University of Oxford

Physics Department

GENERAL RELATIVITY AND COSMOLOGY

EXAM PAPER

2015

SOLUTION NOTES

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GR Exam 2015

A13484W1

B5: GENERAL RELATIVITY AND COSMOLOGYSaturday, 20 June20159.30 am11.30 am

1. Show that if you extremize the action for a test particle $S = -\int \mathrm{d}\lambda\, g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta$ (where λ is the affine parameter and $g_{\alpha\beta}$ the metric), you will obtain the correct expressions for the connection coefficients for a general metric. [4]

Consider the "global rain" metric,

$$\mathrm{d}s^2 = -c^2\left(1-\frac{r_\mathrm{S}}{r}\right)\mathrm{d}\bar{t}^2 + 2\sqrt{\frac{r_\mathrm{S}}{r}}\,c\,\mathrm{d}\bar{t}\,\mathrm{d}r + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) \ ,$$

where $(\bar{t}, r, \theta, \phi)$ are space-time coordinates and r_S is a constant. Show that the non-zero components of the inverse metric are

$$g^{\bar{t}\bar{t}} = -\frac{1}{c^2} \ , \ g^{\bar{t}r} = \frac{1}{c}\sqrt{\frac{r_\mathrm{S}}{r}} \ , \ g^{rr} = \left(1-\frac{r_\mathrm{S}}{r}\right) \ , \ g^{\theta\theta} = \frac{1}{r^2} \ , \ g^{\phi\phi} = \frac{1}{r^2\sin^2\theta} \ .$$

Using the definition of the connection coefficients (or otherwise), show that the radial geodesic equation is

$$\ddot{r} - \frac{1}{2r_\mathrm{S}}\left(\frac{r_\mathrm{S}}{r}\right)^2\dot{r}^2 + \frac{c^2}{2r_\mathrm{S}}\left(1-\frac{r_\mathrm{S}}{r}\right)\left(\frac{r_\mathrm{S}}{r}\right)^2\dot{\bar{t}}^2 - \frac{c}{r_\mathrm{S}}\left(\frac{r_\mathrm{S}}{r}\right)^{\frac{5}{2}}\dot{r}\dot{\bar{t}}$$
$$- \left(1-\frac{r_\mathrm{S}}{r}\right)r\dot{\theta}^2 - \left(1-\frac{r_\mathrm{S}}{r}\right)r\sin^2\theta\,\dot{\phi}^2 = 0 \ .$$

[10]

Compare what happens to this metric at $r = r_\mathrm{S}$ with what happens to the Schwarzschild metric in the usual coordinates. By looking only at light-like radial geodesics, explain why, if $r < r_\mathrm{S}$, photons always fall inwards [hint: show that $\mathrm{d}r/\mathrm{d}\bar{t} < 0$]. [6]

Consider a change of coordinates such that $\bar{t} = \bar{t}(r, t)$ where

$$\frac{\partial \bar{t}}{\partial t} = 1 \ ,$$
$$\frac{\partial \bar{t}}{\partial r} = \sqrt{\frac{r_\mathrm{S}}{r}}\left(1-\frac{r_\mathrm{S}}{r}\right)^{-1}\frac{1}{c} \ .$$

Rewrite the global rain metric in terms of t and r, and show that it is equivalent to the Schwarzschild metric. [5]

2. Consider a conformally flat space-time with metric given in Cartesian coordinates by

$$\mathrm{d}s^2 = \mathrm{e}^{\frac{2\varphi}{c^2}} \, \eta_{\alpha\beta} \, \mathrm{d}x^\alpha \mathrm{d}x^\beta \ ,$$

where φ is a scalar function of space-time coordinates and $\eta_{\alpha\beta}$ is the Minkowski metric. Show that the connection coefficients take the form

$$\Gamma^\mu{}_{\alpha\beta} = \frac{1}{c^2} \left[\partial_\beta \varphi \, \delta^\mu{}_\alpha + \partial_\alpha \varphi \, \delta^\mu{}_\beta - \partial^\mu \varphi \, \eta_{\alpha\beta} \right] \ ,$$

where $\partial^\mu = \eta^{\mu\nu} \, \partial_\nu$. [7]

