A13484W1

SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B5: GENERAL RELATIVITY & COSMOLOGY

TRINITY TERM 2018

Saturday, 16 June, 9.30 am – 11.30 am

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

Expressed in terms of the Christoffel symbols $\Gamma^{\alpha}_{\beta\gamma}$, the Riemann tensor has components

$$R^{\gamma}{}_{\alpha\beta\delta} = \frac{\partial}{\partial x^{\delta}}\Gamma^{\gamma}_{\alpha\beta} - \frac{\partial}{\partial x^{\beta}}\Gamma^{\gamma}_{\alpha\delta} + \Gamma^{\epsilon}_{\alpha\beta}\Gamma^{\gamma}_{\delta\epsilon} - \Gamma^{\epsilon}_{\alpha\delta}\Gamma^{\gamma}_{\beta\epsilon}.$$

The Ricci tensor is defined as $R_{\alpha\beta} \equiv R^{\gamma}{}_{\alpha\beta\gamma}$. When expressed in a basis in which the metric tensor $g_{\alpha\beta}$ is diagonal, it can be written as

$$R_{\alpha\beta} = \frac{1}{2}\frac{\partial^2}{\partial x^{\alpha}\partial x^{\beta}}\ln|g| - \frac{\partial\Gamma^{\gamma}_{\alpha\beta}}{\partial x^{\gamma}} + \Gamma^{\delta}_{\alpha\gamma}\Gamma^{\gamma}_{\beta\delta} - \frac{1}{2}\Gamma^{\gamma}_{\alpha\beta}\frac{\partial}{\partial x^{\gamma}}\ln|g|,$$

where $|g|$ is the modulus of the determinant of $g_{\alpha\beta}$.

The Einstein equation is

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R + \Lambda g_{\alpha\beta} = -\frac{8\pi G}{c^4}T_{\alpha\beta},$$

where $R \equiv g^{\alpha\beta}R_{\alpha\beta}$ is the Ricci scalar, Λ is the cosmological constant and $T_{\alpha\beta}$ is the energy–momentum tensor.

For the line element

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + R^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right]$$

the Einstein equation can be written as the pair of equations

$$\begin{aligned}
\ddot{R} &= -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)R + \frac{1}{3}\Lambda c^2 R,\\
\dot{R}^2 &= \frac{8\pi G}{3}\rho R^2 + \frac{1}{3}\Lambda c^2 R^2 - c^2 k,
\end{aligned}$$

where ρ and p are the rest-frame density and pressure, respectively.

1. Albert has constructed a spaceship. Let $x^\alpha = (ct, x, y, z)$ be the coordinates of his spaceship with respect to an inertial frame. Taking the derivative with respect to proper time τ, let $u^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$ be its four-velocity.

(a) State the strong equivalence principle. [2]

(b) Explain why the rate of change of a vector $V^\alpha(x^\beta)$ along a worldline $x^\beta(\lambda)$ is not simply $\mathrm{d}V^\alpha/\mathrm{d}\lambda$: how is it obtained? Hence write down an expression for the components of the spaceship's four-acceleration a^α in terms of the components u^α and quantities that can be obtained by examining the local behaviour of the metric tensor. [3]

(c) By considering the worldline of a massive body at rest, or otherwise, explain why the scalar product $u^\alpha u_\alpha = -c^2$. Use this to show that u^α and a^α are perpendicular and that, for Albert in his spaceship, $a^\alpha a_\alpha = g^2$, where g is the acceleration he measures in his frame. [6]

(d) Albert uses his spaceship to travel to the vicinity of a black hole, around which the line element in coordinates (ct, r, θ, ϕ) is given by

$$-c^2\,\mathrm{d}\tau^2 = \mathrm{d}s^2 = -\left(1 - \frac{2GM}{c^2 r}\right) c^2\,\mathrm{d}t^2 + \left(1 - \frac{2GM}{c^2 r}\right)^{-1} \mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2 \sin^2\theta\,\mathrm{d}\phi^2,$$

where M is the mass of the black hole. Starting from rest a large distance away, he lets his spaceship fall in towards the black hole and then escapes by accelerating radially away. Write down the formal equations that relate the ct- and r-components of Albert's four-acceleration a^α to the components of his four-velocity u^α. Show that the expression for the radial component of his acceleration can be written as

$$a^r = \frac{\mathrm{d}u^r}{\mathrm{d}\tau} + \frac{GM}{r^2}.$$ [6]

[*You may assume that* $\Gamma^r_{00} = \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)$ *and* $\Gamma^0_{r0} = -\Gamma^r_{rr} = \frac{GM}{c^2 r^2}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}$.]

(e) Albert's spacecraft has a maximum acceleration of g. At this maximum acceleration the radial component of his four-acceleration turns out to be

$$a^r = \frac{\mathrm{d}u^r}{\mathrm{d}\tau} + \frac{GM}{r^2} = \pm g \left[\left(1 - \frac{2GM}{c^2 r}\right) + \left(\frac{u^r}{c}\right)^2\right]^{1/2}.$$

Identify and write down the equations that lead to the right-hand side of this expression. [Do not solve these equations.] Albert's colleague, Isaac, suggests that the smallest radius Albert should try to reach is $r = \sqrt{GM/g}$. Can Albert return home if he reaches this radius? [8]

2. Consider a spacetime in which the line element is of the form

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + f^2(r)\,\mathrm{d}\phi^2 + \mathrm{d}z^2, \qquad (\star)$$

where spacetime points are labelled by coordinates (ct, r, ϕ, z) and f is a function that depends only on the coordinate r.

