SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B5: GENERAL RELATIVITY & COSMOLOGY

TRINITY TERM 2014

Saturday, 21 June, 9.30 am – 11.30 am

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Page 2 contains physics formulae and data for this paper.
The questions start on page 3.

Expressed in terms of the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$, the Riemann tensor has components

$$R^{\gamma}{}_{\alpha\beta\delta} = \frac{\partial}{\partial x^{\delta}}\Gamma^{\gamma}_{\alpha\beta} - \frac{\partial}{\partial x^{\beta}}\Gamma^{\gamma}_{\alpha\delta} + \Gamma^{\epsilon}_{\alpha\beta}\Gamma^{\gamma}_{\delta\epsilon} - \Gamma^{\epsilon}_{\alpha\delta}\Gamma^{\gamma}_{\beta\epsilon}.$$

The Ricci tensor is defined as $R_{\alpha\beta} \equiv R^{\gamma}{}_{\alpha\beta\gamma}$. When expressed in a basis in which the metric tensor $g_{\alpha\beta}$ is diagonal, it can be written as

$$R_{\alpha\beta} = \frac{1}{2}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}}\ln|g| - \frac{\partial\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\delta}_{\alpha\gamma}\Gamma^{\gamma}_{\beta\delta} - \frac{1}{2}\Gamma^{\gamma}_{\alpha\beta}\frac{\partial}{\partial x^{\gamma}}\ln|g|,$$

where $|g|$ is the modulus of the determinant of $g_{\alpha\beta}$.

The Einstein equation is

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = -\frac{8\pi G}{c^4}T_{\alpha\beta},$$

where $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar, Λ is the cosmological constant and $T_{\alpha\beta}$ is the energy–momentum tensor.

For the line element

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right]$$

the Einstein equation can be written as the pair of equations

$$\begin{aligned}\ddot{R} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{1}{3}\Lambda c^2 R,\\ \dot{R}^2 &= \frac{8\pi G}{3}\rho R^2 + \frac{1}{3}\Lambda c^2 R^2 - c^2 k,\end{aligned}$$

where ρ and p are the rest-frame density and pressure, respectively.

1. (a) State the equivalence principle, considering a free-falling lab in a gravitational field. Put together a coordinate frame in our free falling lab, the coordinates of which are ξ^α. Imagine a particle at rest in this frame. Its equation of motion is

$$\frac{\mathrm{d}^2\xi^\alpha}{\mathrm{d}\tau^2} = 0.$$

Define the proper time τ in this equation, using the free-falling coordinates and the locally flat metric $\eta_{\alpha\beta}$. [4]

(b) We want to make a measurement on this particle and soon realize it would be convenient to move to a different set of coordinates, which we call x^α. Starting from the equation of motion in the free-falling coordinates, and making use of the Einstein summation convention, the chain rule, and the Kronecker delta, δ^λ_μ, show that the equation of motion (geodesic equation) in the new coordinate system can be written as

$$\frac{\mathrm{d}^2x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\mu\nu}\frac{\mathrm{d}x^\mu \mathrm{d}x^\nu}{\mathrm{d}\tau \mathrm{d}\tau} = 0,$$

where

$$\Gamma^\lambda_{\mu\nu} = \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{\partial x^\lambda}{\partial \xi^\alpha}.$$

What symmetry is immediately seen in Γ? [5]

(c) Write down the invariant line element in your new coordinates x^α, introducing the metric tensor and defining it to be

$$g_{\mu\nu} = \eta_{\alpha\beta}\frac{\partial\xi^\alpha \partial\xi^\beta}{\partial x^\mu \partial x^\nu}.$$

Given the definition of the affine connection Γ from above, show that

$$\frac{\partial g_{\mu\nu}}{\partial x^\lambda} = g_{\rho\nu}\Gamma^\rho_{\lambda\mu} + g_{\mu\rho}\Gamma^\rho_{\lambda\nu}.$$

[7]

(d) Use the symmetry that you identified in the affine connection to show that the affine connection can be determined entirely using the metric tensor in your new chosen coordinates x, as

$$\Gamma^\sigma_{\mu\lambda} = \frac{g^{\nu\sigma}}{2}\left(\frac{\partial g_{\mu\nu}}{\partial x^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial x^\mu} - \frac{\partial g_{\lambda\mu}}{\partial x^\nu}\right).$$

[3]

(e) Define U^λ as the vector $\mathrm{d}x^\lambda/\mathrm{d}\tau$. Use the geodesic equation you have derived above to define the *covariant derivative.* What is the significance of the covariant derivative in GR? Argue why the covariant derivative must be a mixed, rank-2 tensor. [6]

Q1: 25
Section:25

2. (a) An isolated, spherically symmetric, non-rotating mass is floating in space. Using the Lagrange procedure or otherwise, write down the geodetic equations in (t, r, θ, ϕ) coordinates (also designated as 0, 1, 2, and 3). Explain how the geodesic equation for θ is easily satisfied. How does the four-momentum p_α relate to the geodesic curves? [6]

(b) When is it appropriate to choose proper time as the affine parameter? By choosing an appropriate affine parameter for both massless and massive particles, use the geodetic equations for t and ϕ, and the 0 and 3 components of p_α to explain the physical meaning of the constants (call them k for the former and h for the latter). [4]

(c) For non-null geodesics, the definition of the line element may be used as one of the geodesic equations,

$$g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = c^2,$$

where $\dot{x^\nu}$ represents $\mathrm{d}x^\nu/\mathrm{d}\tau$. In this case, show that the energy equation is given by

$$\dot{r}^2 + \frac{h^2}{r^2}\left(1 - \frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2(k^2 - 1).$$