The Ricci tensor is given by

$$R_{\nu\beta} \equiv \partial_\mu \Gamma^\mu{}_{\beta\nu} - \partial_\beta \Gamma^\mu{}_{\mu\nu} + \Gamma^\mu{}_{\mu\epsilon} \Gamma^\epsilon{}_{\nu\beta} - \Gamma^\mu{}_{\epsilon\beta} \Gamma^\epsilon{}_{\nu\mu} \ .$$

Assume that $\varphi/c^2 \ll 1$, and show that the Einstein tensor takes the form

$$G_{\alpha\beta} = \frac{2}{c^2} \left(\partial_\mu \partial^\mu \varphi \, \eta_{\alpha\beta} - \partial_\alpha \partial_\beta \varphi \right) \ .$$

[6]

Rewrite the metric into spherical coordinates, (t, r, θ, ϕ), and assuming that φ is a function of r only, show that for an equatorial orbit the geodesic equations for t and ϕ take the form

$$\mathrm{e}^{\frac{2\varphi}{c^2}} \, \dot{t} = d \ \text{ and } \ \mathrm{e}^{\frac{2\varphi}{c^2}} \, r^2 \dot{\phi} = \ell \ ,$$

where $\dot{f} \equiv \mathrm{d}f/\mathrm{d}\lambda$ for any function $f(\lambda)$ (λ is the affine parameter), and d and ℓ are integration constants. Write down an expression for the null condition for photons in this metric. [5]

Assume that $\varphi = -GM/r$, where M is a constant and r is the distance from the origin. Show that there is no light deflection around $r = 0$. [7]

3. The Riemann tensor is defined to be

$$R^{\mu}{}_{\nu\alpha\beta} \equiv \partial_\alpha \Gamma^{\mu}{}_{\beta\nu} - \partial_\beta \Gamma^{\mu}{}_{\alpha\nu} + \Gamma^{\mu}{}_{\alpha\epsilon}\Gamma^{\epsilon}{}_{\nu\beta} - \Gamma^{\mu}{}_{\epsilon\beta}\Gamma^{\epsilon}{}_{\nu\alpha} \ ,$$

where $\Gamma^{\mu}{}_{\beta\nu}$ is the connection coefficient tensor. Show that

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu = R^{\mu}{}_{\nu\alpha\beta} V^\nu \qquad (1)$$

for any contravariant vector V^μ. [7]

A "Killing" vector, U^μ, satisfies the condition $\nabla_\mu U_\nu + \nabla_\nu U_\mu = 0$. We define the commutator between two vectors to be

$$W^\mu \equiv [U, V]^\mu = U^\nu \nabla_\nu V^\mu - V^\nu \nabla_\nu U^\mu \ .$$

Using equation 1 (and the symmetry of the Riemann tensor, $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$), show that the commutator of two Killing vectors is also a Killing vector. [8]

Consider a tensor $T_{\mu\nu} = \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^\sigma \nabla_\sigma \phi$, where ϕ is a scalar function of the space-time coordinates. Show that

$$\nabla^\mu T_{\mu\nu} = R_\nu{}^\sigma \nabla_\sigma \phi \ . \qquad [6]$$

Assume now that $\nabla^\mu T_{\mu\nu} = 0$. If k^ν is a Killing vector, show that

$$\nabla^\mu (T_{\mu\nu} k^\nu) = 0 \ . \qquad [4]$$

4. Consider an inflationary universe that undergoes three phases of expansion: an initial inflationary phase in which the pressure, P, and the density, ρ, satisfy $P = -\rho c^2$ up until the scale factor $a = a_1$, followed by a radiation phase in which $P = \frac{1}{3}\rho c^2$ up until the scale factor $a = a_2$, followed by a matter phase in which $P = 0$ up until the scale factor $a = 1$. Find an expression for the Hubble rate and deceleration rate, as a function of the scale factor a, in each of these regimes (neglecting all other non-dominant components of the energy density). Solve the Friedman-Robertson-Walker (FRW) equation in each one of the three phases. [7]

