GR Exam 2019

1. In a free-falling lab in a grav. field, locally all physical effects are the same as in an inertial frame in the absence of gravity.

- $\frac{d^2\xi^\alpha}{d\tau^2} = 0$: locally, the metric tensor is Mink $\eta_{\alpha\beta}$ and $-c^2 d\tau^2 = \eta_{\alpha\beta}\, d\xi^\alpha d\xi^\beta$, where τ is the proper time.

- For $\xi^\alpha = \xi^\alpha(x)$, $\frac{d\xi^\alpha}{d\tau} = \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{dx^\mu}{d\tau}$,

$$\frac{d^2\xi^\alpha}{d\tau^2} = \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} = 0$$

$$\Rightarrow \frac{\partial x^\lambda}{\partial \xi^\alpha}\left(\frac{\partial \xi^\alpha}{\partial x^\mu}\frac{d^2x^\mu}{d\tau^2} + \frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\right) = 0$$

$$\frac{d^2x^\lambda}{d\tau^2} + \underbrace{\frac{\partial x^\lambda}{\partial \xi^\alpha}\frac{\partial^2\xi^\alpha}{\partial x^\mu \partial x^\nu}}_{\Gamma^\lambda_{\mu\nu}}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = 0$$

$$\Rightarrow \ddot{x}^\lambda + \Gamma^\lambda_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0.$$

Clearly, $\Gamma^{\lambda}_{\mu\nu} = \Gamma^{\lambda}_{\nu\mu}$.

- $g_{\mu\nu} = \eta_{\alpha\beta} \dfrac{\partial \xi^{\alpha}}{\partial x^{\mu}} \dfrac{\partial \xi^{\beta}}{\partial x^{\nu}}$

$$-c^2 d\tau^2 = \eta_{\alpha\beta}\, d\xi^{\alpha} d\xi^{\beta} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} dx^{\mu} dx^{\nu}$$

$$= g_{\mu\nu}\, dx^{\mu} dx^{\nu}.$$

- $\dfrac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \eta_{\alpha\beta} \dfrac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\lambda}} \dfrac{\partial \xi^{\beta}}{\partial x^{\nu}} +$

$$+ \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}} \frac{\partial^2 \xi^{\beta}}{\partial x^{\nu} \partial x^{\lambda}}$$

Also, $\Gamma^{\lambda}_{\mu\nu} = \dfrac{\partial x^{\lambda}}{\partial \xi^{\alpha}} \dfrac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\nu}}$

$$\Rightarrow \frac{\partial^2 \xi^{\alpha}}{\partial x^{\mu} \partial x^{\lambda}} = \frac{\partial \xi^{\alpha}}{\partial x^{\rho}} \Gamma^{\rho}_{\mu\lambda}$$

$$\frac{\partial^2 \xi^{\beta}}{\partial x^{\nu} \partial x^{\lambda}} = \frac{\partial \xi^{\beta}}{\partial x^{\rho}} \Gamma^{\rho}_{\nu\lambda}$$

$$\frac{\partial g_{\mu\nu}}{\partial x^{\lambda}} = \eta_{\alpha\beta} \frac{\partial \xi^{\alpha}}{\partial x^{\rho}} \Gamma^{\rho}_{\mu\lambda} \frac{\partial \xi^{\beta}}{\partial x^{\nu}} + \eta_{\alpha\beta} \frac{\partial \xi^{\beta}}{\partial x^{\rho}} \Gamma^{\rho}_{\nu\lambda} \frac{\partial \xi^{\alpha}}{\partial x^{\mu}}$$

$$= g_{\rho\nu}\Gamma^{\rho}_{\mu\lambda} + g_{\rho\mu}\Gamma^{\rho}_{\nu\lambda} =$$

$$= g_{\rho\nu}\Gamma^{\rho}_{\lambda\mu} + g_{\rho\mu}\Gamma^{\rho}_{\lambda\nu} .$$

