \gamma + e^-$

$$k + p_e = k' + p'_e$$

$$(k + p_e - k')^2 = p_e'^2 = -m_e^2 c^2$$

$$\underbrace{k^2}_{=0} + 2k(p_e - k') + \underbrace{p_e^2}_{=-m_e^2c^2} - 2p_e k' + \underbrace{k'^2}_{=0} = -m_e^2 c^2$$

$$\cancel{2}kp_e - \cancel{2}kk' - \cancel{2}p_e k' = 0$$

In the Lab frame: $k^\mu = \left(\frac{\omega}{c}, \bar{k}\right)$, $k'^\mu = \left(\frac{\omega'}{c}, \bar{k}'\right)$, $p_e = \left(\frac{\mathcal{E}}{c}, \bar{P}\right)$, with $\hbar = 1$, $|\bar{k}| = \omega/c$, $|\bar{k}'| = \omega'/c$, and $\bar{e}$ stationary ($p_e = (m_e c, \bar{0})$)

$$k p_e = -m_e \omega$$

$$k k' = -\omega\omega'/c^2 + \omega\omega'/c^2 \cos\varphi$$

$$p_e k' = -m_e \omega'$$

$$\Rightarrow -m_e\omega + \frac{\omega\omega'}{c^2}(1-\cos\varphi) + m_e\omega' = 0$$

$$m_e(\omega - \omega') = \frac{\omega\omega'}{c^2}(1-\cos\varphi)$$

With $\lambda\nu = c$, $\nu = 2\pi\omega$,

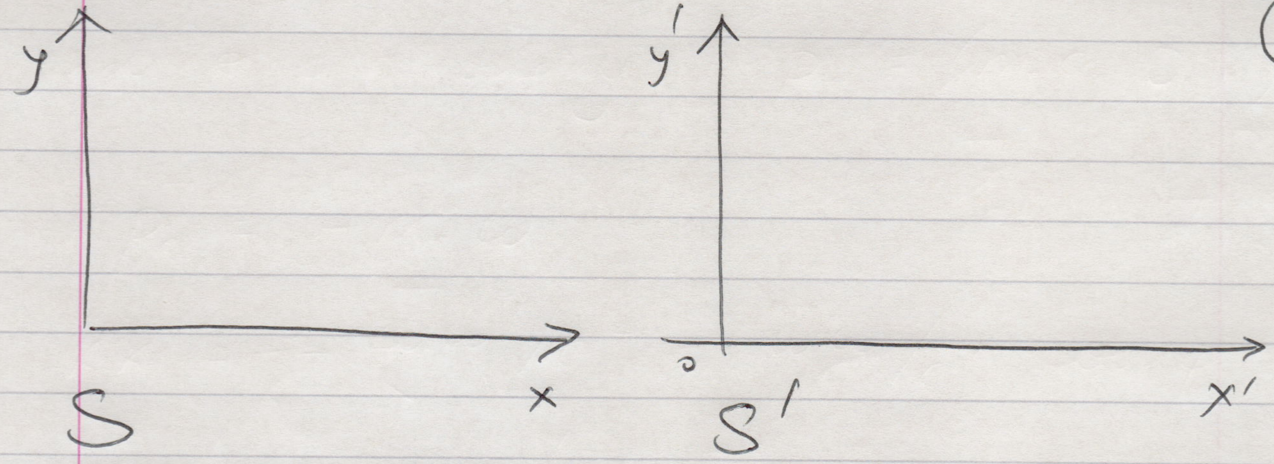
$$2\pi\omega\lambda = c \Rightarrow \omega = \frac{c}{2\pi\lambda}$$

$$\frac{1}{\omega'} - \frac{1}{\omega} = \frac{1}{mc^2}(1-\cos\varphi)$$

$$\frac{2\pi\lambda'}{c} - \frac{2\pi\lambda}{c} = \frac{1-\cos\varphi}{mc^2}$$

$$\boxed{\Delta\lambda = \lambda' - \lambda = \frac{1-\cos\varphi}{2\pi mc}}.$$

(b) Treating photons ~~as~~ point particles, with $c\,dt = dx$,



$$x_1^{\mu} = (ct,\ ct,\ 0,\ 0)$$

$$x_2^{\mu} = (ct,\ ct - x_0,\ 0,\ 0)$$

$$x_1'^{\mu} = (ct',\ ct',\ 0,\ 0)$$

$$x_2'^{\mu} = (ct',\ x',\ 0,\ 0)$$

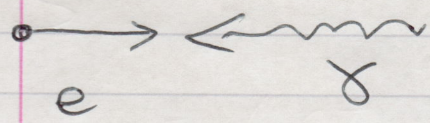
Lor. transf: $\begin{cases} ct' = \gamma(ct - \beta x) \\ x' = \gamma(x - \beta ct) \end{cases}$

