GR Exam 2016

/ Solution notes - A. Starinets /

Q1.

$$-c^2 d\tau^2 = -\left(1-\frac{2GM}{c^2r}\right)c^2dt^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}dr^2 +$$
$$+ r^2 d\theta^2 + r^2\sin^2\theta \, d\varphi^2$$

The metric is explicitly t- and φ-indep., but actually it has a larger symmetry group since $d\Omega^2 = d\theta^2 + \sin^2\theta \, d\varphi^2$ is rot.-invar. E.o.m. for θ:

$$\frac{d}{d\tau}\left(\frac{\partial \mathcal{L}}{\partial \dot\theta}\right) = \frac{\partial \mathcal{L}}{\partial \theta}, \quad \mathcal{L} = g_{\mu\nu}\dot x^\mu \dot x^\nu$$

$$\Rightarrow \frac{d}{d\tau}\left(2r^2\dot\theta\right) = r^2 \sin 2\theta \, \dot\varphi^2$$

$\boxed{\theta = \pi/2 \text{ is a self-consistent solution.}}$

For $\dot t$ and $\dot\varphi$ we find

$$\frac{d}{d\tau}\left[\left(1-\frac{2GM}{c^2r}\right)\dot t\right] = 0$$

$$\Rightarrow \boxed{\left(1-\frac{2GM}{c^2r}\right)\dot t = k = \text{const}}$$

$$\frac{d}{d\tau}\left[r^2\dot{\varphi}\right] = 0 \Rightarrow \boxed{r^2\dot{\varphi} = h = \text{const}}$$

Use $\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = -c^2$ (here the affine param. is τ)

$$-\left(1-\frac{2GM}{c^2r}\right)c^2\dot{t}^2 + \left(1-\frac{2GM}{c^2r}\right)^{-1}\dot{r}^2 + r^2\dot{\varphi}^2 = -c^2$$

or

$$-\left(1-\frac{2GM}{c^2r}\right)^{-1}c^2k^2 + \frac{h^2}{r^2} + \left(1-\frac{2GM}{c^2r}\right)^{-1}\dot{r}^2 + c^2 = 0$$

$$\boxed{\dot{r}^2 = -c^2\left(1-\frac{2GM}{c^2r}\right) - \frac{h^2}{r^2}\left(1-\frac{2GM}{c^2r}\right) + c^2k^2}$$

or $$\frac{\mu\dot{r}^2}{2} + V_{eff} = \frac{c^2(k^2-1)}{2}, \quad \mu \equiv 1,$$

$$V_{eff} = \frac{c^2}{2}\left(\cancel{1} - \frac{2GM}{c^2r}\right) + \frac{h^2}{2r^2}\left(1-\frac{2GM}{c^2r}\right).$$

$$2\dot{r}\ddot{r} = \left(-\frac{2GM}{r^2} + \frac{2h^2}{r^3} - \frac{3\cdot 2GMh^2}{c^2r^4}\right)\dot{r}$$

$$r^2\dot{\varphi} = h \Rightarrow d\varphi = \frac{h}{r^2}d\tau$$

③

$$\dot{r} = \frac{dr}{d\tau} = h\frac{dr}{r^2 d\varphi} = \frac{h}{r^2}r'$$

$$\ddot{r} = -\frac{2h}{r^3}\dot{r}r' + \left(\frac{h}{r^2}\right)^2 r'' =$$

$$= -\frac{2h}{r^3}\frac{h}{r^2}r'^2 + \frac{h^2}{r^4}r''$$

$$u = 1/r \qquad r' = -\frac{1}{u^2}u'$$

$$r'' = -\frac{1}{u^2}u'' + \frac{2}{u^3}u'^2$$

$$\ddot{r} = -\frac{2h^2u^5\,u'^2}{u^4} + h^2u^4\left(\frac{2}{u^3}u'^2 - \frac{u''}{u^2}\right) =$$

$$= -2h^2uu'^2 + 2h^2uu'^2 - h^2u^2u'' =$$

$$= -h^2u^2u''$$

So, $-h^2u^2u'' = -GMu^2 + h^2u^3 - \frac{3GMh^2}{c^2}u^4$

or $\quad u'' = \frac{GM}{h^2} - u + \frac{3GMu^2}{c^2}$

or $$\boxed{u'' + u = A + Bu^2, \quad A = \frac{GM}{h^2}, \quad B = \frac{3GM}{c^2}}$$