We have:

$$\partial_\lambda g_{\mu\nu} = g_{\rho\nu}\Gamma^{\rho}_{\lambda\mu} + g_{\rho\mu}\Gamma^{\rho}_{\lambda\nu}$$

$$\partial_\nu g_{\lambda\mu} = g_{\rho\mu}\Gamma^{\rho}_{\nu\lambda} + g_{\rho\lambda}\Gamma^{\rho}_{\nu\mu}$$

$$\partial_\mu g_{\nu\lambda} = g_{\rho\lambda}\Gamma^{\rho}_{\mu\nu} + g_{\rho\nu}\Gamma^{\rho}_{\mu\lambda}$$

So, $\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu} =$

$$= g_{\rho\nu}\Gamma^{\rho}_{\lambda\mu} + \cancel{g_{\rho\mu}\Gamma^{\rho}_{\lambda\nu}} + \cancel{g_{\rho\lambda}\Gamma^{\rho}_{\mu\nu}} + g_{\rho\nu}\Gamma^{\rho}_{\mu\lambda}$$

$$- \cancel{g_{\rho\mu}\Gamma^{\rho}_{\nu\lambda}} - \cancel{g_{\rho\lambda}\Gamma^{\rho}_{\nu\mu}} = 2 g_{\rho\nu}\Gamma^{\rho}_{\mu\lambda}$$

$$\Rightarrow \Gamma^{\sigma}_{\mu\lambda} = \frac{1}{2} g^{\sigma\nu}\left(\partial_\lambda g_{\mu\nu} + \partial_\mu g_{\nu\lambda} - \partial_\nu g_{\lambda\mu}\right).$$

- $U^\lambda = dx^\lambda / d\tau \Rightarrow$ the geodesic eq. is

$$\frac{dU^\lambda}{d\tau} + \Gamma^{\lambda}_{\rho\sigma} U^\rho U^\sigma = 0$$

For $U^\lambda = U^\lambda(x(\tau))$, we have

$$\frac{\partial U^\lambda}{\partial x^\nu}\frac{dx^\nu}{d\tau} + \Gamma^\lambda_{\rho\nu} U^\rho U^\nu = 0$$

$$\Rightarrow U^\nu \left(\partial_\nu U^\lambda + \Gamma^\lambda_{\nu\rho} U^\rho \right) = 0$$

or $U^\nu \nabla_\nu U^\lambda = 0$, where

$\nabla_\nu U^\lambda = \partial_\nu U^\lambda + \Gamma^\lambda_{\nu\rho} U^\rho$ is the covariant deriv. of U^λ.

A cov. deriv. generalises the notion of ordinary (partial) derivative ∂_ν to curvilinear coordinates with non-trivial $g_{\mu\nu}(x)$, where $\partial x'^\mu / \partial x^\nu$ is not a constant. Such a generalisation is necessary for the derivative to be a tensor,

$$\nabla'_\nu U'^\lambda = \frac{\partial x^\rho}{\partial x'^\nu} \frac{\partial x'^\lambda}{\partial x^\sigma} \nabla_\rho U^\sigma .$$

By construction, $\nabla_\nu U^\lambda$ is a tensor of rank $(1,1)$: its contraction with a vector (U^ν) gives a vector (0).

2. The metric is the Schwarzschild metric

$$ds^2 = -c^2\left(1 - 2\frac{GM}{c^2 r}\right)dt^2 + \frac{dr^2}{1 - \frac{2GM}{c^2 r}} +$$

$$+ r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

(a) Geodesic eqs can be found by using E-L eqs with $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$:

$$\frac{d}{d\tau}\frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu} \quad : \text{(can use } \tau \text{ or affine parameter } \lambda\text{)}:$$

$$\mu = 0: \quad \frac{d}{d\tau}\left(-2c^2\left(1 - 2\frac{GM}{c^2 r}\right)\dot{t}\right) = 0$$

$$\Rightarrow \left(1 - \frac{2GM}{c^2 r}\right)\dot{t} = K = \text{const}$$

$$\mu = 1: \quad \frac{d}{d\tau}\left(2\dot{r}\left(1 - \frac{2GM}{c^2 r}\right)^{-1}\right) = 2r\dot{\theta}^2 +$$

