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SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B3: V. GENERAL RELATIVITY AND COSMOLOGY

TRINITY TERM 2013

Saturday, 15 June, 9.30 am – 11.30 am

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

1. In a universe with a null cosmological constant, consider the space-time metric

$$ds^2 = -c^2\left(1 + 2\frac{\Phi}{c^2}\right)dt^2 + \left(1 - 2\frac{\Psi}{c^2}\right)(dx^2 + dy^2 + dz^2) \ ,$$

where Φ and Ψ are functions of t and $\vec{r} = (x, y, z)$, and assume throughout that $\Phi/c^2 \ll 1$ and $\Psi/c^2 \ll 1$. If $(x^0,\ x^1,\ x^2,\ x^3)$=$(ct,\ x,\ y,\ z)$, show that the connection coefficients for this space-time are

$$\begin{aligned}
\Gamma^0{}_{00} &= \frac{\partial_0\Phi}{c^2} \ , \qquad \Gamma^0{}_{0i} = \frac{\partial_i\Phi}{c^2} \ , \qquad \Gamma^0{}_{ij} = -\frac{\partial_0\Psi}{c^2}\delta_{ij} \ , \\
\Gamma^k{}_{00} &= \frac{\partial_k\Phi}{c^2} \ , \qquad \Gamma^k{}_{0i} = -\frac{\partial_0\Psi}{c^2}\delta^k{}_i \ , \qquad \Gamma^k{}_{ij} = -\frac{1}{c^2}(\partial_j\Psi\delta_{ik} + \partial_i\Psi\delta_{kj} - \partial_k\Psi\delta_{ij}) \ .
\end{aligned}$$

[9]

Write down the energy-momentum tensor for a pressureless, perfect fluid (in its own rest frame) and show that $T^0{}_0 = -\rho c^2$ and $T^i{}_j = 0$. Then calculate the Ricci tensor

$$R_{\nu\beta} \equiv \partial_\mu\Gamma^\mu{}_{\beta\nu} - \partial_\beta\Gamma^\mu{}_{\mu\nu} + \Gamma^\mu{}_{\mu\epsilon}\Gamma^\epsilon{}_{\nu\beta} - \Gamma^\mu{}_{\epsilon\beta}\Gamma^\epsilon{}_{\nu\mu}$$

and Ricci scalar for this space-time. [7]

Show that the Einstein equations with such an energy-momentum tensor are

$$\begin{aligned}
G^0{}_0 &= -\frac{2}{c^2}\nabla^2\Psi = -\frac{8\pi G}{c^2}\rho \ , \\
G^i{}_j &= \frac{1}{c^2}\left[2\partial_0^2\Psi\delta^i{}_j + \nabla^2(\Phi - \Psi)\delta^i{}_j + \partial^i\partial_j(\Psi - \Phi)\right] = 0 \ ,
\end{aligned}$$

where $\nabla^2 \equiv \partial^i\partial_i$ is the spatial Laplacian. [4]

Write down the geodesic equation for a massive particle in this metric. What further approximation do you have to make in order that the spatial part of the geodesic equation becomes equivalent to

$$\frac{d^2\vec{r}}{dt^2} = -\vec{\nabla}\Phi \ \ ?$$

Under what conditions do the Einstein equations you derived above give the Newton-Poisson equation for empty space? [5]

2. If f is a scalar, V^μ a vector and ∇_μ is the covariant derivative, prove that

$$\begin{aligned}(\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)f &= 0\ , \\ (\nabla_\alpha\nabla_\beta - \nabla_\beta\nabla_\alpha)V^\mu &= R^\mu{}_{\nu\alpha\beta}V^\nu\ ,\end{aligned}$$

where the Riemann tensor is

$$R^\mu{}_{\nu\alpha\beta} \equiv \partial_\alpha\Gamma^\mu{}_{\beta\nu} - \partial_\beta\Gamma^\mu{}_{\alpha\nu} + \Gamma^\mu{}_{\alpha\epsilon}\Gamma^\epsilon{}_{\nu\beta} - \Gamma^\mu{}_{\epsilon\beta}\Gamma^\epsilon{}_{\nu\alpha}\ .$$

[7]

Given a vector potential, A_μ, explain briefly why we define the Faraday tensor and Maxwell's equations in any space-time to be

$$\begin{aligned}F_{\mu\nu} &= \nabla_\mu A_\nu - \nabla_\nu A_\mu\ , \\ \nabla^\mu F_{\mu\nu} &= -\mu_0 J_\nu\ ,\end{aligned}$$

where J_ν is the 4-current density. Show that, if $\nabla^\mu A_\mu = 0$, then the vector potential satisfies

$$\nabla^\mu\nabla_\mu A_\nu - R^\alpha{}_\nu A_\alpha = -\mu_0 J_\nu\ .$$

[7]

Now consider the metric

$$ds^2 = a^2(\eta)\left[-c^2 d\eta^2 + dx^2 + dy^2 + dz^2\right]\ ,$$

where $a(\eta) = -1/\eta$ and $-\infty < \eta < 0$. Show that the connection coefficients for this theory, where $(x^0, x^1, x^2, x^3) = (c\eta, x, y, z)$ and $a' \equiv \partial_0 a$, are

$$\begin{aligned}\Gamma^0{}_{00} &= \frac{a'}{a}\ , \qquad \Gamma^0{}_{0i} = 0\ , \qquad \Gamma^0{}_{ij} = \frac{a'}{a}\delta_{ij}\ , \\ \Gamma^k{}_{00} &= 0\ , \qquad \Gamma^k{}_{0i} = \frac{a'}{a}\delta^k{}_i\ , \qquad \Gamma^k{}_{ij} = 0\ .\end{aligned}$$

