B2

SYMMETRY AND RELATIVITY

EXAM PAPER

2016

SOLUTION NOTES

(PROBLEMS 3 & 4)

A13481W1

SECOND PUBLIC EXAMINATION

Honour School of Physics – Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B2. SYMMETRY AND RELATIVITY

TRINITY TERM 2016

Wednesday, 15 June, 2.30 pm – 4.30 pm

Answer **five** *questions with at least* **one** *from each section:*

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

The following notation is used throughout the paper: capital bold letters (e.g. $\mathbf{U}$) indicate 4-vectors; lower case bold letters (e.g. $\mathbf{v}$) indicate 3-vectors. Note, however, that the symbols $\mathbf{E}$, $\mathbf{B}$ and $\mathbf{A}$ in questions 3 and 4 indicate 3-vectors: the electric and magnetic fields and vector potential respectively.

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1. For a particle of mass m moving along a world-line in an inertial reference frame, S, define the proper time, τ, 4-velocity, $\mathbf{U}$ and 4-acceleration, $\mathbf{A}$. Using the relation for the γ-factor, $\gamma(v) = 1/\sqrt{1 - v^2/c^2}$ of the particle, $\mathrm{d}\gamma/\mathrm{d}t = \gamma^3(\mathbf{v} \cdot \mathbf{a}/c^2)$, where $\mathbf{v}$ and $\mathbf{a}$ are the particle 3-velocity and 3-acceleration, correspondingly, find $\mathbf{A}$ in terms of γ, $\mathbf{v}$ and $\mathbf{a}$.

Find the invariants of 4-velocity and 4-momentum. Evaluate explicitly the scalar product, $\mathbf{U} \cdot \mathbf{A}$. [5]

Two events in S are characterized by 4-coordinates, $\mathbf{D} = (ct_d, \mathbf{x}_d)$ and $\mathbf{B} = (ct_b, \mathbf{x}_b)$, where $\mathbf{x}_d$ and $\mathbf{x}_b$ are 3-vectors. Write down the condition for these events to be connected by a time-like interval. In such a case, can we find an inertial frame, S', where the two events are occurring simultaneously? Explain. What is the physical meaning of the condition $\mathbf{D} \cdot \mathbf{B} = 0$? [5]

Define proper acceleration and pure force. A particle undergoing hyperbolic motion has a worldline given by $x^2 - t^2 = L^2$, where L is a constant and the speed of light is $c = 1$. Find the particle's speed, v, the Lorentz factor, $\gamma(v)$ and the acceleration, a, as functions of x and show that $a\gamma^3$ is constant. [5]

Now consider the motion of a particle of mass m, initially at rest under the influence of a constant 3-force, $\mathbf{f}$. Find $\beta \equiv \mathbf{v}/c$ and the Lorentz factor $\gamma(t)$ of the particle as a function of time t. Sketch the graphs of $\gamma(t)$ and $\beta(t)$. [5]

An electron is accelerated from rest through a gap of $L = 10$ m by an electric field of strength $5\,\mathrm{MV\ m^{-1}}$ that is constant throughout the gap. Find γ and β at the other end of the gap. How much would β change if the accelerating field was reduced by 20%? How long does it take for the electron to reach the other end of the gap? [5]

2. A frame S′ is moving relative to the laboratory frame, S, with velocity $\mathbf{v} = (v_x, 0, 0)$. A particle of mass m moves in S with velocity $\mathbf{u} = (u_x, u_y, 0)$. Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$ in S. Using $\mathbf{u} = \mathbf{u}_{||} + \mathbf{u}_{\perp}$, where $\mathbf{u}_{||}$ is the component of the particle's velocity in the direction of motion of S′ and $\mathbf{u}_{\perp}$ is the component perpendicular to it, show that the angle θ' in frame S′ is given by

$$\tan\theta' = \frac{u\sin\theta}{\gamma_v(u\cos\theta - v)}\,,$$

where $\gamma_v \equiv \gamma(v)$ is the Lorenz factor. [5]