$$+ 2r\sin^2\theta\,\dot{\varphi}^2 -$$

$$- \dot{t}^2 \frac{2GM}{r^2} - \dot{r}^2\left(1 - \frac{2GM}{c^2 r}\right)^{-2}\frac{2GM}{c^2 r^2}$$

$$\Rightarrow \ddot{r}\left(1 - \frac{2GM}{c^2 r}\right)^{-1} - \dot{r}^2\left(1 - \frac{2GM}{c^2 r}\right)^{-2}\frac{GM}{c^2 r^2} -$$

$$- r\dot{\theta}^2 - r\sin^2\theta\,\dot{\varphi}^2 + \frac{GM}{r^2}\dot{t}^2 = 0$$

$\mu = 2: \quad \frac{d}{d\tau}\left(2r^2\dot{\theta}\right) = r^2 2\sin\theta\cos\theta\,\dot{\varphi}^2$

$\Rightarrow r^2\ddot{\theta} + 2r\dot{r}\dot{\theta} - r^2\sin\theta\cos\theta\,\dot{\varphi}^2 = 0$

$\mu = 3: \quad \frac{d}{d\tau}\left(2r^2\sin^2\theta\,\dot{\varphi}\right) = 0$

$\Rightarrow r^2\sin^2\theta\,\dot{\varphi} = h = \text{const.}$

- Setting $\theta = \pi/2$ satisfies eq for θ.
- $p^\mu = mU^\mu$, $U^\mu = dx^\mu/d\tau$ is the tangent vector to $x^\mu = x^\mu(\tau)$ curve.

(6) For massive particles, one can choose proper time as an affine param.

We have $p^0 = mU^0 \Rightarrow p_0 = mU_0 =$

$= m g_{00} U^0 = mc\, g_{00}\,\dot{t}$

$\Rightarrow$ E.o.m. implies that $\dot{p}_0 = 0$.

$\Rightarrow p_0 = K = \text{const}$ (along geodesics).

At $r \to \infty$, the metric becomes Mink,

$t \to \tau$, $p_0 \to mc(-c^2) = K = -\mathcal{E}_\infty c$, where $\mathcal{E}_\infty$ is the particle's rest energy ($\mathcal{E}_\infty = mc^2$) at $r \to \infty$. (If the particle is not stationary at $r \to \infty$, then $dt/d\tau = \gamma$ and $K = -\mathcal{E} \cdot c$, where $\mathcal{E} = \gamma mc^2$.)

Similarly, $p^3 = mU^3 = m\dot{\varphi}$, $p_3 = g_{33} p^3 =$
$= mr^2\dot{\varphi} \Rightarrow p_3 = mh = \text{const}$
$\Rightarrow$ at $r \to \infty$ (with $\tau \to t$) h is the angular momentum per unit mass.

Note that the specific proportionality factors can always be adjusted, the eoms only tell us that

$$r^2\dot{\varphi} = \text{const} \quad \text{and} \quad \left(1 - \frac{2GM}{c^2 r}\right)\dot{t} = \text{const},$$

with deriv. taken w.r.t. affine parameter.

• Note that $p_\mu u^\mu_{obs} = \text{inv}$. In the inertial frame (e.g. at $r \to \infty$), $p^\mu = \left(\frac{\mathcal{E}}{c}, \bar{p}\right)$, $u^\mu_{obs} = (\gamma c, \gamma \bar{v}) = (c, \bar{0})$; $p_\mu u^\mu_{obs} = -c^2 \frac{\mathcal{E}}{c} c = -\mathcal{E}c^2$
$= K \cdot c$.