Explain why the expansion rate in such a universe is slower when $a < a_1$ than in a universe where there is *no* initial inflationary phase (i.e. a universe where there is no period of inflation for $a < a_1$ and that has exactly the same expansion rate as a function of the scale factor for $a > a_1$). [Hint: assume continuity in the Hubble rate at $a = a_1$ for the inflationary universe]. [5]

Assume that the initial scale factor of the universe is $a_{\rm in}$. Find an expression for the age of the universe in terms of an integral over the Hubble rate. Taking the limit $a_{\rm in} \to 0$, show that the inflationary universe must be older than the non-inflationary universe. [6]

Find an expression for the particle horizon in each one of the phases of the inflationary universe. Comparing the physical size of a length scale of fixed comoving size with the particle horizon, explain the qualitative difference between what happens for $a < a_1$ and $a > a_1$. [7]

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(Solution notes: A. Starinets)

Q1:

1) $S = -\int d\lambda \, g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \qquad \delta S = 0 - ?$

$$\delta S = S[x+\delta x] - S[x] =$$

$$= -\int d\lambda \left\{ g_{\alpha\beta}(x+\delta x) (\dot{x}+\delta\dot{x})^\alpha (\dot{x}+\delta\dot{x})^\beta \right\} - S[x] =$$

$$= -\int d\lambda \Big[\left(g_{\alpha\beta}(x) + \partial_\mu g_{\alpha\beta}(x) \cdot \delta x^\mu + \ldots \right) \left(\dot{x}^\alpha + \delta\dot{x}^\alpha \right) \times$$

$$\times \left(\dot{x}^\beta + \delta \dot{x}^\beta \right) \Big] - S[x] =$$

$$= -\int d\lambda \Big[\partial_\mu g_{\alpha\beta} \, \delta x^\mu \dot{x}^\alpha \dot{x}^\beta + g_{\alpha\beta} \, \delta\dot{x}^\alpha x^\beta +$$

$$+ g_{\alpha\beta} \dot{x}^\alpha \delta\dot{x}^\beta \Big] + O(\delta x^2) =$$

$$= -\int d\lambda \left[\partial_\mu g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \, \delta x^\mu + 2 g_{\alpha\beta} \dot{x}^\alpha \delta \dot{x}^\beta \right] =$$

$$= -2g_{\alpha\beta}\dot{x}^{\alpha}\delta x^{\beta}\Big|_{x}^{x} - \int d\lambda\,\Big[\partial_\mu g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}\delta x^{\mu} -$$

$$-\left(\frac{d}{d\lambda}2g_{\alpha\beta}\dot{x}^{\alpha}\right)\delta x^{\beta}\Big] = 0$$

With b.c. $\delta x(x) = 0$:

$$\delta S = -\int d\lambda\left[\partial_\mu g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} - 2\frac{d}{d\lambda}\left(g_{\alpha\mu}\dot{x}^{\alpha}\right)\right]\delta x^{\mu} = 0$$

$$\Rightarrow \partial_\mu g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} - 2\partial_\nu g_{\alpha\mu}\dot{x}^{\nu}\dot{x}^{\alpha} - 2g_{\alpha\mu}\ddot{x}^{\alpha} = 0$$

$$\ddot{x}^{\sigma} - \frac{1}{2}g^{\mu\sigma}\partial_\mu g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} + g^{\mu\sigma}\partial_\nu g_{\alpha\mu}\dot{x}^{\nu}\dot{x}^{\alpha} = 0$$

or

$$\ddot{x}^{\sigma} + \frac{1}{2}g^{\sigma\mu}\left(\partial_\beta g_{\alpha\mu} + \partial_\alpha g_{\beta\mu} - \partial_\mu g_{\alpha\beta}\right)(\dot{x}^{\alpha}\dot{x}^{\beta}) = 0$$

i.e. $\ddot{x}^{\alpha} + \Gamma^{\sigma}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0$, with

$$\boxed{\Gamma^{\sigma}_{\alpha\beta} = \frac{1}{2}g^{\sigma\mu}\left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right)}$$

$$ds^2 = -c^2\left(1-\frac{r_s}{r}\right)d\bar{t}^2 + 2\sqrt{\frac{r_s}{r}}\,c\,d\bar{t}\,dr +$$

$$+\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

This metric is off-diagonal in its $r-\bar{t}$ part.