[3]

(d) Using this and the geodetic equation for ϕ (the definition of h from above, show that the orbital shape follows from:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}\phi^2} + u = \frac{GM}{h^2} + \frac{3GM}{c^2}u^2$$

where $u = 1/r$. The form of this equation is the same as the Newtonian equation, with an additional term. Which of the terms corresponds to the relativistic correction, and under what circumstances is it significant? [5]

(e) The Newtonian Lagrangian equation of motion of a particle in a central potential is

$$\frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 + V_{\text{eff}}(r) = E,$$

V_{eff} being the effective potential and E the total energy per unit mass. Using the energy equation you derived earlier, write down the equation for the effective potential in GR for such a system from the time frame of somebody moving with the particle in orbit. [4]

For circular orbits, show that the extrema occur at

$$r = \frac{h}{2GM}\left(h \pm \sqrt{h^2 - 12\frac{G^2M^2}{c^2}}\right).$$

[3]

Q2: 25
Section:50

3. (a) In the early 2000s, astronomers discovered a pulsar in a binary system, with an orbital period $T = 0.10225156248$ days. Give one reason why a GR expert should take notice of this result. [1]

(b) Exactly 1000 days later, the orbital period was re-measured to be shorter by 10^{-9} days. Within those 1000 days, it was established that the binary companion was also a pulsar. Pulsar masses mostly fall under a tight distribution, centred around $1.4 M_{\odot}$. Assume the eccentricity of the system to be negligible. What is the approximate radius of the circular orbit? [3]

(c) The quadrupole formula for gravitational radiation at large r is

$$\bar{h}^{ij} = \frac{2G}{c^6 r} \frac{\mathrm{d}^2 I^{ij}}{\mathrm{d}t'^2}.$$

Define $\bar{h}$ and t'. [3]

(d) The quadrupole moment tensor for this system is given by $I^{ij} = Mc^2 x^i x^j$, where M is the total mass of the system (*i.e.*, $2M_{\text{pulsar}}$). Remembering that pulsars are very small in size, what does x^i and x^j signify in that formula? [3]

(e) Write the components of $\bar{h}$ as a function of t'. What is the frequency of the gravitational wave emitted? State your answer in Hz to 3 significant figures. [9]

(f) There is actually a tiny eccentricity in this system, which allowed us to fully describe the system and all of its orbital parameters, including the masses of the two pulsars. Why does the measurement of the derivative of the orbital period mentioned above test general relativity? [6]

Q3: 25
Section:75

4. The Einstein Field Equation is normally written

$$R_{\mu\nu} - \frac{g_{\mu\nu}R}{2} = -\frac{8\pi G}{c^4}T_{\mu\nu} \tag{1}$$

In 1917, Einstein proposed that for applications to cosmology, this equation should be amended to

$$R_{\mu\nu} - \frac{g_{\mu\nu}R}{2} = -\frac{8\pi G}{c^4}T_{\mu\nu} + \Lambda g_{\mu\nu} \tag{2}$$

where Λ is a constant: the cosmological constant. The original reasons for adding this term don't concern us here, but the equation turns out to be a good description of the real Universe.

(a) State the Bianchi Identities is the form of covariant divergence and prove that equation (2) obeys them. Why is this physically significant? [3]

(b) Recall that the stress tensor $T_{\mu\nu}$ of a perfect fluid of rest frame energy density ρ and rest frame pressure P is given by

$$T_{\mu\nu} = Pg_{\mu\nu} + (P + \rho/c^2)U_\mu U_\nu$$

What are the U_ρ quantities? If our cosmological metric is given by

$$-c^2\mathrm{d}\tau^2 = -c^2\mathrm{d}t^2 + R^2(t)\left(\frac{\mathrm{d}r^2}{1 - r^2/a^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right)$$

give precise expressions for *both* U^λ and U_λ, with $\lambda = t$ and $\lambda = r$. [3]

(c) Show that equation (2) takes the form of equation (1) provided that ρ and P are changed to ρ' and P' with

$$\rho' = \rho + \rho_V, \quad P' = P - \rho_V c^2,$$

where ρ_V is a constant which you should explicitly evaluate. What is the physical interpretation of ρ_V? [4]

(d) Prove that the identity

$$\frac{\ddot{R}}{R} = H^2 + \dot{H}$$

holds for any FRW universe, where H is the Hubble parameter, $H = \dot{R}/R$. [3]

(e) For a flat universe with ordinary (nonrelativistic) matter and cosmological constant (our universe), show that the Hubble parameter satisfies its own differential equation:

$$2\dot{H} + 3H^2 = \Lambda c^2$$

[4]

(f) Show that if $H \propto 1/t$ at early times,

$$H = A \coth(3At/2), \quad A^2 = \Lambda c^2/3.$$

Comment on the late time behaviour of the Hubble parameter. [4]

[*Hint: note that* $p \int \mathrm{d}x/(p^2 - x^2) = \coth^{-1}(px)$.]

(g) Astronomers are interested in the possibility that ρ_V may not be *exactly* constant for all times. If at time t_0, $R = R_0$, and $\rho_V \propto R^\epsilon$, show that, *in a flat universe like ours,* for any finite $\epsilon > 0$, $R(t)$ becomes infinite at some finite time. (You may neglect the effect of ordinary matter after t_0.) [4]

Q4: 25
Section:100

Total Marks in Paper 100