(c) $g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} = g_{\mu\nu}\frac{dx^{\mu}}{d\tau}\frac{dx^{\nu}}{d\tau} = -c^2$

$$-c^2\left(1-\frac{2GM}{c^2 r}\right)\dot{t}^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2 +$$

$$+ r^2\dot{\varphi}^2 = -c^2$$

$$-c^2\left(1-\frac{2GM}{c^2 r}\right)^{-1}k^2 + \left(1-\frac{2GM}{c^2 r}\right)^{-1}\dot{r}^2 +$$

$$+ \frac{h^2}{r^2} = -c^2$$

$$\dot{r}^2 - k^2c^2 + \frac{h^2}{r^2}\left(1-\frac{2GM}{c^2 r}\right) = -c^2\left(1-\frac{2GM}{c^2 r}\right)$$

$$\boxed{\dot{r}^2 + \frac{h^2}{r^2}\left(1-\frac{2GM}{c^2 r}\right) - \frac{2GM}{r} = c^2(k^2-1)}$$

(d) $u = 1/r \qquad du = -dr/r^2$

$r^2\dot{\varphi} = h \quad , \quad \dot{r} = \frac{dr}{d\tau} \quad ,$

$$\frac{d\varphi}{d\tau} = h/r^2 \Rightarrow \dot{r} = \frac{h}{r^2}\frac{dr}{d\varphi} = -h\frac{du}{d\varphi}$$

$$\Rightarrow h^2u'^2 + h^2u^2\left(1-\frac{2GM}{c^2}u\right) - 2GMu =$$

$$= c^2(k^2-1)$$

$$h^2 2u'u'' + 2h^2 uu' - 3 \cdot \frac{2GM}{c^2} h^2 u u^{2\prime} - 2GMu' = 0$$

$$\Rightarrow \quad u'' + u - \frac{3GM}{c^2} u^2 = \frac{GM}{h^2}$$

i.e. $$\boxed{u''_{\varphi\varphi} + u = \frac{GM}{h^2} + \frac{3}{2}\frac{2GM}{c^2} u^2}$$

The last term is rel. correction (note c^2), it should be compared with u: it is significant when $\frac{2GM}{c^2} u \sim 1$.

(e) The eq. derived in (c) is

$$\frac{\dot{r}^2}{2} - \frac{GM}{r} + \frac{h^2}{2r^2}\left(1 - \frac{2GM}{c^2 r}\right) = \frac{c^2(k^2-1)}{2}$$

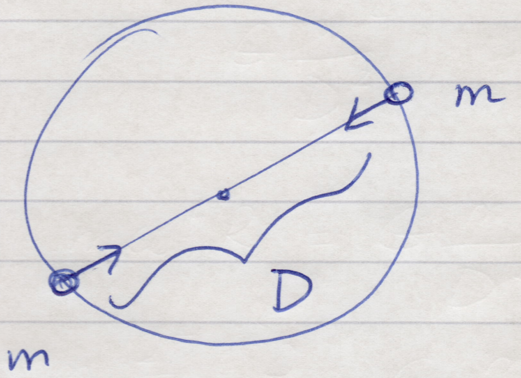
$$\Rightarrow V_{eff} = -\frac{GM}{r} + \frac{h^2}{2r^2} - \frac{GMh^2}{c^2 r^3}$$

Circular orbits ($r = \text{const}$) correspond to $V'_{eff} = 0 \Rightarrow r^2 - \frac{h^2}{GM} r + \frac{3h^2}{c^2} = 0$

$$\Rightarrow r_\pm = \frac{h^2}{2GM}\left(1 \pm \sqrt{1 - \frac{12G^2M^2}{c^2h^2}}\right).$$

3. (a) Orbital period short => objects are moving fast => rel. effects likely to be significant (rel. + grav. = GR).

(b) With school physics:



$$F = G\frac{m^2}{D^2}$$

$$m\omega^2 \frac{D}{2} = G\frac{m^2}{D^2}$$

$$\Rightarrow \boxed{\left(\frac{2\pi}{T}\right)^2 = \frac{2Gm}{D^3}} \qquad D = 2R$$

We have $m \simeq 1.4\, M_\odot$. Also, in the Sun-Earth system $\frac{GM_\odot}{a^3} = \left(\frac{2\pi}{T_\oplus}\right)^2$ $\quad T_\oplus \simeq 365$ days