$$\Rightarrow \quad x_1 \to x_1' \begin{cases} ct' = \gamma(ct - \beta ct) = \gamma ct(1-\beta) \\ ct' = \gamma(ct - \beta ct) = \gamma ct(1-\beta) \end{cases}$$

$$x_2 \to x_2' : \begin{cases} ct' = \gamma(ct - \beta ct + \beta x_0) \\ x' = \gamma(ct - x_0 - \beta ct \end{cases}$$

$$\Rightarrow \Delta x' = ct' - x' = \gamma\left(ct - \beta ct + \beta x_0 - ct + x_0 + \beta ct\right) = \gamma x_0 (1+\beta),$$

where $\gamma = 1/\sqrt{1 - v_x^2/c^2}$.

(c) 

$\gamma + e^- \rightarrow \gamma' + e^-$

$$p_e + K = p_e' + K'$$

<u>In Lab frame:</u>

$$p_e = \left(\frac{\mathcal{E}}{c}, \bar{p}\right)$$

$$p_e' = \left(\frac{\mathcal{E}'}{c}, \bar{p}'\right)$$

$$K = \left(\frac{\omega}{c}, \bar{K}\right)$$

$$K' = \left(\frac{\omega'}{c}, \bar{K}'\right)$$

We have $(p_e + K - K')^2 = p_e'^2 = -m_e^2 c^2$

$$\underbrace{p_e^2}_{= -m_e^2 c^2} + 2p_e(K - K') + \underbrace{K^2}_{=0} - 2KK' + \underbrace{K'^2}_{=0} = -m_e^2 c^2$$

$$\Rightarrow p_e K - p_e K' - KK' = 0.$$

The collision is head-on; this

simplifies things (consider 1-dim case only):

$$p_e K = -\frac{\mathcal{E}}{c}\frac{\omega}{c} + \bar{p}\bar{k} = -\frac{\mathcal{E}\omega}{c^2} - |\bar{p}|\frac{\omega}{c}$$

$$p_e K' = -\frac{\mathcal{E}\omega'}{c^2} + |\bar{p}|\frac{\omega'}{c}$$

$$K K' = -\frac{\omega\omega'}{c^2} - \frac{\omega\omega'}{c^2}$$

$$-\frac{\mathcal{E}\omega}{c^2} - |\bar{p}|\frac{\omega}{c} = -\frac{\mathcal{E}\omega'}{c^2} + |\bar{p}|\frac{\omega'}{c} - \frac{2\omega\omega'}{c^2}$$

$\Rightarrow$

$$-\gamma m_e \omega - \gamma m_e \beta \omega = -\gamma m_e \omega' + \gamma\beta m_e \omega' - 2\omega\omega'/c^2$$

since $\mathcal{E} = \gamma m_e c^2$ and $|\bar{p}| = \gamma\beta m_e c$.

$$\Rightarrow \omega' = \frac{\gamma m_e c^2 (1+\beta)\omega}{2\omega + \gamma m_e c^2 (1-\beta)}$$

Since $\beta = (1 - 1/\gamma^2)^{1/2} \approx 1 - \frac{1}{2\gamma^2}$, $\gamma \gg 1$,

we have $1+\beta \approx 2$, $1-\beta \approx \frac{1}{2\gamma^2}$.

Also, in our situation $\omega \ll mc^2/\gamma$, since with $\beta \simeq 0.999$ we have

$\gamma \simeq 22$; $\lambda \sim 8\,\mu m \sim 0.15\,eV$

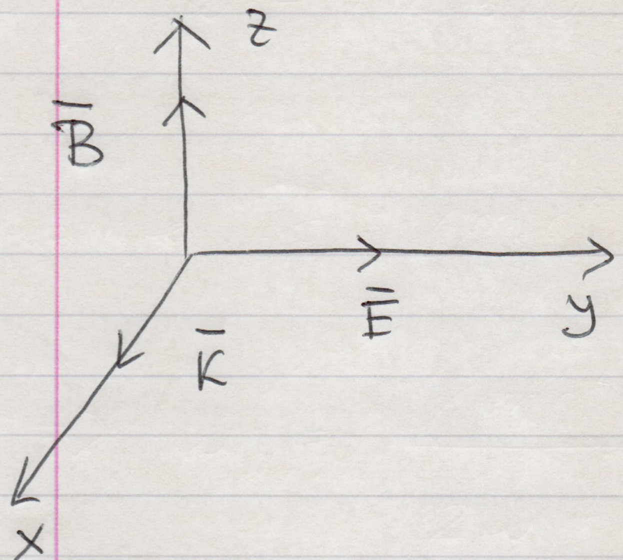
and $m_e c^2 \simeq 0.5\,MeV$.