Astronaut : $r = R \quad \theta = \pi/2 \quad \varphi = 0$

Proper distance ; $dl_0 = R\,d\varphi$,

Proper time : $d\tau_0 = \left(1 - \frac{2GM}{c^2R}\right)^{1/2} dt$,

where

$$\begin{cases} d\varphi = \frac{h}{R^2}\, d\tau \\ dt = \left(1 - \frac{2GM}{c^2R}\right)^{-1} k\, d\tau \end{cases}$$

for circ. orbit param. by τ.

So

$$v_{circ} = \frac{dl_0}{d\tau_0} = \frac{h/R \cdot d\tau}{\left(1-\frac{2GM}{c^2R}\right)^{-1/2} k \cdot d\tau} = \frac{h\left(1-\frac{2GM}{c^2R}\right)^{\frac{1}{2}}}{k \cdot R}$$

For circ. orbit $\dot{r} = 0$

$$\frac{GM}{h^2} = \frac{1}{R} - \frac{3GM}{c^2R^2} = \frac{1}{R}\left(1 - \frac{3GM}{c^2R}\right)$$

$$h^2 = \frac{GMR}{1 - 3GM/c^2R}$$

$$c^2k^2 = c^2\left(1 - \frac{2GM}{c^2R}\right) + \frac{h^2}{R^2}\left(1 - \frac{2GM}{c^2R}\right)$$

$$k^2 = \left(1 - \frac{2GM}{c^2R}\right) + \frac{GM/Rc^2}{1 - 3GM/c^2R}\left(1 - \frac{2GM}{c^2R}\right) =$$

$$= \left(1 - \frac{2GM}{c^2R}\right)\left(1 + \frac{GM/c^2R}{1 - 3GM/c^2R}\right) =$$

$$= \frac{\left(1 - 2GM/c^2R\right)^2}{1 - 3GM/c^2R}.$$

So, $v_{circ} = \frac{h}{k \cdot R}\left(1 - 2GM/c^2R\right)^{1/2} =$

$$= \frac{\sqrt{GMR}}{\sqrt{1 - \frac{2GM}{c^2R}}} \frac{1}{R} \Rightarrow$$

$$\boxed{v_{circ}/c = \sqrt{\frac{GM/c^2R}{1 - 2GM/c^2R}}}$$

One orbit $\Delta\varphi = 2\pi \Rightarrow$

$$2\pi R^2/h = \Delta\tau$$

$$\Rightarrow \Delta t = \left(1 - \frac{2GM}{c^2R}\right)^{-1} \frac{k}{h} 2\pi R^2$$

For the astronaut: proper time

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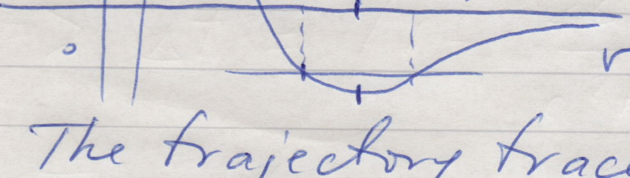
$$\Delta T = \left(1 - \frac{2GM}{c^2R}\right)^{1/2} \cdot \left(1 - \frac{2GM}{c^2R}\right)^{-1} \frac{K}{h} 2\pi R^2 =$$