$a \simeq 150 \cdot 10^6$ km

$$\Rightarrow D^3 = \frac{GmT^2}{2\pi^2} = \frac{1.4\, a^3\, 4\pi^2}{2\pi^2}\frac{T^2}{T_\oplus^2}$$

$$\Rightarrow \left(\frac{2R}{a}\right)^3 = 2.8\left(\frac{T}{T_\oplus}\right)^2 \Rightarrow R = \frac{a}{2}(2.8)^{\frac{1}{3}}\left(\frac{T}{T_\oplus}\right)^{2/3}$$

$$\Rightarrow R \simeq 0.46 \cdot 10^6 \text{ km.}$$

(c) $$\bar{h}^{ij} = \frac{2G}{c^6 r} \frac{d^2 I^{ij}}{dt'^2}$$

Here $t' = t - r/c$ (r - distance from source).

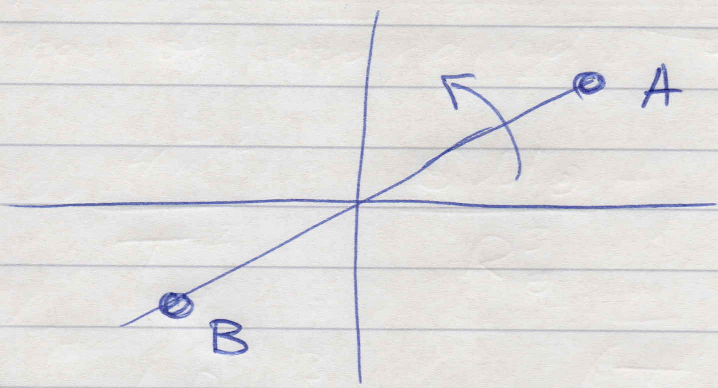
The metric is expanded around flat Mink: $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, all indices are raised/lowered with $\eta_{\mu\nu}$,

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h, \text{ where } h \equiv h^\mu_\mu.$$

(d) $$I^{ij} = Mc^2 x^i x^j$$

Here x^i are spatial coordinates of the components of the system w.r.t. CMF.

(e) In the x'-y' plane



$$x^i_A = (R\cos\omega t', R\sin\omega t', 0)$$

$$x^i_B = (-R\cos\omega t', -R\sin\omega t', 0)$$

Then $I^{ij} = Mc^2R^2 \begin{pmatrix} 2\cos^2\omega t' & 2\sin\omega t'\cos\omega t' & 0 \\ 2\sin\omega t'\cos\omega t' & 2\sin^2\omega t' & 0 \\ 0 & 0 & 0 \end{pmatrix}$

and $\bar{h}^{ij} = \frac{2G}{c^6 r}\ddot{I}^{ij}$ The entries of the matrix can be conveniently written as $2\cos^2\omega t' = 1 + \cos 2\omega t'$,

$2\sin^2\omega t' = 1 - \cos 2\omega t'$ and

$2\sin\omega t'\cos\omega t' = \sin 2\omega t'$, so

$$\bar{h}^{ij} = -\frac{2GMR^2 4\omega^2}{c^4 r}\begin{pmatrix} \cos 2\omega t' & \sin 2\omega t' \\ \sin 2\omega t' & -\cos 2\omega t' \end{pmatrix}$$

The frequency $\Omega = 2\omega = 4\pi/T = 2\pi\nu$

$\Rightarrow \nu \sim 1.42\cdot 10^{-3}\,Hz/2\pi \Rightarrow \nu \sim 2.26\cdot 10^{-4}\,Hz$

(f) The system loses energy by emitting grav. waves $\Rightarrow$ GR allows to compute change of T due to this loss and compare with observations.

4.

(a) $$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu} + \Lambda g_{\mu\nu}$$

Bianchi : $\nabla^\mu \left(R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right) = 0$

Since $\nabla^\mu T_{\mu\nu} = 0$ (covar. conserv. of energy and momentum) and $\nabla^\mu g_{\mu\nu} = 0$ (in GR, connection $\Gamma^\lambda_{\mu\nu}$ is the one satisfying this eq), the LHS and RHS are compatible. (An earlier version, $R_{\mu\nu} = -\frac{8\pi G}{c^4} T_{\mu\nu}$, is incompatible with $\nabla^\mu T_{\mu\nu} = 0$.)