(a) Show that the only non-zero Christoffel symbols are $\Gamma^r_{\phi\phi} = -ff'$ and $\Gamma^\phi_{r\phi} = \Gamma^\phi_{\phi r} = f'/f$, where primes denote derivatives with respect to r. Hence show that the energy–momentum tensor of the distribution of matter that produces this line element is given by

$$T^{\alpha\beta} = \frac{c^4}{8\pi G}\frac{f''}{f}\mathrm{diag}(-1, 0, 0, 1).$$

You may assume that the only non-zero components of the Ricci tensor are R_{rr} and $R_{\phi\phi}$. [13]

(b) The line element around a certain object is of the form

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + \mathrm{d}r^2 + (1-\lambda)^2 r^2\,\mathrm{d}\phi^2 + \mathrm{d}z^2$$

for $r > R$, where λ is a constant. For $r < R$, the line element has the form of equation $(\star)$ for some unknown $f(r)$, with $f'(r) \to 0$ as $r \to 0$. Write down an expression for the object's mass per unit length in terms of this unknown function f. Find an expression for this mass per unit length in terms of λ, justifying any assumptions you make about the behaviour of $f(r)$ at $r = R$. [8]

(c) Suppose that at least one such object exists somewhere in the observable universe, but with a very small radius R. By considering the motion of test particles in the vicinity of such an object (or otherwise), explain why the object is difficult to detect. What astronomical observations could in principle yield evidence for its presence? [4]

3. (a) Starting from how one measures the distance between the earth and the sun, and working up though the cosmic distance ladder, explain how distances to galaxies are estimated. [6]

(b) Assuming a line element of the form

$$ds^2 = -c^2\,dt^2 + R^2(t)\,[dx^2 + dy^2 + dz^2],$$

explain why spectral lines emitted from distant galaxies are observed to be red shifted with respect to their local, rest-frame values. Outline how such measurements allow us to estimate the present value of $\dot{R}/R$. [5]

(c) Describe one observational method that can be used to constrain the second derivative $\ddot{R}$. Why does knowledge of $\dot{R}/R$ and $\ddot{R}/R$ suffice to constrain the future fate of the universe? [5]

(d) An advanced, extremely patient civilisation in our own Galaxy discovers a similar civilisation in another galaxy at redshift $z = 1$. Our Galactic neighbours send a message to their distant peers. Assuming that the universe is flat and matter dominated (i.e., an Einstein–de Sitter universe), estimate how long they have to wait for a reply. If each civilisation waits for a reply to its last message before sending a new one, is there any fundamental limit to the number of messages that can be exchanged? [6]

(e) In reality the universe appears to be flat, yet is not matter dominated. How is this possible? Does this impose a limit on the number of messages that can be exchanged? Justify your answer. [3]

4. Consider a comoving region of the universe a few seconds after the big bang, when the temperature was $T = 5 \times 10^9\,\mathrm{K}$. According to standard thermodynamics, the entropy of the region is given by

$$S = \frac{1}{T}(U + PV + \sum_i \mu_i N_i),$$

where U, P and V are its internal energy, pressure and volume, respectively, N_i is the total number of particles that belong to the i^{th} particle species and μ_i is their chemical potential.

(a) Explain why $\mu_i = 0$ for the photons within this region. What other particle species move relativistically within the region? Why may we assume that $\mu_i = 0$ for these species? [3]

(b) The number of particles of species i within phase-space volume $\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}$ is

$$f_i\,\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p} = \frac{g_i}{(2\pi\hbar)^3}\frac{\mathrm{d}^3\mathbf{x}\,\mathrm{d}^3\mathbf{p}}{\mathrm{e}^{(\epsilon_i(\mathbf{p})-\mu_i)/k_{\mathrm{B}}T} \pm 1},$$

where g_i is the degeneracy of the particle species, $\epsilon_i(\mathbf{p})$ is the energy of a particle of species i that has momentum $\mathbf{p}$ and is approximately equal to $|\mathbf{p}|c$ for relativistic particles, and the plus sign in the denominator holds in the case that the particles are fermions, the minus sign when they are bosons. Obtain expressions for N_i as a function of $(kT/\hbar c)$ in each case. What are the corresponding contributions, U_i and P_i, to the region's internal energy and pressure respectively? [3]

(c) How does the entropy of each species depend upon the temperature? Show further that the ratio of the entropy of the electrons in the region to that of the photons is $\frac{7}{8}$. [5]

$$\left[\textit{You may assume that } \int_0^\infty \frac{x^3\,\mathrm{d}x}{\mathrm{e}^x - 1} = \frac{\pi^4}{15}, \quad \int_0^\infty \frac{x^2\,\mathrm{d}x}{\mathrm{e}^x - 1} = 2.404\right.$$
$$\left.\textit{and } \quad \frac{1}{\mathrm{e}^x + 1} = \frac{1}{\mathrm{e}^x - 1} - \frac{2}{\mathrm{e}^{2x} - 1}.\right]$$

(d) Explain why, as the region expands, there comes a point at which the photon temperature rises. Calculate the factor by which it increases. [5]

(e) Estimate the present-day temperature and number density of the cosmic neutrino background. Using the fact that the universe is close to the critical density, obtain an upper bound on the sum of the masses of all three neutrino species. Express your answer in eV/c^2. Assume that the Hubble constant $H_0 = 72\,\mathrm{km\,s^{-1}Mpc^{-1}}$. [9]