Thus, $\omega' \approx 4\gamma^2 \omega$, i.e. $\lambda' \approx \dfrac{\lambda}{4\gamma^2}$.

$\lambda' \sim 4 \cdot 10^{-3}\,\mu m$.

(d) The photon is characterised by

$k^\mu = \left(\frac{\omega}{c}, \frac{\omega}{c}, 0, 0 \right)$.



$\bar{k} \sim \bar{E} \times \bar{B}$

$\bar{E} \perp \bar{B}$

$\Rightarrow \bar{B} = (0, 0, B_z)$

$\bar{E} = (0, E_y, 0)$.

$$\bar{E} = -\nabla \varphi - \frac{\partial \bar{A}}{\partial t}$$

$$\bar{B} = \bar{\nabla} \times \bar{A}$$

$$A^\mu = (A^0; \bar{A}) = (\varphi/c, \bar{A}).$$

One can use gauge invariance

$A_\mu \to A_\mu - \partial_\mu f$ to choose a

convenient form for A^μ, e.g. the one

satisfying e.o.m. $-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\bar{A} + \nabla^2\bar{A} = 0$,

$A^0 = 0$, $\operatorname{div}\bar{A} = 0$.

E.g. $A_y = A_{y0}\cos(\omega t - k_x x)$ gives

$$E_y = A_{y0}\,\omega \sin(\omega t - k_x x)$$

$$\bar{B} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ 0 & A_y & 0 \end{vmatrix}$$

$$B_z = A_{y0} k_x \sin(\omega t - k_x x)$$

$$A^\mu = (0,\ 0,\ A_{y0}\cos(\omega t - k_x x),\ 0).$$

Note that $\omega t - k_x x = -k_\mu x^\mu$ is Lor-invariant.

(e) $\bar{v} = (0, 0, v_z)$

$A^\mu = (\phi/c, 0, 0, 0)$, $\phi = \phi_0 \cos k_u z$,

$k_u = 2\pi/d$

In the Lab frame S we find

- $\bar{E} = -\nabla\phi - \partial\bar{A}/\partial t \Rightarrow$

$E_z = \phi_0 k_u \sin k_u z$, $E_x = 0$, $E_y = 0$.

- $\bar{B} = \bar{\nabla} \times \bar{A} = 0$

In the S' frame (rest frame of the electron):

$\bar{E}'_{\parallel} = \bar{E}_{\parallel}$ $\quad \bar{E}'_{\perp} = \gamma(\bar{E}_{\perp} + \bar{v} \times \bar{B})$

$\bar{B}'_{\parallel} = \bar{B}_{\parallel}$ $\quad \bar{B}'_{\perp} = \gamma(B_{\perp} - \bar{v} \times \bar{E}/c^2)$

$\Rightarrow \bar{E}'_{\parallel} \neq 0$, $\bar{E}'_{\perp} = 0$

$\bar{E}' = (0, 0, \phi_0 k_u \sin k_u z)$

$\bar{B}'_{\parallel} = 0$ and $\bar{B}'_{\perp} = 0$, since $\bar{v} \parallel \bar{E}$.

This is not an EM wave, since $\bar{E}' \neq 0$, $\bar{B}' = 0$.

4. (a) Polar 3-vectors change sign under $P: \vec{x} \to -\bar{x}$.

E.g. velocity $\bar{v}$ is a polar vector.

Axial vectors do not change sign.

E.g. angular vel. $\bar{\omega}$ is an axial vector:

$$\bar{v} = [\bar{\omega} \times \bar{r}], \quad \bar{v} \text{ and } \bar{r}$$

are polar vectors $\Rightarrow$ $\bar{\omega}$ is axial.

E.o.m. of charge in electric field

$\bar{F} = m\bar{a} = q\bar{E}$ Since $\bar{a}$ is polar vector, $\bar{E}$ is also polar.

In magnetic field, $\bar{F} = m\bar{a} = q\,\bar{v} \times \bar{B}$

$\Rightarrow$ $\bar{B}$ is axial.

(b) $J^\mu = (\rho c, \vec{J})$, $\bar{J} = \rho\bar{v}$.

- Continuity eq: $\partial_\mu J^\mu = 0$ is

$\frac{\partial \rho}{\partial t} + \operatorname{div} \bar{J} = 0$ (the usual 3d charge conservation).