In frame S an electron moves in a uniform magnetic field, $\mathbf{B} = (0, 0, B)$, along a spiral trajectory defined by the Larmor radius, R, and a constant longitudinal velocity, $\mathbf{v}_z \,||\, \mathbf{B}$. The initial 4-momentum of the electron is $\mathbf{P}_0$. The ratio between longitudinal and transverse components of the electron's 3-momentum is $\mathbf{p}_{||}/\mathbf{p}_{\perp} = 1$, while the electron's Lorentz factor is $\gamma = 17$. At $t = t_0$ a constant electric field, $\mathbf{E} = (0, 0, E_z)$, is applied in such a way that it decelerates the electron. After propagating 1 m along the z direction, the electron's longitudinal velocity drops to zero, $v_z = 0$. Find the strength, E_z, of the electric field applied and the factor γ at the end of the trajectory. Sketch the dependence of the Larmor radius on z. Ignore any radiation effects by the electron. [7]

[Hint: Use the fact that the electron's rest energy in eV is $m_0c^2 = 0.511$ MeV and $\gamma = 1 + eV/m_0c^2$, where V is the voltage applied.]

Two photons of the same angular frequency, ω, and with 4-momenta $\mathbf{P}_1$ and $\mathbf{P}_2$, move in the stationary frame S. The first photon moves along the x direction and has $v_y = 0$ while the second photon moves at some angle θ to the x direction. Both photons have $v_z = 0$. Find:

(a) the rest energy of the system as a function of ω and θ; and

(b) the velocity of the centre of mass frame relative to the lab frame as a function of θ, making a sketch of the θ-dependence.

[7]

Photons with wave 4-vector $\mathbf{K}$ are radiated by a stationary laser towards a beam of electrons. Each electron in the beam has energy W and velocity v along the x direction. A single photon scatters off an electron. In the lab frame, S, find the maximum photon energy after the scattering. If the electron's energy is 2 GeV and the photon's wavelength is 1 cm, calculate the scattered photon's wavelength. [6]

3. Name three phenomena, each of which can be detected via the studies of emitted or reflected light in astrophysics, that would represent a direct test of Special Relativity. Explain what the relativistic Doppler effect is and derive the equation for the photon's frequency shift from ω to ω' between reference frames S and S$'$, respectively, moving with relative velocity $\mathbf{v}$. The photon is emitted at an angle θ with respect to $\mathbf{v}$ in S. [6]

A plane monochromatic electromagnetic wave with angular frequency ω and wave 3-vector $\mathbf{k}$ propagates in a uniform medium with refractive index n. Define the phase and group velocities, v_{ph} and v_{gr}, and show that $v_{gr}v_{ph} = c^2$ for $n = 1$. [4]

An electron of mass m_e and a proton of mass M_p are moving in the lab frame S in opposite directions but toward each other with velocities, $\mathbf{v} = (0, 0, \pm v_z)$, respectively. At some moment the particles collide and a photon is emitted in the direction perpendicular to the z axis. After the collision, the electron and the proton are moving as a single particle. The wavelength of the photon as measured by an observer in the laboratory frame is λ. Find the minimal total energy of the electron and proton, E_{tot}, before the collision that would allow emission of such a photon. [7]

A plane, linearly polarized electromagnetic wave propagates in the z direction through a uniform medium with the refractive index $n > 1$. The electric and magnetic fields of the wave are given by

$$E_y = E_0 \cos(\omega t - kz), \quad B_x = B_0 \cos(\omega t - kz) \quad \text{and} \quad |\mathbf{E}_0 \times \mathbf{B}_0| = 1.$$

Find the vector potential, $\mathbf{A}$, the scalar potential, φ and the wave 4-vector, $\mathbf{K}$, in the Lorentz gauge, $\partial^\mu A_\mu = 0$. Is it possible to find a reference frame in which either the electric or magnetic field of the wave defined above would vanish? If yes, find the frame's velocity relative to the lab frame. [8]

4. Write down the relation between the electric and magnetic fields, **E** and **B**, and the components of the 4-vector potential, A^{μ}. Define the field strength tensor, $F^{\alpha\beta} = \partial^{\alpha}A^{\beta} - \partial^{\beta}A^{\alpha}$ and show that its components satisfy $\partial^{c}F^{ab} + \partial^{a}F^{bc} + \partial^{b}F^{ca} = 0$. Define the 4-current, J_{ν}, and show that two out of the four Maxwell's equations can be written in the form $\partial^{\mu}F_{\mu\nu} = J_{\nu}$.

