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SECOND PUBLIC EXAMINATION

Honour School of Physics Part B: 3 and 4 Year Courses

Honour School of Physics and Philosophy Part B

B3: V. GENERAL RELATIVITY AND COSMOLOGY

TRINITY TERM 2012

Wednesday, 13 June, 9.30 am – 11.00 am

Answer **two** *questions.*

Start the answer to each question in a **fresh book.**

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners expect to assign to each part of the question.

Do NOT turn over until told that you may do so.

1. The space-time metric around the Earth in cartesian coordinates is approximately given by

$$\mathrm{d}s^2 = -c^2\left(1-\frac{2GM}{rc^2}\right)\mathrm{d}t^2 + \left(1+\frac{2GM}{rc^2}\right)(\mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2)\ ,$$

where M is the mass of the Earth, $\vec{r}=(x,y,z)$, $r^2=\vec{r}\cdot\vec{r}$ and $GM/(rc^2)\ll 1$. Show that the geodesics in this space time satisfy

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\tau}\left[2c^2\left(1-\frac{2GM}{rc^2}\right)\dot{t}\right] &= 0\ ,\\
\frac{\mathrm{d}}{\mathrm{d}\tau}\left[2\left(1+\frac{2GM}{rc^2}\right)\dot{\vec{r}}\right] + c^2\dot{t}^2\,\vec{\nabla}\left(1-\frac{2GM}{rc^2}\right) - (\dot{\vec{r}}\cdot\dot{\vec{r}})\,\vec{\nabla}\left(1+\frac{2GM}{rc^2}\right) &= 0\ ,
\end{aligned}$$

where τ is proper time, an overdot is the derivative with respect to τ and $\vec{\nabla}$ is the three dimensional gradient in flat space.

Write down all the non-zero connection coefficients for this metric.

A satellite space station orbiting the earth follows a circular orbit $x^\mu_S(\tau)$ given by $x = R\cos\omega\tau$, $y = R\sin\omega\tau$ and $z=0$. What is the orbital period? [11]

If, as assumed, $GM/(rc^2)\ll 1$ we have that part of the Riemann curvature tensor is given by

$$R^i{}_{0j0} \simeq \partial_j\Gamma^i{}_{00} - \partial_0\Gamma^i{}_{0j} = \frac{1}{2}\partial_i\partial_j g_{00}\ .$$

Explain why you expect this from the definition of $R^i{}_{0j0}$ and the magnitude of the connection coefficients. Show that

$$R^i{}_{0j0} = -\frac{GM}{c^2r^3}\left(\delta_{ij} - 3\frac{x^ix^j}{r^2}\right)\ .$$

[8]

An astronaut drifts off, so that she is just a few metres away from the satellite, onto another circular orbit and takes the space-time path $x^\mu_A(\tau)$. The relative space separation, $\Delta^i = x^i_A(\tau) - x^i_S(\tau)$ obeys the equation for geodesic deviation.

$$\frac{1}{c^2}\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\Delta^i + R^i{}_{0j0}\Delta^j = 0\ .$$

Without deriving this equation from scratch, explain its origins and the approximations that have been used to get it into the above form. Explain, without solving the geodesic deviation equation, why you expect the magnitude of Δ^i to grow in time. [6]

2. Consider the metric

$$ds^2 = -c^2\alpha\,dt^2 + \alpha^{-1}dr^2 + r^2d\theta^2 + r^2\sin^2\theta\,d\phi^2\ ,$$

where $\alpha = 1 - r^2/r_S^2$. Show that geodesics in this space time satisfy:

$$\begin{aligned}
\frac{d}{d\lambda}\left(2c^2\alpha\dot{t}\right) &= 0\ ,\\
\frac{d}{d\lambda}\left(2r^2\dot{\theta}\right) - 2r^2\sin\theta\cos\theta\dot{\phi}^2 &= 0\ ,\\
\frac{d}{d\lambda}\left(2r^2\sin^2\theta\dot{\phi}\right) &= 0\ ,\\
\frac{d}{d\lambda}\left(\frac{2\dot{r}}{\alpha}\right) + c^2\alpha'\dot{t}^2 + \frac{\alpha'}{\alpha^2}\dot{r}^2 - 2r\dot{\theta}^2 - 2r\sin^2\theta\dot{\phi}^2 &= 0\ ,
\end{aligned}$$

where λ is an affine parameter and $\alpha' = \frac{d\alpha}{dr}$. Use these geodesic equations to write down all the non-zero connection coefficients for this space time. [10]

Consider a light ray along the radial direction and set $\theta = \pi/2$ and $\phi = 0$. Show that the geodesic equations can be integrated to give

$$\begin{aligned}
\alpha\dot{t} &= 1\ ,\\
\dot{r} &= c\ ,
\end{aligned}$$

where we have set $\dot{t} = 1$ at $r = 0$. Integrate these equations to show that a light ray emitted at $t = 0$ from $r = 0$ follows a path

$$r = r_S\tanh\left(\frac{ct}{r_S}\right)\ .$$

[9]