- Lorentz force:

$$\bar{F} = q\bar{E} + q\,\bar{v}\times\bar{B}$$

$\bar{f} = \bar{F}/V = \rho\bar{E} + \bar{J}\times\bar{B}$, where ρ and $\bar{J}$ are charge and current density, resp.

Covariant expr. for Lorentz force – must be linear in $\bar{E}$, $\bar{B}$ and J^μ

$\Rightarrow$ $F^\mu = F^{\mu\nu} J_\nu$, where $F^{\mu\nu}$ is the field strength tensor

$$F^{\mu\nu} = \begin{pmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{pmatrix}$$

Checking spatial comp. $F^i = F^{i0} J_0 + F^{ik} J_k$, e.g. $F^1 = F^{10} J_0 +$

$+ F^{12} J_2 + F^{13} J_3$

$J_0 = -\rho c$, $F^{10} = -E_x/c$

$F^x = \rho E_x + \rho(v_y B_z - v_z B_y)$: OK

Same for F^y, F^z.

$F^0 = F^{0i} J_i = \frac{E_i}{c} \rho v_i = \frac{\rho}{c} \bar{E} \cdot \bar{v}$

$= \frac{\bar{f} \cdot \bar{v}}{c}$: proport. to power (work per unit time by external field on the charges in the volume considered).

(c) $A^\mu = (0, A_0 \cos kx, A_0 \sin kx, 0)$,

$x^\mu = (ct, x, y, z)$, $k^\mu = (\frac{\omega}{c}, 0, 0, k_z)$.

$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$.

$kx = -\omega t + k_z z$. $F^{\alpha\beta} = -F^{\beta\alpha}$,

$F^{\alpha\beta} = 0$ when $\alpha = \beta$. Also, A^μ only depends on t and z.

(30)

$$F^{\alpha\beta} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & +B_y & -B_x & 0 \end{bmatrix}$$

$$F^{01} = \partial^0 A^1 - \underbrace{\partial^1 A^0}_{=0} = \partial^0 A^1 = -\frac{\partial A^1}{c\,\partial t},$$

since $\partial^0 = \frac{\partial}{\partial x_0}$, $x_0 = -x^0 = -ct$.

$$F^{01} = -\frac{1}{c}\partial_t A_0 \cos(-\omega t + k_z z) =$$

$$= -\frac{\omega}{c} A_0 \sin(-\omega t + k_z z) = E_x/c$$

$$\Rightarrow E_x = -\omega A_0 \sin kx$$

$$F^{02} = \frac{\omega}{c} A_0 \cos(-\omega t + k_z z) = E_y/c$$

$$\Rightarrow E_y = \omega A_0 \cos kx$$

$$F^{03} = 0 \Rightarrow E_z = 0.$$

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = 0 \Rightarrow B_z = 0$$

$$F^{13} = \partial^1 A^3 - \partial^3 A^1 = -\frac{\partial}{\partial z} A_0 \cos kx =$$

$$= A_0 k_z \sin kx = -B_y .$$

$$F^{23} = \partial^2 A^3 - \partial^3 A^2 = -\frac{\partial}{\partial z} A_0 \sin kx =$$

$$= -A_0 k_z \cos kx = B_x .$$

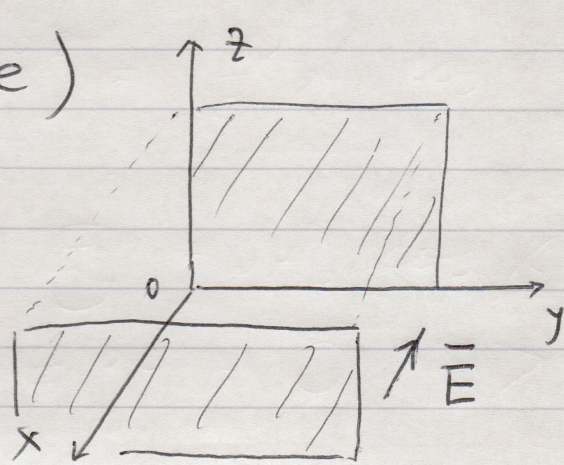
(d) $S \rightarrow S'$

$\bar{E}'_{\parallel} = \bar{E}_{\parallel}$ $\quad \bar{B}'_{\parallel} = \bar{B}_{\parallel}$ ($\parallel$ means parallel to $\bar{v}$ in S)

$$\bar{E}'_{\perp} = \gamma(\bar{E}_{\perp} + \bar{v} \times \bar{B}_{\perp})$$

$$\bar{B}'_{\perp} = \gamma\left(\bar{B}_{\perp} - \frac{\bar{v} \times \bar{E}_{\perp}}{c^2}\right).$$