Show that the Lorentz force acting on the unit volume of charge density, ρ, can be written as $f_{\mu} = J^{\nu}F_{\nu\mu}$. What is the physical meaning of the f_0 component of this 4-vector? [7]

Consider a unit vector, $U^{\mu}_{W} = W^{\mu}/(m_0 c|\mathbf{s}|)$, parallel to the Pauli-Lubanski spin vector, $W^{\mu} = (\mathbf{s}\cdot\mathbf{p}, E\mathbf{s}/c)$. Show without direct calculations that U^{μ}_{W} and the 4-momentum are orthogonal to each other, i.e. $U^{\mu}_{W}P_{\mu} = 0$. [Hint: Use the rest frame.] [5]

A free relativistic electron (with $\gamma = 100$) moves (a) in a vacuum (refractive index $n = 1$), or (b) in a uniform medium (refractive index $n = 2$). Explain whether the electron emits a photon in either of the cases listed. If the electron in case (a) or case (b) can emit a photon, at what angle with respect to the electron's trajectory is the photon emitted? [6]

A straight wire with the charge density ρ moves with velocity $\mathbf{v}$ along the z axis. Find the electric and magnetic fields, **E** and **B**, generated by the wire in the frames:

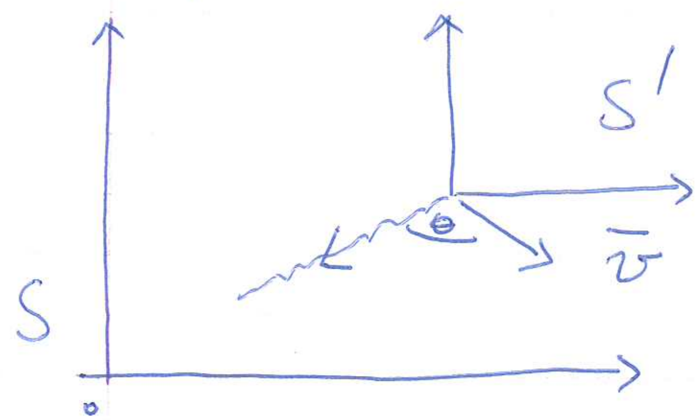
(i) S$'$ which is co-moving with the wire;

(ii) the laboratory (stationary) frame S.

A second identical wire having the same charge density and separated by the distance, d, from the first wire moves parallel to the first wire with the same velocity. Is it possible to find such a frame that the forces between wires can be described as purely magnetic? Prove the statement. [7]

- Stellar aberration
- Relativ. Doppler effect
- Radio time delay
- Headlight effect

Doppler effect:



Photon in S':

$$k'^{\mu} = \left(\frac{\omega'}{c}, \bar{k}'\right)$$

Photon in S:

$$k^{\mu} = \left(\frac{\omega}{c}, \bar{k}\right)$$

Lor. transf. $S \to S'$:

$k^2 = 0$
$k'^2 = 0$

$$k'^0 = \gamma\left(k^0 - \bar{\beta}\cdot\bar{k}\right)$$

$$\Rightarrow \frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \frac{v}{c}|\bar{k}|\cos\theta\right) =$$

$$= \gamma\frac{\omega}{c}\left(1 - \frac{v}{c}\cos\theta\right), \quad \text{since } |\bar{k}| = \frac{\omega}{c}.$$

$$\Rightarrow \boxed{\omega = \frac{\omega'}{\gamma\left(1 - \frac{v}{c}\cos\theta\right)}}.$$