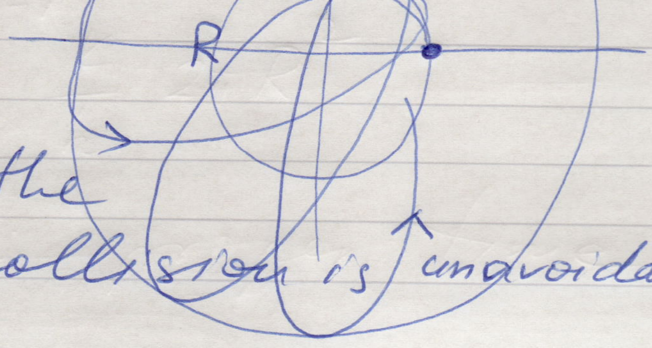
$$= \left(1 - \frac{2GM}{c^2R}\right)^{-\frac{1}{2}} 2\pi R^2 \frac{1 - 2GM/c^2R}{\sqrt{GMR}} =$$

$$= \frac{2\pi R}{c} \sqrt{\frac{1 - 2GM/c^2R}{GM/c^2R}}$$

$$\boxed{\Delta T = \frac{2\pi R}{c} \sqrt{\frac{1 - 2GM/c^2R}{GM/c^2R}}}$$

If a satellite is launched with $v = v_{circ} + \Delta v$, its orbit will be a precessing ellipse, recall and that $V_{eff} \neq 1/r$ or r^2.





The trajectory traces the circle $r = R$, so the collision is unavoidable.

Q2 $T^{ab} = \left(\rho + \frac{P}{c^2}\right) u^a u^b + g^{ab} P$

T^{ab} is conserved, $\partial_a T^{ab} = 0$,

in curved space-time $\Rightarrow \nabla_a T^{ab} = 0$

G_{ab} is built out of available tensors related to g_{ab} : g_{ab} and R_{ab} (terms such as $g_{ab} R^2$ etc would introduce new dimensionful couplings but are possible).

The combination $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$ has the property $\nabla^a G_{ab} = 0$, so

$G_{ab} \sim T_{ab}$ would be OK as e.o.m. coupling matter ~~to~~ gravity.

$$ds^2 = -e^{2\Phi(r)} d(ct)^2 + e^{2\Lambda(r)} dr^2 + r^2 d\Omega^2$$

All off-diag. comp. of G_{ab} are zero $\Rightarrow T_{ab}$ is diag. $\Rightarrow$ fluid is at rest, $u^1 = 0$, $u^2 = 0$, $u^3 = 0$,

$$u_\mu u^\mu = g_{00}(u^0)^2 = -c^2$$

$$\Rightarrow -e^{+2\Phi(r)}(u^0)^2 = -c^2$$

$$u^0 = c\, e^{-\Phi(r)}, \quad u_0 = g_{00} u^0 = -c\, e^{+\Phi}.$$

$G_{ab} = -\frac{8\pi G}{c^4} T_{ab}$ (we use lecturer's sign conventions)

$$G_{00} = -\frac{1}{r^2} e^{2\Phi} \frac{d}{dr}\left[r\left(1-e^{-2\Lambda}\right)\right] =$$

$$= -\frac{8\pi G}{c^4} T_{00} = -\frac{8\pi G}{c^4} \rho u_0^2 =$$

$$= -\frac{8\pi G}{c^4} \rho c^2 e^{2\Phi}$$

$$\frac{d}{dr}\left[r\left(1-e^{-2\Lambda}\right)\right] = \frac{8\pi G\rho}{c^2} r^2$$

$$r\left(1-e^{-2\Lambda}\right) = \frac{8\pi G}{c^2}\int_0^r \rho(r) r^2\, dr =$$

$$= \frac{2G}{c^2} M(r)$$

$$\Rightarrow \boxed{e^{-2\Lambda} = 1 - \frac{2GM(r)}{c^2 r}}$$

$$G_{11} = \frac{1}{r^2} e^{2\Lambda} - \frac{1}{r^2} - \frac{2}{r}\phi' = -\frac{8\pi G}{c^4} T_{11} =$$

$$= -\frac{8\pi G}{c^4} e^{2\Lambda} P$$

$$\frac{2}{r}\phi' = \frac{1}{r^2} e^{2\Lambda} + \frac{8\pi G}{c^4} P e^{2\Lambda} - \frac{1}{r^2}$$

$$\frac{2}{r}\phi' e^{-2\Lambda} = \frac{1}{r^2}\left(1 - e^{-2\Lambda}\right) + \frac{8\pi G}{c^4} P$$