The inverse metric is defined by $G^{-1}G = \underline{I}$

or $g^{\alpha\mu} g_{\mu\beta} = \delta^{\alpha}_{\beta}$.

In matrix form:

$$G = \begin{bmatrix} -c^2\left(1-\frac{r_s}{r}\right) & \sqrt{\frac{r_s}{r}}\,c & 0 & 0 \\ c\sqrt{\frac{r_s}{r}} & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\sin^2\theta \end{bmatrix}$$

Since for $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ the inverse is

$A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, we get (note that

G has block-diagonal form)

$G = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$, $\det A = -c^2\left(1-\frac{r_s}{r}\right) - c^2\frac{r_s}{r} = -c^2$

$$G^{-1} = \begin{bmatrix} -\frac{1}{c^2} & \frac{1}{c}\sqrt{\frac{r_s}{r}} & 0 & 0 \\ \frac{1}{c}\sqrt{\frac{r_s}{r}} & \left(1-\frac{r_s}{r}\right) & 0 & 0 \\ 0 & 0 & 1/r^2 & 0 \\ 0 & 0 & 0 & 1/r^2\sin^2\theta \end{bmatrix}.$$

To get the radial geodes. eq., can use

$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu$ and E-L eqs

$$\frac{d}{d\lambda}\left(\frac{\partial \mathcal{L}}{\partial \dot{r}}\right) = \frac{\partial \mathcal{L}}{\partial r}$$

$$\frac{d}{d\lambda}\left[2\dot{r} + 2\sqrt{\frac{r_s}{r}}\, c\,\dot{t}\right] = -c^2\frac{r_s}{r^2}\dot{t}^2 - \frac{c\sqrt{r_s}}{r^{3/2}}\dot{t}\dot{r}$$

$$+2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\varphi}^2$$

$$2\ddot{r} - \cancel{\sqrt{r_s}\,c\,\frac{1}{r^{3/2}}\dot{r}\dot{t}} + 2\sqrt{\frac{r_s}{r}}\,c\,\ddot{t} =$$

$$= -\frac{c^2}{r_s}\left(\frac{r_s}{r}\right)^2\dot{t}^2 - \cancel{\frac{\sqrt{r_s}\,c}{r^{3/2}}\dot{r}\dot{t}} + 2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\varphi}^2$$

This can be combined with eom for $\dot{t}$:

$$\frac{d}{d\lambda}\left(-2c^2\left(1-\frac{r_s}{r}\right)\dot{t}+2\sqrt{\frac{r_s}{r}}\,c\dot{r}\right)=0$$

$$\Rightarrow\ -2c^2\left(1-\frac{r_s}{r}\right)\ddot{t}-2c^2\frac{r_s}{r^2}\dot{t}\dot{r}+2\sqrt{\frac{r_s}{r}}\,c\ddot{r}-$$

$$-\sqrt{r_s}\,c\frac{1}{r^{3/2}}\dot{r}^2=0$$

$$\ddot{t}=-\frac{2c^2}{2c^2}\frac{r_s}{r^2}\left(1-\frac{r_s}{r}\right)^{-1}\dot{t}\dot{r}+\frac{1}{c}\sqrt{\frac{r_s}{r}}\left(1-\frac{r_s}{r}\right)^{-1}\ddot{r}$$

$$-\frac{\sqrt{r_s}}{2cr^{3/2}}\left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2$$

$$\ddot{r}+\sqrt{\frac{r_s}{r}}\,c\ddot{t}+\frac{c^2}{2r_s}\left(\frac{r_s}{r}\right)^2\dot{t}^2-r\dot{\theta}^2-r\sin^2\theta\dot{\varphi}^2$$
$$=0$$