- phase velocity

Elm waves $\sim e^{-i\omega t + i\bar{k}\bar{x}}$

Const. phase: $-\omega t + \bar{k}\bar{x} = \text{const}$

Can choose direction along $\bar{k}$ in isotropic medium (e.g. choose z along $\bar{k}$)

$\Rightarrow -\omega t + k z = \text{const}, \quad k = |\bar{k}|.$

$-\omega\, dt + k\, dz = 0 \Rightarrow v_{ph} = \frac{dz}{dt} = \frac{\omega}{|\bar{k}|}$

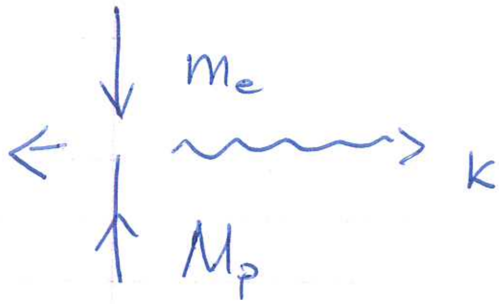
In medium, $k^{\mu} = \left(\frac{\omega n}{c}, \bar{k}\right), \quad k^2 = 0$

$|\bar{k}| = \omega n / c \Rightarrow v_{ph} = \frac{\omega}{|\bar{k}|} = c/n.$

- Group velocity: wave packet with $v_{gr} = d\omega / dk, \quad \frac{d\omega}{dk} = c/n$

$\Rightarrow v_{gr} \cdot v_{ph} = c^2/n^2 = c^2$ for $n = 1$.

- $\bar{v} = (0, 0, \pm v_z)$



$$p_e + P_p = p' + k$$

<u>Before</u>: $p_e = \left(\frac{\mathcal{E}_e}{c}, \bar{p}_e \right) = (\gamma m_e c, -\gamma m_e v_z)$

$$P_p = \left(\frac{\mathcal{E}_p}{c}, \bar{P}_p \right) = (\gamma M_p c, \gamma M_p v_z)$$

Here $\gamma = \left(1 - v_z^2/c^2\right)^{-1/2}$

$$\mathcal{E}_{TOT} = \gamma (m_e + M_p) c^2 \equiv \gamma \mathcal{M}_{TOT} c^2.$$

<u>After</u>: $k^\mu = \left(\frac{2\pi c}{\lambda}, \frac{2\pi c}{\lambda}, 0, 0 \right)$

$$p'^\mu = \left(\gamma' M_{TOT} c, \gamma' M_{TOT} v_x', 0, \gamma' M_{TOT} v_z' \right)$$

So, $\gamma v_z (M_p - m_e) = \gamma' (M_p + m_e) v_z'$

$$0 = \gamma' (M_p + m_e) v_x' + \frac{2\pi c}{\lambda} = 0$$

$$\gamma'^2 = \left(1 - v_x'^2/c^2 - v_z'^2/c^2\right)^{-1}$$

$$\begin{cases} \gamma' v_z' = \gamma v_z \dfrac{M_p - m_e}{M_p + m_e} \\[2ex] \gamma' v_x' = -\dfrac{2\pi c}{\lambda (M_p + m_e)} \end{cases}$$

$$\gamma'^2 = 1 + \frac{1}{c^2}\gamma^2 v_z^2 \left(\frac{M_p - m_e}{M_p + m_e}\right)^2 + \left(\frac{2\pi \cancel{c}}{\lambda}\right)^2 \frac{1}{(M_p + m_e)^2}$$

<u>Also</u>, $\gamma (m_e + M_p) c = \dfrac{2\pi c}{\lambda} + \gamma' (m_e + M_p) c$

$$\gamma' (m_e + M_p) c = \gamma (m_e + M_p) c - \frac{2\pi c}{\lambda}$$

$$\boxed{\gamma' = \gamma - \frac{2\pi}{\lambda (m_e + M_p)}}$$

So, we have the following eq. relating λ and v_z:

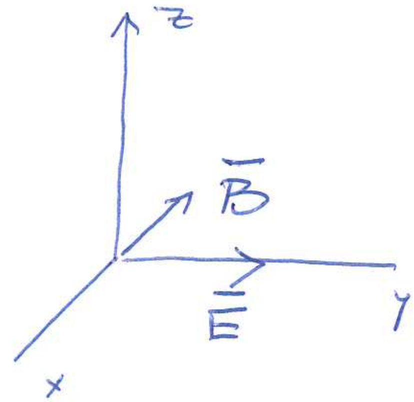
$$\left(\gamma - \frac{2\pi}{\lambda\,(m_e+M_p)}\right)^2 = 1 + (\gamma^2-1)\left(\frac{M_p-m_e}{M_p+m}\right)^2 + \left(\frac{2\pi}{\lambda}\right)^2\frac{1}{(M_p+m_e)^2}.$$

For a given λ, this allows to find γ

$\Rightarrow \mathcal{E}_{TOT} = \gamma\,(m_e+M_p)\,c^2.$

- $E_y = E_0 \cos(\omega t - kz)$

$B_x = -B_0 \cos(\omega t - kz)$



$$A^\mu = (\phi/c, \bar{A})$$

$$k^\mu = \left(\frac{\omega}{c}, 0, 0, k\right)$$

$$X^\mu = (ct, x, y, z)$$

$$\Box A^\mu = 0 \quad \text{(eom)}$$

$$\bar{B} = \text{curl}\,\bar{A} \qquad \bar{B} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ A_x & A_y & A_z \end{vmatrix}$$

$$B_x = \cancel{\partial_y A_z} - \partial_z A_y = -B_0 \cos(\omega t - kz)$$

$$B_y = -\partial_x A_z + \partial_z A_x = 0$$

$$B_z = \partial_x A_y - \partial_y A_x = 0$$

$$A_y = -\frac{B_0}{k} \sin(\omega t - kz) \qquad A_x = 0 \quad A_z = 0$$

$$\bar{E} = -\nabla\phi - \partial\bar{A}/\partial t \Rightarrow \phi = 0,$$

$$E_x = 0, \quad E_y = -\partial_t A_y =$$

$$= + \frac{\omega B_0}{k} \cos(\omega t - kz) = E_0 \cos(\omega t - kz)$$

$$\omega/k = c/n \quad \Rightarrow \quad E_0 = c B_0 / n.$$

$$|\bar{E}_0 \times \bar{B}_0| = 1 \qquad c B_0^2 / n = 1$$

$$B_0^2 = n/c \qquad B_0 = \sqrt{n/c} \qquad E_0 = \sqrt{c/n}$$

$$\boxed{A^\mu = \left(0, 0, \frac{1}{\sqrt{\omega k}} \sin(k^\mu x_\mu), 0\right)}$$

Invariants of electromag. field:

$$\alpha \sim \bar{E} \cdot \bar{B}, \quad \mathcal{D} \sim \bar{E}^2/c^2 - \bar{B}^2.$$

Elm wave is a solution of Maxwell's eq. with $\alpha = 0$, $\mathcal{D} = 0$

$\Rightarrow$ in $\forall$ frame $E' \perp B'$, $E'/c = B'$,

E' and B' are always non-zero.

Problem 4

$$A^\mu = (\Phi/c, \bar{A})$$

$$\left.\begin{array}{l} \bar{B} = \text{curl}\,\bar{A} = \bar{\nabla}\times\bar{A} \\ \bar{E} = -\nabla\phi - \partial_t\bar{A} \end{array}\right\}$$

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \quad \text{(definition)}$$

Note: $F^{\alpha\beta} = -F^{\beta\alpha}$

<u>Relation to $\bar{E}, \bar{B}$:</u> $F^{0i} = E^i/c$

$$F^{ij} = \varepsilon^{ijk}B^k$$

$$F^{\alpha\beta} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -\frac{E_x}{c} & 0 & B_z & -B_y \\ -\frac{E_y}{c} & -B_z & 0 & B_x \\ -\frac{E_z}{c} & B_y & -B_x & 0 \end{bmatrix}$$