(e)



$\bar{E} = (-E, 0, 0)$

$E = 100$ MV/m

$d = 0.1$ m

$\bar{v}_0 = 0$

$$\frac{dp_x}{dt} = |e| E = -|e| E_x = f_x$$

$$\bar{f} = -\nabla \varphi \Rightarrow \varphi = -|e| E x$$

Conservation of energy:

$$\gamma mc^2 + \varphi = \text{const} = mc^2 \quad \left(\begin{matrix}\text{initial}\\ \text{cond.}\end{matrix}\right)$$

$$\Rightarrow (\gamma - 1) mc^2 - |e| E x = 0$$

$$\Rightarrow \gamma - 1 = \frac{|e| E d}{mc^2} \Rightarrow \gamma - 1 = \frac{10 \text{ MeV}}{0.5 \text{ MeV}} = 20$$

$\gamma = 21.$ $\quad \mathcal{E} = \gamma mc^2 \Rightarrow \mathcal{E} = 10.5 \text{ MeV}.$

$\mathcal{E}_{kin} = \mathcal{E} - mc^2 = 10 \text{ MeV}.$

ii) It is possible – by applying magnetic field e.g. in the y direction

Then e.o.m. $(\bar{B} = (0, B, 0))$:

$$\frac{dp_x}{dt} = -|e| E + |e| v_z B$$

$$\frac{dp_y}{dt} = 0 \qquad \frac{dp_z}{dt} = -|e| v_x B$$

With $\bar{B}$ perpendicular to the initial velocity of the particle (i.e. $\bar{B} \perp \bar{E}$, since $\bar{v}_0 = 0$ and the initial velocity is the one particle gains by moving along $\bar{E}$), the Lorentz force $-|e|\bar{v}\times\bar{B}$ will be perpendicular to the initial direction of motion in the electric field => will deviate the particle.

- We can switch to frame S', where $\bar{E}' = 0$. Then in S' <u>only</u> magnetic field is present. Velocity $\bar{V}$ of S' should obey $\bar{E}_\perp + \bar{V} \times \bar{B} = 0$, i.e. with $\bar{E}_\perp = (-E, 0, 0)$ and $\bar{B} = (0, B, 0)$ we should have $\bar{V} = (0, 0, -E/B)$.

Then $\bar{B}' = (0, B', 0)$, where

$$B' = \gamma B \left(1 - E^2/B^2c^2\right)$$

Here $\gamma = \dfrac{1}{\sqrt{1 - V_z^2/c^2}}$, $V_z = -\dfrac{E}{B}$

$$\Rightarrow \frac{1}{\gamma^2} = 1 - \frac{E^2}{c^2B^2} > 0, \quad \text{so } B' = B/\gamma.$$

Now, in constant magnetic field only (i.e. in the frame S' moving along z dir. of S with $V_z = -E/B$), the motion is along the circle in the x'-z' plane ($\bar{B}'$ is along y') with the radius $R = \dfrac{p'_\perp}{|e|B'}$, where $p'_\perp = \gamma m v'_\perp$ is the magnitude of the momentum in x'-z' plane. In S', the initial velocity of the particle is V_z (since in S it is zero) $\Rightarrow$

$$\Rightarrow p'_\perp = \gamma(V_z)\, m\, |V_z|.$$

$$\Rightarrow R = \frac{\gamma m E}{B\,|e|\,\gamma B\,(1 - E^2/B^2c^2)} =$$

$$= \frac{mc^2}{|e|E}\,\frac{E^2}{B^2c^2}\,\frac{1}{1 - E^2/B^2c^2} =$$

$$= \frac{mc^2}{|e|E}\,\gamma^2\left(1 - 1/\gamma^2\right) < d/2 \quad , \text{ where}$$

the last condition means the diameter of the circle in S' is $2R < d$: note that S' moves w.r.t. S along z, so distances in x-dir. are not Lor.-contracted. Thus,

$$\gamma^2 < 1 + \frac{|e|Ed}{2mc^2} \equiv \gamma_*$$

This can be written as condition on B, since $\gamma^2 = \left(1 - E^2/c^2B^2\right)^{-1}$:

$$B > B_* = \frac{E}{c}\sqrt{1 + \frac{2mc^2}{|e|Ed}} .$$

With numbers given, $\gamma_*^2 = 11$,

$B_* \simeq 1.05 \, E/c \simeq 0.35 \, T$.

The trajectory in S will be a drift along z :