GR Exam 2018

1. $x^\alpha = (ct, x, y, z)$, $u^\alpha = dx^\alpha/d\tau$

(a) Strong equivalence principle: one cannot distinguish by the outcome of <u>any</u> physical experiment between a local freely falling ref frame and inertial ref frame in Mink space-time

(b) Computing $dV^\alpha/d\lambda$ involves operation $V^\alpha(\lambda + d\lambda) - V^\alpha(\lambda)$. In flat space, where vector is characterized by length and direction, this operation is well defined (indep. of λ). In curved space, it has to be generalised, to connect vector spaces at $\lambda + d\lambda$ and λ. Similarly, one can see that $\frac{dV^\alpha}{dx^\beta}$ is not a tensor in curved space (assuming V^α is: $V'^\alpha = \frac{\partial x'^\alpha}{\partial x^\gamma} V^\gamma$, then $dV'^\alpha/dx'^\beta = \frac{\partial x'^\alpha}{\partial x^\gamma} \frac{dV^\gamma}{dx^\delta} \frac{\partial x^\delta}{\partial x'^\beta}$ + sec. deriv. term)

$$\frac{DV^\alpha}{d\lambda} = \frac{dV^\alpha}{d\lambda} + \underbrace{\Gamma^\alpha_{\beta\gamma}}_{\text{connection coeffi}} \frac{dx^\beta}{d\lambda} V^\gamma$$

For acceleration: $V^\alpha = u^\alpha = dx^\alpha / d\tau$

$$\frac{DU^\alpha}{d\tau} = \frac{du^\alpha}{d\tau} + \Gamma^\alpha_{\beta\gamma} u^\beta u^\gamma$$

or $a^\alpha = \ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma$

(c) For the massive body at rest,

$$x^\mu = (c\tau, 0, 0, 0)$$

$$\left.\begin{aligned} u^\mu &= (c, 0, 0, 0) \\ u_\mu &= (-c, 0, 0, 0) \end{aligned}\right\} \quad u_\mu u^\mu = -c^2$$

Or $u^\mu u_\mu = g_{\mu\nu} u^\mu u^\nu = g_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} =$

$= \frac{ds^2}{d\tau^2} = \frac{-c^2 d\tau^2}{d\tau^2} = -c^2$ (invar. - same in all frames).

Taking covar. deriv. of $u^\mu u_\mu = -c^2$

$$\frac{D}{d\tau}(u^\mu u_\mu) = 2a^\mu u_\mu = 0 \Rightarrow a^\mu u_\mu = 0$$

In ship's own frame, $a^\mu = (0, g, 0, 0)$,
$a^\mu a_\mu = g^2$ (this is invar. = same in any frame).

(d) $ds^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2 dt^2 +$

$$+ \left(1 - \frac{2GM}{c^2 r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2$$

$$a^0 = \frac{du^0}{d\tau} + \Gamma^0_{\beta\gamma} u^\beta u^\gamma =$$

$$= \frac{du^0}{d\tau} + \Gamma^0_{r0} u^r u^0 + \Gamma^0_{0r} u^0 u^r =$$

$$= \frac{du^0}{d\tau} + 2\Gamma^0_{r0} u^r u^0 =$$

$$= \frac{du^0}{d\tau} + 2 \frac{GM}{c^2 r^2} \left(1 - \frac{2GM}{c^2 r}\right)^{-1} u^r u^0 .$$

$$a^r = \frac{du^r}{d\tau} + \Gamma^r_{00} (u^0)^2 + \Gamma^r_{rr} (u^r)^2 =$$

$$= \frac{du^r}{d\tau} + \frac{GM}{c^2 r^2} \left(1 - \frac{2GM}{c^2 r}\right) (u^0)^2 -$$

$$- \frac{GM}{c^2 r^2} \left(1 - \frac{2GM}{c^2 r}\right)^{-1} (u^r)^2 .$$

We also have $u^\alpha u_\alpha = -c^2$:

$$g_{00}(u^0)^2 + g_{rr}(u^r)^2 =$$

$$= -\left(1-\frac{2GM}{c^2 r}\right)(u^0)^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1}(u^r)^2 =$$

$$= -c^2$$

Inserting this into expr. for a^r, we get

$$a^r = \frac{du^r}{d\tau} + \frac{GM}{r^2} .$$

(e) We have 4 non-trivial var: u^0, u^r and a^0, a^r and 3 eqs relating them: $u_\mu u^\mu = -c^2$, $u_\mu a^\mu = 0$, $a^\mu a_\mu = g^2$ => can find rel. between e.g. a^r and u^r. Explicitly:

$$\begin{cases} g_{00}(a^0)^2 + g_{rr}(a^r)^2 = g^2, \\ g_{00}\, u^0 a^0 + g_{rr}\, u^r a^r = 0, \\ g_{00}(u^0)^2 + g_{rr}(u^r)^2 = -c^2. \end{cases}$$

Eliminating u^0, a^0 from these eqs, we find

$$(a^r)^2 = g^2\left(g^{rr} + \frac{(u^r)^2}{c^2}\right),$$ with

$g^{rr} = 1 - 2GM/c^2 r$ for Schwarzschild.