<u>Maxwell's eqs:</u> $\partial_\beta F^{\alpha\beta} = \mu_0 J^\alpha$

4 eqs (in components) corresp. to

$\text{div}\,\bar{E} = \rho/\varepsilon_0$ and $\text{curl}\,\bar{B} = \mu_0 \bar{j} + \varepsilon_0\mu_0 \frac{\partial \bar{E}}{\partial t}$

Indeed, since $J^\mu = (\rho c, \bar{j})$, where $\rho, \bar{j}$ are the charge density and current density in the lab frame, with $\partial_\mu J^\mu = \frac{\partial \rho}{\partial t} + \mathrm{div}\,\bar{j} = 0$, we have for $\alpha = 0$:

$$\partial_\beta F^{0\beta} = \mu_0 J^0 .$$

i.e. $\partial_0 \underset{\parallel \atop 0}{F^{00}} + \partial_i F^{0i} = \mu_0 J^0 = \mu_0 c \rho$

$\Rightarrow \partial_i E^i = \mu_0 c^2 \rho = \rho / \varepsilon_0 .$

And similarly for $\alpha = 1, 2, 3$ (show this!)

- The other half of Maxwell's eqs

$$\mathrm{div}\,\bar{B} = 0, \quad \mathrm{curl}\,\bar{E} = -\partial \bar{B}/\partial t,$$

corresp. to Bianchi identity

$$\partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0 .$$

This identity is satisfied due to anti-symmetry of $F_{\alpha\beta}$ and $\partial_\alpha \partial_\beta = \partial_\beta \partial_\alpha$!

$$\partial_\alpha\left(\partial_\beta A_\gamma - \partial_\gamma A_\beta\right) + \partial_\gamma\left(\partial_\alpha A_\beta - \partial_\beta A_\alpha\right)$$
$$+ \partial_\beta\left(\partial_\gamma A_\alpha - \partial_\alpha A_\gamma\right) =$$

$$= \cancel{\partial_\alpha \partial_\beta A_\gamma} - \cancel{\partial_\alpha \partial_\gamma A_\beta} + \cancel{\partial_\gamma \partial_\alpha A_\beta} - \cancel{\partial_\gamma \partial_\beta A_\alpha} +$$
$$+ \cancel{\partial_\beta \partial_\gamma A_\alpha} - \cancel{\partial_\beta \partial_\alpha A_\gamma} = 0 .$$

For example:

$$\partial^1 F^{23} + \partial^3 F^{12} + \partial^2 F^{31} = 0$$

is $\partial^x B^x + \partial^z B^z + \partial^y B^y = 0$

i.e. $\operatorname{div} \bar{B} = 0$.

- Lorentz force (density) $\bar{f} = \bar{F}/V$

$$\bar{f} = \rho \bar{E} + \bar{j} \times \bar{B} \qquad \left(\begin{array}{l} \rho = Q/V \\ \bar{j} = \bar{J}/V \end{array} \right)$$

can be written in covariant form as

$$\underline{f_\mu = F_{\mu\nu} J^\nu}$$ (note a typo in the order of indices in the problem)

Indeed, $f_i = F_{i\nu} J^\nu = F_{i0} J^0 +$

$+ F_{ij} J^j = \frac{E_i}{\cancel{c}} \cdot \rho \cancel{c} + \varepsilon_{ijk} B_k j_j$

The f_0 component is

$$f_0 = F_{0\nu} J^\nu = \underset{\overset{\shortparallel}{0}}{F_{00}} J^0 + F_{0i} J^i =$$

$$= - \bar{E} \cdot \bar{J} / c \ .$$

Since $\bar{j} = n q \bar{v}$, $f_0 \sim \bar{f} \cdot \bar{v}$, i.e. this is power density or the work done by external field on charges inside the volume.

Consider $U_w^\mu = W^\mu / m_0 c |\bar{s}|$, where

$W^\mu = (\bar{s} \cdot \bar{p}, E\bar{s}/c)$ is the Pauli-Lubanski vector

$U_w^\mu \cdot P_\mu = 0$?

