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- Many-body rel. system. Can have $T \gg mc^2$
- $a(t)$ vs $T(t)$
- freeze-out : out of equil. → reaction stops
- In equil: $\mu_{\alpha_1} + \mu_{\alpha_2} + \dots = \mu_{\beta_1} + \mu_{\beta_2} + \dots$

$$\alpha_1 + \alpha_2 + \dots \to \beta_1 + \beta_2 + \dots$$

$$\mu_\gamma = 0$$

$$\mu_{e^+} + \mu_{e^-} = 0 \Rightarrow \mu_{e^+} = -\mu_{e^-}$$

B, L_e, L_μ, L_τ, Q

$$\boxed{dN_i^{F/B} = \frac{g_i}{(2\pi\hbar)^3} \frac{1}{e^{\frac{\varepsilon-\mu}{kT}} \pm 1} d^3p\, d^3x}$$

Consider ultrarel. particles with $\mu = 0$

~~Photons~~ Bosons: $\left(\begin{matrix} \text{e.g.} \\ \mu = 0 \end{matrix}\right)$ $\varepsilon = pc$ (Rel) $g = \cancel{\;}$

$$dN_B = \frac{2V}{(2\pi\hbar)^3} \frac{4\pi p^2\, dp}{e^{\frac{pc}{kT}} - 1}$$

$$N_B = \frac{8\pi V}{(2\pi\hbar)^3} \int_0^\infty \frac{\bar{p}^2\, d\bar{p}}{e^{\bar{p}\cancel{c/kT}}-1} \left(\frac{kT}{c}\right)^3$$

$$= \frac{8\pi V}{(2\pi\hbar)^3} \left(\frac{kT}{c}\right)^3 \underbrace{\int_0^\infty \frac{x^2\,dx}{e^x-1}}_{= 2\zeta(3)} = \frac{2(kT)^3 \zeta(3)}{\pi^2 \hbar^3 c^3}$$

$$N_B = \frac{2\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \sim T^3 .$$

Rel. fermions with g_i, $\varepsilon = pc$

$$dN_f = \frac{g_i V}{(2\pi\hbar)^3} \frac{4\pi p^2\, dp}{e^{\frac{\varepsilon}{kT}}+1}$$

$$N_f = \frac{g_i V \cdot 4\pi}{(2\pi\hbar)^3} \int_0^\infty \frac{p^2\,dp}{e^{pc/kT}+1} = \frac{g_i V\cdot 4\pi}{(2\pi\hbar)^3} \left(\frac{kT}{c}\right)^3 \overbrace{\int_0^\infty \frac{x^2\,dx}{e^x+1}}^{\frac{3}{2}\zeta(3)}$$

$$N_f = \frac{g_i V 3}{4\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \zeta(3) \sim T^3.$$

<u>Moral</u>: Number density

- rel. Bosons: $N_B = \frac{g\,\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3$

- rel fermions: $N_F = \frac{g\,\zeta(3)}{\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \cdot \frac{3}{4}$

<u>Energy density</u>: rel. fermi / Bose gas

$$d\varepsilon_{B/F} = \frac{g_i}{(2\pi\hbar)^3}\,\frac{\varepsilon}{e^{\frac{\varepsilon-\mu}{kT}} \mp 1}\, d^3p\, d^3x$$

For $\varepsilon = pc \Rightarrow d^3p = \frac{4\pi\varepsilon^2}{c^3}d\varepsilon$

Let $\mu = 0$:

$$\varepsilon_{B/F} = \frac{g_i V}{(2\pi\hbar c)^3} 4\pi \int_0^\infty \frac{\varepsilon^3\, d\varepsilon}{e^{\varepsilon/kT} \mp 1}$$

$$\rho_{B/F} c^2 = \frac{\varepsilon_{B/F}}{V_3} = \frac{g_i \, 4\pi}{(2\pi \hbar c)^3} (kT)^4 \int_0^\infty \frac{x^3 \, dx}{e^x \mp 1}$$

$$\frac{\pi^4}{15} \quad \text{B}$$

$$\frac{7\pi^4}{120} \quad \text{F}$$

$$\rho_B c^2 = \frac{g_i \pi^2 (kT)^4}{30 (\hbar c)^3} \quad \text{B}$$

$$\rho_F c^2 = \frac{g_i \pi^2 (kT)^4 \, 7}{2 \cdot 120 (\hbar c)^3} = \frac{7}{8} g_i \frac{\pi^2 (kT)^4}{30 (\hbar c)^3} \quad \text{F}$$

g_i : spin d.o.f. number
= 2 for photons;
~~= 2s+1 for massive particles of spin s~~ (= 2 for electrons)
= 1 for neutrinos (Weyl)