$$\ddot{r}+\frac{r_s}{r}\left(1-\frac{r_s}{r}\right)^{-1}\ddot{r}-\sqrt{\frac{r_s}{r}}\,c\frac{r_s}{r^2}\left(1-\frac{r_s}{r}\right)^{-1}\dot{t}\dot{r}-$$

$$-\frac{r_s}{2r^2}\left(1-\frac{r_s}{r}\right)^{-1}\dot{r}^2+\frac{c^2}{2r_s}\left(\frac{r_s}{r}\right)^2\dot{t}^2-r\dot{\theta}^2-$$

$$-r\sin^2\theta\,\dot{\varphi}^2=0$$

$$\Rightarrow \ddot{r} - \frac{c}{r_s}\left(\frac{r_s}{r}\right)^{5/2}\dot{t}\dot{r} - \frac{1}{2r_s}\left(\frac{r_s}{r}\right)^2\dot{r}^2 +$$

$$+ \frac{c^2}{2r_s}\left(1-\frac{r_s}{r}\right)\left(\frac{r_s}{r}\right)^2\dot{t}^2 - r\left(1-\frac{r_s}{r}\right)\dot{\theta}^2 -$$

$$- r\sin^2\theta\left(1-\frac{r_s}{r}\right)\dot{\varphi}^2 = 0$$

For Schwarzschild metric, g_{tt} and g_{rr} are singular (coord. singularity), here coord. sing. only for $g_{\bar{t}\bar{t}}$.
(Think also about g^{-1} in both cases.)

Light-like radial geodesic $ds^2 = 0$:

$$-c^2\left(1-\frac{r_s}{r}\right)\dot{\bar{t}}^2 + 2\sqrt{\frac{r_s}{r}}\,c\,\dot{\bar{t}}\dot{r} + \dot{r}^2 = 0$$

$$\Rightarrow \left(\frac{dr}{d\bar{t}}\right)^2 = c^2\left(1-\frac{r_s}{r}\right) - 2\sqrt{\frac{r_s}{r}}\,c\,\frac{dr}{d\bar{t}}$$

For $r < r_s$ must have $\frac{dr}{d\bar{t}} < 0$ to have l.h.s. positive.

$$\bar{t} = \bar{t}(r,t)$$

$$ds^2 = -c^2\left(1 - r_s/r\right)\left(\frac{\partial \bar{t}}{\partial r}dr + \frac{\partial \bar{t}}{\partial t}dt\right)^2 + dr^2$$

$$+ 2\sqrt{\frac{r_s}{r}}\, c\, dr\left(\frac{\partial \bar{t}}{\partial r}dr + \frac{\partial \bar{t}}{\partial t}dt\right) + r^2 d\Omega^2 =$$

$$= (1+A)\,dr^2 + B\,dt^2 + C\,dr\,dt + r^2 d\Omega^2,$$

where

$$A = -c^2\left(1 - r_s/r\right)\left[\left(\frac{\partial \bar{t}}{\partial r}\right)^2\right] + 2\sqrt{\frac{r_s}{r}}\, c\left[\frac{\partial \bar{t}}{\partial r}\right] =$$

$$= -\cancel{c^2}\left(1 - \frac{r_s}{r}\right)\frac{1}{\cancel{c^2}}\frac{r_s}{r}\left(1 - \frac{r_s}{r}\right)^{-2} + 2c\sqrt{\frac{r_s}{r}}\,\frac{1}{c}\sqrt{\frac{r_s}{r}}\left(1 - \frac{r_s}{r}\right)^{-1}$$

$$= -\frac{r_s}{r}\left(1 - \frac{r_s}{r}\right)^{-1} + 2\,\frac{r_s}{r}\left(1 - \frac{r_s}{r}\right)^{-1} =$$

$$= \frac{r_s}{r}\left(1 - \frac{r_s}{r}\right)^{-1} \Rightarrow 1 + A = \left(1 - \frac{r_s}{r}\right)^{-1}.$$

$$B = -c^2\left(1 - \frac{r_s}{r}\right)\left(\frac{\partial \bar{t}}{\partial t}\right)^2 = -c^2\left(1 - \frac{r_s}{r}\right).$$

$$C = -c^2\left(1-\frac{r_s}{r}\right) 2\frac{\partial \bar{t}}{\partial r}\frac{\partial \bar{t}}{\partial t} + 2\sqrt{\frac{r_s}{r}}c\,\frac{\partial \bar{t}}{\partial t} =$$

$$= -2c\sqrt{\frac{r_s}{r}} + 2c\sqrt{\frac{r_s}{r}} = 0$$

$\Rightarrow$ ds^2 in r, t coord. is the Schwarzschild metric.