Indeed, $U_w^\mu = \left(\frac{\bar{p} \cdot \bar{s}}{m_0 c |\bar{s}|}, \frac{E\bar{s}}{m_0 c^2 |\bar{s}|} \right)$

In particle's rest frame, $p^\mu = (m_0 c, \bar{0})$

and $U_w^\mu = \left(0, \frac{\bar{s}}{|\bar{s}|}\right) \Rightarrow$

$$U_w^\mu P_\mu = -\underset{\overset{\shortparallel}{0}}{U_w^0} P^0 + \bar{U}_w \underset{\overset{\shortparallel}{0}}{\bar{P}} = 0$$

Since Mink. scalar product is invar, $U_w^\mu P_\mu = 0$ in any frame.

- Electron with $\gamma = 100$ moves in

a) vac with $n=1$.

$$p = p' + k, \quad \text{where } k^\mu = \left(\frac{\omega}{c}, \bar{k}\right)$$

photon's 4-vector of momentum

and p, p' are 4-momenta of the electron before/after emission.

$$\underset{m^2c^2}{\underset{\parallel}{p^2}} = \underset{m^2c^2}{\underset{\parallel}{p'^2}} + 2pk + \underset{0}{\underset{\parallel}{k^2}} \Rightarrow pk = 0$$

With $p' = \left(\frac{\mathcal{E}'}{c}, \bar{p}'\right)$, we have

$$-\frac{\mathcal{E}'}{c}\frac{\omega}{c} + \bar{p}' \cdot \bar{k} = 0 \qquad |\bar{k}| = \omega/c$$

or $\frac{\mathcal{E}'}{c} = |\bar{p}'| \cos\varphi$ $\qquad$ φ - angle between $\bar{p}', \bar{k}$.

$$\Rightarrow \cos\varphi = \sqrt{1 + \frac{m^2c^4}{p^2c^2}} > 1 \text{ (impossible)}.$$

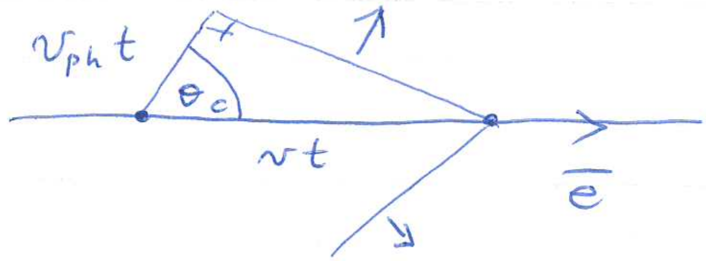
b) a medium with $n = 2$.

In this case, the electron is not free.

Light in medium has phase vel.

$v_{ph} = c/n < c$

Electron moving with $v > c/n$ can emit photon (Cherenkov radiation)



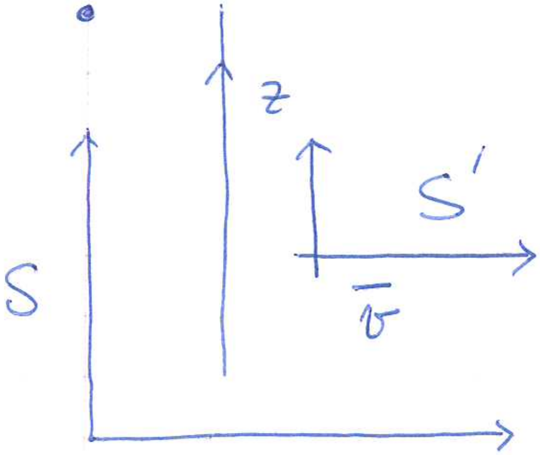
$$v \cos\theta_c = v_{ph} \Rightarrow \cos\theta_c = v_{ph}/v$$

$$v = c\sqrt{1 - 1/\gamma^2}$$

$$\cos\theta_c = \frac{\cancel{c}}{n \cdot \cancel{c}\sqrt{1-1/\gamma^2}} = \frac{1}{n(\sqrt{1-1/\gamma^2})} \approx 0.5$$

$\theta_c \approx 60°$.