[4]

Consider a scalar field that satisfies the equation

$$\nabla^\mu\nabla_\mu\phi = 0$$

on this space-time. Show that a solution of this equation can take the form

$$\phi(\eta, \vec{r}) = [A + B\eta]\exp(-\mathrm{i}c|k|\eta + \mathrm{i}\vec{k}\cdot\vec{r})\ ,$$

where A and B are constants and $\vec{k}$ is a constant vector with $|k|^2 = \vec{k}\cdot\vec{k}$. Find the relationship between A and B. [7]

3. Consider a homogenous and isotropic universe with scale factor $a(t)$ satisfying the Friedman-Robertson-Walker (FRW) equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_{\rm M} + \rho_{\rm R}) - \frac{kc^2}{a^2} \ ,$$

where $\rho_{\rm M}$ is the energy density in dust and $\rho_{\rm R}$ is the energy density of radiation. Show how $\rho_{\rm M}$ and $\rho_{\rm R}$ depend on a. Re-express the FRW equation in terms of the fractional energy density in dust today, $\Omega_{\rm M,0}$, and in radiation today, $\Omega_{\rm R,0}$. If we define conformal time, η, through $dt = a d\eta$, show that for $k = 0$ the scale factor is given by

$$a(\eta) = \sqrt{\Omega_{\rm R,0}}(H_0\eta) + \frac{1}{4}\Omega_{\rm M,0}(H_0\eta)^2 \ ,$$

where $a_0 \equiv 1$. [6]

Find an expression for the conformal time at dust-radiation equality, $\eta_{\rm eq}$, in terms of $\Omega_{\rm M,0}$, $\Omega_{\rm R,0}$ and H_0. [3]

From now on consider a universe with no radiation and with $k > 0$. Show that the FRW equation can be solved by

$$\begin{aligned} a &= \frac{A}{k}\sin^2\left(\frac{k^{1/2}c\eta}{2}\right) \ , \\ t &= \frac{A}{2ck^{3/2}}\left[k^{1/2}c\eta - \sin(k^{1/2}c\eta)\right] \ , \end{aligned}$$

and find the value of the constant A. Find an expression for the fractional energy density as a function of time, $\Omega_{\rm M}(\eta)$, and the Hubble constant as a function of time, $H(\eta)$, and plot them as functions of time. [9]

Find an expression for the maximum scale factor, $a_{\rm max}$, in terms of $\Omega_{\rm M,0}$, and for the lifetime of such a universe, $t_{\rm life}$, in terms of $\Omega_{\rm M,0}$ and H_0. Find an expression for the deceleration parameter, q_0, in terms of $\Omega_{\rm M,0}$. Do cosmological observations support this kind of behaviour in our Universe? [7]

4. Consider a homogeneous and isotropic universe with flat geometry, whose only content is a homogeneous scalar field, $\varphi(t)$, with energy density and pressure given by

$$\begin{aligned} \rho &= \frac{1}{2}\dot{\varphi}^2 + V(\varphi) \; , \\ P &= \frac{1}{2}\dot{\varphi}^2 - V(\varphi) \; , \end{aligned}$$

where $V(\varphi) = V_0 \exp(-\lambda \frac{\varphi}{M_{\rm Pl}})$, $\dot{\varphi} \equiv \frac{d\varphi}{dt}$, V_0 and λ are real constants, $M_{\rm Pl} = (8\pi G)^{-1/2}$, and throughout this question we will take $c = 1$. The scalar field obeys the equation of motion

$$\ddot{\varphi} + 3\frac{\dot{a}}{a}\dot{\varphi} - \lambda \frac{V_0}{M_{\rm Pl}} \exp(-\lambda \frac{\varphi}{M_{\rm Pl}}) = 0 \;\; ,$$

and the Friedman-Robertson-Walker equation is

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho \;\; . \qquad (1)$$

Show that if $V_0 = 0$, the scalar field has an equation of state parameter $w \equiv P/\rho = 1$, and find out how ρ depends on a. [3]

Now take $V_0 \neq 0$ and show that $a(t) = t^p$ and $\varphi = BM_{\rm Pl} \ln(t/t_0)$ are solutions to the Friedman-Robertson-Walker and the scalar field evolution equations with $p = 2/\lambda^2$ and, if $\lambda \leq \sqrt{6}$, find expressions for B and t_0 in terms of λ, V_0 and $M_{\rm Pl}$. [6]

Find an expression for the equation of state parameter, w, and the deceleration parameter, q_0, as a function of λ. Is it possible to have $w < -1$? For what values of λ is the particle horizon infinite? For these values of λ, find an expression for the event horizon and the age of the universe in terms of the Hubble constant, H_0, and λ. [8]

Now consider instead a flat universe filled with a perfect fluid with completely arbitrary, constant w. Find a general, exact expression for the luminosity distance $D_{\rm L}(z)$ as a function of redshift. Expand $D_{\rm L}(z)$ to 2$^{\rm nd}$ order in z; using this expression, or otherwise, find an expression for the deceleration parameter, q_0. Show that, when $w < -1$, the scale factor will blow up, i.e. $a \to \infty$, in a finite time. [8]