1. $$S = -\int d\lambda\, g_{\alpha\beta}(x)\,\dot{x}^{\alpha}\dot{x}^{\beta}$$

$$\frac{d}{d\lambda}\left(g_{\gamma\beta}\dot{x}^{\beta}\right) = \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\dot{x}^{\alpha}\dot{x}^{\beta}$$

$$g_{\gamma\beta}\ddot{x}^{\beta} + \frac{\partial g_{\gamma\beta}}{\partial x^{\sigma}}\dot{x}^{\sigma}\dot{x}^{\beta} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\dot{x}^{\alpha}\dot{x}^{\beta} = 0$$

$$\ddot{x}^{\varkappa} + g^{\varkappa\gamma}\left(\frac{\partial g_{\gamma\beta}}{\partial x^{\alpha}} - \frac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\right)\dot{x}^{\alpha}\dot{x}^{\beta} = 0$$

$$\ddot{x}^{\varkappa} + \frac{1}{2}g^{\varkappa\gamma}\left(\frac{\partial g_{\alpha\gamma}}{\partial x^{\beta}} + \frac{\partial g_{\beta\gamma}}{\partial x^{\alpha}} - \frac{\partial g_{\alpha\beta}}{\partial x^{\gamma}}\right)\dot{x}^{\alpha}\dot{x}^{\beta} = 0$$

$$\cancel{\ddot{x}^{\varkappa} + \tfrac{1}{2}} \qquad \ddot{x}^{\varkappa} + \Gamma^{\varkappa}_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta} = 0$$

- Extremize the action
- (Affine parameter; otherwise $\int F\left(\sqrt{g_{\alpha\beta}\dot{x}^{\alpha}\dot{x}^{\beta}}\right)$ will give a different eqn.)

But OK

$$\begin{pmatrix} -c^2\left(1-\frac{r_s}{r}\right) & \sqrt{\frac{r_s}{r}}\,c \\ \sqrt{\frac{r_s}{r}}\,c & 1 \end{pmatrix}$$

$$\det = -c^2\left(1-\frac{r_s}{r}\right) - c^2\frac{r_s}{r} = -c^2$$

$$\begin{pmatrix} -\frac{1}{c^2} & \frac{1}{c}\sqrt{\frac{r_s}{r}} \\ \frac{1}{c}\sqrt{\frac{r_s}{r}} & \left(1-\frac{r_s}{r}\right) \end{pmatrix} \quad \text{OK}$$

Diag: $\frac{1}{r^2}$ $\quad \frac{1}{r^2\sin^2\theta}$ OK.

$$g_{\alpha\beta}\,\dot{x}^\alpha \dot{x}^\beta = -c^2\frac{d\tau^2}{d\lambda^2} \quad (\lambda \overset{\text{or}}{=} \tau)$$

$$g_{\alpha\beta}\,\frac{dx^\alpha dx^\beta}{d\tau\, d\tau}\left(\frac{d\tau}{d\lambda}\right)^2 = -\frac{c^2 d\tau^2}{\cancel{d\tau^2}}\,\cancel{\frac{d\tau^2}{d\lambda^2}}$$

$$g_{tt}\dot{t}^2 + 2g_{rt}\dot{r}\dot{t} + g_{rr}\dot{r}^2 + g_{\theta\theta}\dot{\theta}^2 + g_{\phi\phi}\dot{\phi}^2 = -c^2\left(\frac{d\tau}{d\lambda}\right)^2$$