2. $ds^2 = -c^2 dt^2 + dr^2 + f^2(r)\, d\varphi^2 + dz^2$ (*)

(a) To compute $\Gamma^\alpha_{\beta\gamma}$, write

$$\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu = -\dot{x}^{0\,2} + \dot{r}^2 + f^2(r)\,\dot{\varphi}^2 + \dot{z}^2$$

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu}\right) = \frac{\partial \mathcal{L}}{\partial x^\mu} \qquad \text{E-L eqs.}$$

can be compared with $\ddot{x}^\alpha + \Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma = 0$ to find $\Gamma^\alpha_{\beta\gamma}$.

1) $\mu = 0$: $\quad -2 \dfrac{d}{d\lambda}(\dot{x}^0) = 0$

$\Rightarrow \Gamma^0_{\beta\gamma} = 0 \quad \forall \beta, \gamma.$

2) $\dfrac{d}{d\lambda}(2\dot{r}) = 2ff'\dot{\varphi}^2$

$\Rightarrow \ddot{r} - ff'\dot{\varphi}\dot{\varphi} = 0$

$\Rightarrow \Gamma^r_{\varphi\varphi} = -ff'$, all other $\Gamma^r_{\beta\gamma} = 0$.

3) $\dfrac{d}{d\lambda}\left[(2\dot{\varphi}) f^2\right] = 0$

$\Rightarrow f^2 \ddot{\varphi} + 2ff'\dot{\varphi}\dot{r} = 0$

$$\ddot{\varphi} + 2\frac{f'}{f}\dot{\varphi}\dot{r} = 0$$

$\Rightarrow \Gamma^{\varphi}_{\varphi r} = \Gamma^{\varphi}_{r\varphi} = f'/f$, all other $\Gamma^{\varphi}_{\beta\gamma} = 0$.

4) $\frac{d}{d\lambda}(2\dot{z}) = 0 \Rightarrow \Gamma^{z}_{\beta\gamma} = 0 \ \forall \beta, \gamma$.

To compute Ricci, use the expression given (for a diag. metric):

$$R_{\alpha\beta} = \frac{1}{2}\partial^2_{\alpha\beta}\ln|g| - \partial_\gamma \Gamma^{\gamma}_{\alpha\beta} + \Gamma^{\delta}_{\alpha\gamma}\Gamma^{\gamma}_{\beta\delta} - \frac{1}{2}\Gamma^{\gamma}_{\alpha\beta}\partial_\gamma \ln|g|,$$

where $g = \det g_{\mu\nu} = -f^2(r)$.

$$R_{rr} = \frac{ff'' - f'^2}{f^2} + \Gamma^{\varphi}_{r\varphi}\Gamma^{\varphi}_{r\varphi} = f''/f$$

$$R_{\varphi\varphi} = -\partial_r \Gamma^{r}_{\varphi\varphi} + 2\Gamma^{r}_{\varphi\varphi}\Gamma^{\varphi}_{\varphi r} - \Gamma^{r}_{\varphi\varphi}\frac{f'}{f}$$

$$= ff''.$$

Ricci scalar:

$$R = g^{rr} R_{rr} + g^{\varphi\varphi} R_{\varphi\varphi} = 2f''/f .$$

Es eqs

$$\underbrace{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R}_{} = -\frac{8\pi G}{c^4} T^{\mu\nu}$$

$$\begin{pmatrix} f''/f & & & \\ & 0 & & \\ & & 0 & \\ & & & -f''/f \end{pmatrix} = -\frac{8\pi G}{c^4} T^{\mu\nu}$$

$$\Rightarrow T^{\mu\nu} = \frac{c^4}{8\pi G} \frac{f''}{f} \operatorname{diag}(-1, 0, 0, 1).$$

(b) $ds^2 = -c^2 dt^2 + dr^2 + (1-\lambda)^2 r^2 d\varphi^2 + dz^2$

for $r > R$, here $\lambda = \text{const}$ is NOT an affine param.; and (*) for $r < R$.

With $T^{\mu\nu} = (\rho + P/c^2) u^\mu u^\nu + g^{\mu\nu} P$,

$$T^{00} = \rho c^2 = -\frac{c^4}{8\pi G} \frac{f''}{f}$$

$f(r) = (1-\lambda) r$ for $r > R$ $\Rightarrow f'' = 0$

$\Rightarrow \rho = 0$, $P = 0$ outside.