A lighthouse at $r = r_S/2$ sends a light pulse with period $\Delta\tau_E$ to a stationary observer at $r = 0$. Find the period of observed light pulse at $r = 0$ and calculate the redshift. What happens if r is infinitesimally close to r_S? [6]

3. Consider a metric of the form

$$\mathrm{d}s^2 = -c^2\,\mathrm{d}t^2 + a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2\right] ,$$

where k is a constant and a is the scale factor of the Universe. Write down the Friedmann-Robertson-Walker equation for this metric with all possible terms that might play a role.

Specialise for a completely empty Universe with no cosmological constant. What value must k take in terms of the Hubble constant, H_0? [5]

Describe in detail the geometry of its space-like surfaces, the explicit dependence of a on t, the value of the deceleration parameter, q_0, and the age of the Universe in terms of H_0.

Now consider a Universe which is almost empty with a small amount of energy density in dust, ρ_M. Write down the corresponding Friedmann-Robertson-Walker equation and define Ω_M and Ω_K. Find an expression for the deceleration parameter and the redshift, z, at which curvature begins to dominate in terms of Ω_M and Ω_K. [8]

Solve the equations for radial light-like geodesics in such a space time to show that the angular diameter distance is given by

$$D_A = \frac{1}{1+z}\,\frac{c}{\sqrt{\Omega_k}H_0}\sinh\left[\sqrt{\Omega_k}H_0 D_C/c\right] ,$$

where $D_C = \int_t^{t_0} c\,\mathrm{d}t'/a(t')$.

Find an expression for D_C in terms of an integral over the scale factor. It should be solely dependent on Ω_M and H_0. [8]

Show why the angular size of an object at a fixed distance is smaller in an open Universe than in a flat Universe. How can the angular diameter distance be used to measure the geometry and, more generally, the density parameters of this Universe? [4]

4. Consider a flat, homogenous and isotropic universe with scale factor $a(t)$ satisfying the FRW equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_{\rm M} + \rho_\gamma) \; ,$$

where $\rho_{\rm M}$ is the energy density in dust and ρ_γ is the energy density in photons. Assume that $\rho_{\rm M} = \rho_{\rm B} + \rho_{\rm C}$ where $\rho_{\rm B}$ is the energy density in baryons and $\rho_{\rm C}$ is the energy density in cold dark matter. Derive the evolution of ρ_γ and ρ_M as a function of a. Find an expression for the redshift of equality between ρ_γ and ρ_M in terms of the fractional energy densities Ω_γ, Ω_B and Ω_C. [7]

The entropy per baryon of the universe is defined to be $s = n_\gamma/n_B$, where n_γ is the number density of photons and and n_B is the number density of baryons. Explain why we expect $n_\gamma \propto (k_{\rm B}T/\hbar c)^3$ and $\rho_\gamma c^2 \propto \hbar c(k_{\rm B}T/\hbar c)^4$ where $k_{\rm B}$ is Boltzmann's constant, T is the temperature, $\hbar$ is Planck's constant and c is the speed of light. Assuming that $n_\gamma = 0.24\,(k_{\rm B}T/\hbar c)^3$ and $\rho_\gamma c^2 = 0.66\,\hbar c(k_{\rm B}T/\hbar c)^4$, show that

$$s \simeq 0.33 \left[\frac{m_p^4 c^3}{\rho_c \hbar^3}\right]^{1/4} \frac{\Omega_\gamma^{3/4}}{\Omega_B} \; ,$$

where m_p is the mass of the proton and $\rho_c \simeq 10^{-26}\,{\rm kg\,m^{-3}}$ is the critical mass density of the Universe. Estimate the value of s. [7]

Assume that all the baryons in the Universe are either in the form of free protons or in hydrogen atoms. Define the ionization fraction X to be the ratio of the number density of free protons to the total number of baryons. Show that, in thermal equilibrium, X depends on the temperature of the Universe through

$$\frac{1-X}{X^2} \simeq \frac{3.8}{s}\left(\frac{k_{\rm B}T}{M_e c^2}\right)^{\frac{3}{2}} \exp\left(\frac{B}{k_{\rm B}T}\right) \; ,$$

where B is the binding energy of Hydrogen and M_e is the mass of the electron. Explain, without attempting to solve the equation, why $k_{\rm B}T$ is much smaller than B at recombination, when $X \simeq 0.5$. [7]

Explain why the assumption of thermal equilibrium cannot be used to explain the origin of the abundance of Helium nuclei when the universe had a temperature of about $10^9\,{\rm K}$. [4]