$$\frac{2}{r}\phi' e^{-2\Lambda} = \frac{2GM(r)}{c^2 r^3} + \frac{8\pi G P}{c^4}$$

$$\phi' = e^{2\Lambda} \frac{r}{2} \frac{1}{r^3}\left(\frac{2GM(r)}{c^2} + \frac{8\pi G P r^3}{c^4}\right)$$

$$\boxed{\phi' = \left(1 - \frac{2GM(r)}{c^2 r}\right)^{-1} \frac{1}{r^2 c^2}\left(GM(r) + \frac{4\pi G P r^3}{c^2}\right)}$$

With $(\rho c^2 + P)\phi' = -\frac{dP}{dr}$ we have

$$\boxed{\frac{dP}{dr} = -\left(\rho c^2 + P\right)\left(1 - \frac{2GM}{c^2 r}\right)^{-1} \frac{1}{r^2 c^2}\left(GM + \frac{4\pi G P r^3}{c^2}\right)}$$

Constant density star:

$$\bar{M} = \bar{\rho}\bar{V} = \frac{4}{3}\pi R^3 \bar{\rho} \qquad M(r) = \frac{4}{3}\pi\bar{\rho} r^3$$

$$\frac{dP}{dr} = -(\bar{\rho}c^2 + P)\left(1 - \frac{8\pi\bar{\rho} r^2 G}{3c^2}\right)^{-1} \frac{1}{r^2c^2} \times$$

$$\times \left(\frac{4}{3}\pi\bar{\rho} G + \frac{4\pi G P}{c^2}\right) r^3 =$$

$$= -(\bar{\rho}c^2 + P)\left(1 - \frac{8\pi G}{3c^2}\bar{\rho} r^2\right)^{-1} \frac{4\pi G}{3c^4}(\bar{\rho}c^2 + 3P) r$$

$$\int_{P_0}^{0} \frac{dp}{(\bar{\rho}c^2 + P)(\bar{\rho}c^2 + 3P)} = -\frac{4\pi G}{3c^4}\int_0^R \frac{r\,dr}{1 - \frac{8\pi G}{3c^2} r^2 \bar{\rho}}$$

$$\boxed{P_0 = \rho c^2 \frac{1 - \left(1 - \frac{2GM}{c^2R}\right)^{1/2}}{3\left(1 - \frac{2GM}{c^2R}\right)^{1/2} - 1}}$$

Q3.

$$w^2 + x^2 + y^2 + z^2 = A^2 \quad (*)$$

Parametrize: $x = A \sin\chi \sin\theta \cos\varphi$

$y = A \sin\chi \sin\theta \sin\varphi$

$z = A \sin\chi \cos\theta$

$w = A \cos\chi$

Then $(*)$ is satisfied.

Now compute the induced metric from $ds^2 = dw^2 + dx^2 + dy^2 + dz^2$ and param. above:

$$\boxed{ds^2 = A^2\left(d\chi^2 + \sin^2\chi\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right)}$$

Alternatively, introduce the usual spher. coord. for x, y, z and use

$$dw^2 = -d(x^2 + y^2 + z^2) = -d(r^2) =$$

$$= -2r\,dr \Rightarrow$$

$$w\,dw = -r\,dr$$

$$dw = -\frac{r\,dr}{w}$$

$$dw^2 = \frac{r^2 dr^2}{w^2} = \frac{r^2 dr^2}{A^2 - r^2}$$

$$\Rightarrow ds^2 = \frac{r^2 dr^2}{A^2 - r^2} + dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2$$

$$\Rightarrow \boxed{ds^2 = \frac{A^2}{A^2 - r^2} dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\varphi^2.}$$

The two metrics are related by

$$r^2 = A^2 \sin^2\chi \quad (\text{or } r = A \sin\chi).$$

Flat space metric is obviously

$$ds^2 = dr^2 + r^2 d\Omega^2 \quad \left(\text{these are usual spher. coord}\right)$$

It formally corresp. to $A \to \infty$ limit of the S^3 metric above.