To compute mass:

$$m = \int_0^R dr \int_0^{2\pi} f \, d\varphi \int_0^z dz \, \rho(r), \quad \text{with}$$

$$\rho(r) = -\frac{c^2}{8\pi G} f''/f$$

$$m = -\frac{c^2}{4G} z f'(r) \Big|_0^R = \frac{\lambda c^2}{4G} z$$

$\Rightarrow$ mass per unit length z is $\frac{\lambda c^2}{4G}$.

Here we used $f'(r) \to 1$ for $r \to 0$ *
(the assumption given) and continuity of $f'(r)$ at $r = R$ (no surface mass density present).

(c) The metric for $r > R$ is flat

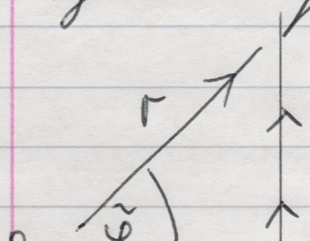
$\Rightarrow$ redefine φ coordinate by

* The condition in the problem, $f'(0) = 0$, I believe is a typo. In Mink, we would have $f = r$, $f'(0) = 1$.

$$(1-\lambda)\, d\varphi = d\tilde{\varphi} \Rightarrow$$

$\tilde{\varphi} = (1-\lambda)\,\varphi$ This is the same as Mink locally, but notice that $\varphi \in [0, 2\pi]$ and $\tilde{\varphi} \in [0, (1-\lambda)2\pi]$ $\Rightarrow$ conical singularity. This affects light propagation. Objects of this type are known as cosmic strings (no evidence of their existence now).

In the metric $ds^2 = -c^2 dt^2 + dr^2 + r^2 d\tilde{\varphi}^2 + dz^2$, light rays are not deflected and $\tilde{\varphi}$



changes from $-\pi/2$ to $\pi/2$, i.e. by $\Delta\tilde{\varphi} = \pi$.

$$\Rightarrow \Delta\varphi = \frac{\pi}{1-\lambda} \simeq \pi(1+\lambda),$$

so there is a deflection by $\delta\varphi = \pi\lambda$ which in principle can be detected (Vilenkin, 1981).

3.

(a) Distances can be found by

- radar signals for nearby objects (in Solar system)
- parallax for nearby stars
- luminosity of stars in a given spectral class (flux $\sim 1/r^2$ can be measured $\Rightarrow r$ using calibration with a star of the same class and given distance).
- Cepheid variable stars; supernovae.

(b) $ds^2 = -c^2 dt^2 + R^2(t)(dx^2 + dy^2 + dz^2)$

Cosmological red-shift - standard derivation (see notes).

$$\Rightarrow \quad z \equiv \frac{\lambda_e - \lambda_{obs}}{\lambda_{obs}} = \frac{R(t_0)}{R(t_e)} - 1$$

Note: $R(t) = R(t_0)\left[1 + (t - t_0)H_0 + \dots\right]$

where $H_0 = \frac{\dot{R}}{R}(t = t_0)$

Compare galaxies at x and $x + dx$:

$$dz = \frac{R(t_0)}{R(t_e)} - \frac{R(t_0)}{R(t_e + dt_e)} =$$

$$= \frac{R(t_0)}{R(t_e)} \left(1 - \frac{R(t_e)}{R(t_e + dt_e)} \right) \simeq$$

$$\simeq \frac{R(t_0)}{R(t_e)} H_0 \, dt_e = H_0 \frac{R(t_0)}{c} dx$$

=> can find H_0 by considering dz/dx.

(c) $$R(t) = R(t_0) \Big[1 + (t - t_0) H_0 + \underbrace{+ \tfrac{1}{2} (t - t_0)^2 \ddot{R} / R_0}_{} + \ldots \Big]$$

sometimes written as

$$- \tfrac{1}{2} q_0 H_0^2 (t - t_0)^2,$$ where

$q(t) = - R\ddot{R} / \dot{R}^2$ is „deceleration parameter".

Can use this e.g. in d_L (see luminosity-distance notes)

$$d_L = \frac{c}{H_0}\left[z + \frac{1}{2}(1-q_0)z^2 + \dots \right]$$

Consider supernovae as standard candles, plot $d_L(z)$ and compare with $d_L(z)$ computed from FRW eq with assumed $\Omega_i \Rightarrow$ get Ω_i by fitting to data.

- Es eqs $\Rightarrow$ 2 indep. eqs

(c) Flat 3d, matter-dom. Universe

$\Rightarrow R(t) \sim t^{2/3}$. Or $R(t) = (t/t_0)^q$

$t_0 = q/H_0$.