Analysis of Cherenkov eff.: see e.g. L-L or Jackson.



In S', the wire is stationary

Gauss' theorem:

$$2\pi r' l' E_r' = l' \rho' / \varepsilon_0$$

for the length l' of the wire. Here $\rho' = \rho_0$: proper charge density. So, in S' we have $\bar{B}' = 0$ and $E_r' = \rho' / 2\pi\varepsilon_0 r'$, where $\rho' = \rho_0$ and $r'^2 = x'^2 + y'^2$.

$$\Rightarrow E_z' = 0, \quad E_x' = \frac{\partial x'}{\partial r'} E_r' = \frac{x'}{r'} E_r' =$$

$$= \frac{\rho' x'}{2\pi\varepsilon_0 r'^2}, \quad E_y' = \frac{y'}{r'} E_r' = \frac{\rho' y'}{2\pi\varepsilon_0 r'^2}.$$

To find fields in S, make Lor. transf.

Note: $\rho' dl' = \rho dl$ (charge conserved)

$\Rightarrow \rho = \gamma \rho' = \gamma \rho_0$ in S

Also, $x' = x$, $y' = y$, $r' = r$.

So, $E_z' = 0$, $E_x' = \dfrac{\rho_0 x}{2\pi\varepsilon_0 r^2}$,

$$E_y' = \frac{\rho_0 y}{2\pi\varepsilon_0 r^2} = \frac{\rho y}{\gamma\, 2\pi\varepsilon_0 r^2}$$ (here ρ is charge dens. in S).

$(r^2 = x^2 + y^2)$

Field transformations: $S \to S'$

$$\bar{E}_\parallel' = \bar{E}_\parallel$$
$$\bar{E}_\perp' = \gamma\left(\bar{E}_\perp + \bar{v} \times \bar{B}\right)$$

$S' \to S$:
$$\begin{cases} \bar{E}_\parallel = \bar{E}_\parallel' \\ \bar{E}_\perp = \gamma\left(\bar{E}_\perp' - \bar{v} \times \underbrace{\bar{B}'}_{=0}\right) \end{cases}$$

$$\Rightarrow \quad E_z = 0$$

$$\left.\begin{aligned} E_x &= \frac{\gamma \rho_0 x}{2\pi\varepsilon_0 r^2} = \frac{\rho x}{2\pi\varepsilon_0 r^2} \\ E_y &= \frac{\gamma \rho_0 y}{2\pi\varepsilon_0 r^2} = \frac{\rho y}{2\pi\varepsilon_0 r^2} \end{aligned}\right\} \text{Lab frame}$$

Magnetic field: $S \to S'$

$$\begin{cases} \bar{B}'_{\parallel} = \bar{B}_{\parallel} \\ \bar{B}'_{\perp} = \gamma\left(\bar{B}_{\perp} - \bar{v}\times\bar{E}/c^2\right) \end{cases}$$

For $S' \to S$:

$$\begin{cases} \bar{B}_{\parallel} = \bar{B}'_{\parallel} \\ \bar{B}_{\perp} = \gamma\left(\bar{B}'_{\perp} + \bar{v}\times\bar{E}'/c^2\right) \end{cases}$$

$$\Rightarrow \text{Lab frame} \begin{cases} B_z = 0 \\ B_x = -\dfrac{\gamma v}{c^2} E'_y = -\dfrac{\rho v y}{2\pi\varepsilon_0 r^2 C^2} \\ B_y = \dfrac{\rho v x}{2\pi\varepsilon_0 r^2 C^2} \end{cases}$$

($r^2 = x^2 + y^2$, ρ is the charge density in S).

- In S', forces are electric, $\bar{B}' = 0$.

In any other frame S'', $\bar{E}''_{\perp} = \tilde{\gamma}\,\bar{E}'_{\perp}$

$\Rightarrow$ impossible to eliminate electric forces.