(b) $T_{\mu\nu} = P g_{\mu\nu} + (\rho + P/c^2) U_\mu U_\nu$

U^μ are 4-vel. of the points of the perfect fluid (no friction etc - i.e. $\partial_\sigma U_\mu = 0$ in local frame) filling the space, $u^\mu = dx^\mu/d\tau$, where x^μ are coord. of those points. We have $u^t = \frac{c\,dt}{d\tau}$

and $U^r = dr/d\tau$. Then $U_t = g_{tt} U^t$ and $U_r = g_{rr} U^r$, i.e. $U_t = -c \frac{dt}{d\tau}$ and $U_r = \frac{R(t)}{\sqrt{1-r^2/a^2}} \frac{dr}{d\tau}$ for the metric given.

(c) See e.g. Section 9.4.1. of the lecture notes: $T'_{\mu\nu} = T_{\mu\nu} - \frac{c^4}{8\pi G} \Lambda g_{\mu\nu}$

Comparing by components with $T_{\mu\nu}$ of a perfect fluid $\Rightarrow$ $\rho' = \rho + \rho_v$, $P' = P - \rho_v c^2$, with $\rho_v = \Lambda c^2/8\pi G$.

(d) $H = \dot{R}/R$

$\dot{H} = \ddot{R}/R - \dot{R}^2/R^2 \Rightarrow$

$$\frac{\ddot{R}}{R} = H^2 + \dot{H}.$$

(e) We have $H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G \rho}{3}$,

where ρ now includes contrib. from Λ

Also, $\dfrac{\ddot{R}}{R} = -\dfrac{4\pi G}{3}\left(\rho + \dfrac{3P}{c^2}\right).$

Then $\dot{H} + H^2 = -\dfrac{H^2}{2} + \Lambda c^2/2$

$\Rightarrow \dot{H} + \frac{3}{2}H^2 = \Lambda c^2/2$

(Note $P_{N/R} \approx 0$).

(f) Integrating the eq. in (e):

$$\dot{H} = \frac{3}{2}\left(A^2 - H^2\right)$$

$$\int \frac{dH}{A^2 - H^2} = \frac{1}{A}\coth^{-1} A^{-1}H = \frac{3}{2}t$$

(with $H \sim 1/t$ at early times –

– note $\coth^{-1} z \sim 1/z$ at $z \to 0$)

$\Rightarrow H(t) = A \coth \dfrac{3At}{2}$, where

$A^2 = \Lambda c^2/3.$

At late times, $H \to A$ since $\coth z \underset{z\to\infty}{\to} 1$.

(g) Big Rip scenario.

With $\rho_v = \alpha R^{\varepsilon}$:

$$\dot{R} = \sqrt{\frac{8\pi G \rho_v}{3}}\, R = \sqrt{\frac{8\pi G \alpha}{3}}\, R^{1+\varepsilon/2}$$

(We ignore any other contributions to ρ.)

Integrating $\dot{R} = \beta R^{1+\varepsilon/2}$, $\beta \equiv \sqrt{\frac{8\pi G\alpha}{3}}$,

we have $$\int_{R_0}^{R} \frac{dR}{R^{1+\varepsilon/2}} = \beta \int_{t_0}^{t} dt$$

$$\Rightarrow R(t) = \left(R_0^{-\varepsilon/2} - \frac{\beta}{2}\varepsilon(t-t_0)\right)^{-2/\varepsilon} =$$

$$= \frac{R_0}{\left[1 - \frac{\beta\varepsilon}{2} R_0^{\varepsilon/2}(t-t_0)\right]^{2/\varepsilon}}.$$

$\Rightarrow R(t)$ blows up at <u>finite</u> time

$$t - t_0 = \frac{2}{\beta\varepsilon} R_0^{-\varepsilon/2} \Rightarrow \underline{\text{Big Rip}}.$$