Sending from $(t_0, 0)$ to (t_1, r_1):

$$r_1 = c\int_{t_e}^{t_0} \frac{dt}{R(t)} = \frac{c\,t_0}{1-q}\left[1 - \left(\frac{1}{1+z}\right)^{1-q}\right].$$

$$c \int_{t_0}^{t_1} \frac{dt}{R(t)} = r_1$$

$$\Rightarrow t_1^{1-q} - t_0^{1-q} = (1-q)\, r_1 / c \, .$$

Also, $t_2^{1-q} - t_1^{1-q} = (1-q)\, r_1 / c$

$$\Rightarrow t_2 = t_0 \left[1 + 2 t_0 \left(1 - \left(\frac{1}{1+z} \right)^{1-q} \right) \right]^{\frac{1}{1-q}} =$$

$$= \frac{2}{3H_0} \left[1 + \frac{4}{3H_0} \left(1 - \left(\frac{1}{1+z} \right)^{1/3} \right) \right]^3 .$$

The n-th exchange:

$$t_{2n}^{1-q} - t_0^{1-q} = \underbrace{2n(1-q)\, r_1 / c}_{\text{finite}}$$

$\Rightarrow$ no limit on n.

(e) Observations seem to imply $\Lambda > 0$

$\Rightarrow$ with $t \to \infty$ only Λ is relevant on the rhs of FRW eqs. $\Rightarrow R(t) \sim e^{\sqrt{\Lambda} t}$

$\Rightarrow$ neighbours disappear behind horizon.

4. $T = 5 \cdot 10^9 K$

$$S = \frac{1}{T}\left(U + PV + \sum \mu_i N_i\right)$$

(a) $\mu_i = 0$ when N_i is not a conserved quantity. At relativ. energies, the number of particles is not a conserved quantity since with $\varepsilon \sim 2m_i c^2 \sim kT$, particles can be created / destroyed.

(b) $$f_i \, d^3x \, d^3p = \frac{g_i}{(2\pi\hbar)^3} \frac{d^3x \, d^3p}{e^{\frac{\varepsilon_i - \mu_i}{kT}} \pm 1}$$

See "Thermal history of the Universe" notes or standard TD.

$$n_i = \frac{g_i}{(2\pi\hbar)^3} \int \frac{4\pi p^2 dp}{e^{pc/kT} \pm 1} = \frac{g_i}{2\pi^2}\left(\frac{kT}{\hbar c}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x \pm 1}$$

$$u_i = \frac{g_i}{(2\pi\hbar)^3} \int \frac{4\pi p^2 \, pc \, dp}{e^{pc/kT} \pm 1} = \frac{g_i}{2\pi^2} \frac{(kT)^4}{(\hbar c)^3} \int_0^\infty \frac{x^3 dx}{e^x \pm 1}$$

$\Rightarrow$ can compute any TD quantity

e.g. $P_i = \frac{1}{3} u_i$ (EOS ultrarel. free matter/rad).

(c) $s_i = \frac{4}{3} \frac{u_i}{T} = \frac{4}{3} \frac{I_2}{I_1} n_i \cdot k$,

$$I_2 = \int_0^\infty \frac{x^3 dx}{e^x \pm 1}, \quad I_1 = \int_0^\infty \frac{x^2 dx}{e^x \pm 1}$$

$g_\gamma = 2$ (d.o.f).

$g_e = 2$ ($= 2s+1$ with $s = 1/2$).

$$\Rightarrow \frac{s_e}{s_\gamma} = 7/8.$$

(d) The reaction $e^+ + e^- \rightleftarrows 2\gamma$

freezes out at $T \sim 10^9 K$, photons are not energetic enough:

$\Rightarrow N_\gamma$ increases. Before that:

$$s_\gamma + s_{e^+} + s_{e^-} = \left(1 + 2 \cdot \frac{7}{8}\right) s_\gamma = \frac{11}{4} s_\gamma$$

S is conserved (no inter.), $s \sim T^3$

$\Rightarrow$ photon T increases by $\left(\frac{11}{4}\right)^{1/3}$.

(e) $T_\gamma = 2.73\ K$ now.

$$T_\nu = 2.73 \left(4/11\right)^{1/3} \sim 1.95\ K$$

$$\rho_c = \frac{3}{8\pi G} H_0^2 \quad (\text{with } k=0 \text{ in FRW})$$

For 3 neutrino species, $g_\nu = 2$,

$$n_{3\nu} = 2.404 \frac{3}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 .$$

With $m_{3\nu} n_{3\nu} = \rho_c$

$$m_{3\nu} = \frac{H_0^2}{\frac{2.404}{\pi^2} 8\pi G \left(\frac{kT}{\hbar c}\right)^3} \sim 12\ eV.$$