Moral: ultrarel. gas ($\varepsilon \simeq pc$)

$$\rho c^2 = g_{eff} \frac{\pi^2}{30} \frac{(kT)^4}{(\hbar c)^3}$$

$$g_{eff} = \sum_{\substack{\text{Bosons} \\ (mc^2 \ll kT)}} g_i + \frac{7}{8} \sum_{\substack{\text{Fermions} \\ (mc^2 \ll kT)}} g_i$$

$$\text{EOS}: P_i = \frac{1}{3} \rho_i c^2 \Rightarrow \quad (w = 1/3)$$

Non-rel. gas: $Mc^2 \gg kT$

$$n_i = g_i \left(\frac{m_i kT}{2\pi \hbar^2} \right)^{3/2} e^{\frac{\mu_i - M_i c^2}{kT}}$$

$$\rho_i c^2 = M_i c^2 n_i$$

$$P_i = n_i kT \ll M_i c^2 n_i = \rho_i c^2 \Rightarrow P \approx 0$$

CMB: $kT_0 a_0 = kT_1 a_1$

$$\frac{\lambda_{obs}}{\lambda_{emit}} = \frac{a(t_0)}{a(t_1)} \qquad \lambda\nu = c$$

$$\frac{\nu_{em}}{\nu_{obs}} = \frac{a_{obs}}{a_{em}} \Rightarrow \nu_1 a_1 = \nu_2 a_2$$

Planck: $e^{h\nu/kT} - 1 \rightarrow$

$$e^{h\nu_1/kT_1} - 1 = e^{h\nu_2/kT_2} - 1$$

form-invariant: $\boxed{T \cdot a = \text{const}}$

$$1+z = \frac{1}{a}$$

Rate: Can we use thermal equilibr? (7)

$l = c\tau$

$H^2 \sim G\rho \qquad G \sim 1/M_p^2$

$\rho \sim T^4$

$\Rightarrow \frac{\dot{a}}{a} \sim \frac{T^2}{M_p} \sim \frac{1}{\text{time}} \sim \Gamma_{exp}$

Rate of reactions: $\Gamma^{-1} \sim \frac{1}{n\sigma}$

(chemical rate - different)

$\sigma \sim \frac{\alpha_{em}^2}{T^2} \qquad \alpha_{em} = e^2/4\pi\hbar c \sim 1/137$

$n \sim T^3 \Rightarrow n\sigma \sim \alpha^2 T$

$$\boxed{\Gamma_{react} \sim \alpha^2 T}$$

~ Equilibrium: $\Gamma_{react} \gg \Gamma_{expansion}$

$\alpha^2 T \gg \frac{T^2}{M_p} \Rightarrow T \ll \alpha^2 M_p \sim \sim 10^{14}$ GeV

$T \ll \alpha^2 M_p$

$\alpha^2 \sim 10^{-4}$ $\quad M_p \sim 10^{19}$ GeV

Non-rel. particles at equilibrium

Equil: $\mu_1 = \mu_2$ (chemical)
Suppose $m_1 \gtrsim m_2$ (but $m_1 \sim m_2$).
Then:

$$\frac{n_1}{n_2} = \frac{g_1}{g_2}\left(\frac{m_1}{m_2}\right)^{3/2} e^{\frac{(m_2 - m_1)c^2}{kT}} \longrightarrow 0$$

as $T \to 0$

$\Rightarrow$ heavier particles (e.g. neutrons vs p) disappear...

$\Rightarrow$ not an equil.

$$\frac{d \ln N_1}{d \ln a} = - \frac{\Gamma}{H} \left[1 - \left(\frac{N_1^{eq}}{N_1} \right)^2 \right]$$

$$N_1^{eq} = n_1^{eq} a^3$$

$$\frac{\Gamma}{H} \gg 1 \Rightarrow \quad N_1 \sim N_1^{eq}$$

$$\frac{\Gamma}{H} \ll 1 \Rightarrow \quad \frac{d \ln N_1}{d \ln a} \simeq 0$$ Chemical freez-out

$N_1 \sim$ const (comoving), i.e.

$n_1 \sim N_1 / a^3$ with time.

at $\Gamma \sim H$.

Abundances of elements: can be measured.