$$\frac{d}{d\lambda}\left(2\dot{r}g_{rr} + 2g_{rt}\dot{t}\right) = \frac{\partial g_{tt}}{\partial r}\dot{t}^2 + 2\frac{\partial g_{rt}}{\partial r}\dot{r}\dot{t} + \frac{\partial g_{rr}}{\partial r}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\phi}^2$$

$$\ddot{r} + \Gamma^r_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 0$$

$$\frac{d}{d\lambda}\left(2\dot{t}g_{tt} + 2g_{rt}\dot{r}\right) = 0$$

$$2\ddot{t}g_{tt} = -2g_{rt}\ddot{r} - 2\frac{\partial g_{tt}}{\partial r}\dot{r}\dot{t} - 2\frac{\partial g_{rt}}{\partial r}\dot{r}^2$$

$$2\ddot{r}g_{rr} + 2\frac{\partial g_{rt}}{\partial r}\dot{r}\dot{t} + 2g_{rt}\ddot{t} =$$

$$= 2g_{rr}\ddot{r} + 2\frac{\partial g_{rt}}{\partial r}\dot{r}\dot{t} + \frac{g_{rt}}{g_{tt}}\left(-2\ddot{r}g_{rt} - 2\frac{\partial g_{tt}}{\partial r}\dot{r}\dot{t} - 2\frac{\partial g_{rt}}{\partial r}\dot{r}^2\right)$$

$$\underbrace{2 g_{rr} \ddot{r} - 2 \frac{g_{rt}^2}{g_{tt}} \ddot{r}} + \cancel{2 \frac{\partial g_{rt}}{\partial r} \dot{r} \dot{t}} -$$

$$- 2 \frac{g_{rt}}{g_{tt}} \frac{\partial g_{tt}}{\partial r} \dot{r} \dot{t} - 2 \frac{g_{rt}}{g_{tt}} \frac{\partial g_{rt}}{\partial r} \dot{r}^2$$

$$= \frac{\partial g_{tt}}{\partial r} \dot{t}^2 + \cancel{2 \frac{\partial g_{rt}}{\partial r} \dot{r} \dot{t}} + \cancel{\frac{\partial g_{rr}}{\partial r} \dot{r}^2}$$

$$+ 2r \dot{\theta}^2 + 2r \sin^2\theta \, \dot{\phi}^2$$

$$2\ddot{r}\left(1 + \frac{r_s}{r\left(1 - \frac{r_s}{r}\right)}\right) =$$

$$= 2\ddot{r}\left(\frac{1}{1 - \frac{r_s}{r}}\right)$$

$$\ddot{r} = \frac{1}{2}\left(1 - \frac{r_s}{r}\right)\Big[2r\dot{\theta}^2 + 2r\sin^2\theta\,\dot{\phi}^2 +$$

$$+ \frac{\partial g_{tt}}{\partial r} \dot{t}^2 + 2 \frac{g_{rt}}{g_{tt}} \frac{\partial g_{rt}}{\partial r} \left(\cancel{\dot{r}\dot{t}} + \dot{r}^2\right)\Big]$$

$$+ 2 \frac{g_{rt}}{g_{tt}} \frac{\partial g_{tt}}{\partial r} \dot{r} \dot{t}$$

Pedro: Prob 1
2014

```
In[1]:= << RGTC/EDCRGTCcode388.m
In[1]:= xCoord = {t, r, θ, φ};
In[5]:= g = {{-c^2 (1 - rs / r), c Sqrt[rs / r], 0, 0}, {c Sqrt[rs / r],
        1, 0, 0}, {0, 0, r^2, 0}, {0, 0, 0, r^2 Sin[θ]^2}};
In[6]:= simpRules = TrigRules;
In[7]:= RGtensors[g, xCoord]
```

$$\text{gdd} = \begin{pmatrix} -c^2\left(1-\frac{rs}{r}\right) & c\sqrt{\frac{rs}{r}} & 0 & 0 \\ c\sqrt{\frac{rs}{r}} & 1 & 0 & 0 \\ 0 & 0 & r^2 & 0 \\ 0 & 0 & 0 & r^2\,\text{Sin}[\theta]^2 \end{pmatrix}$$

$$\text{LineElement} = d[r]^2 + 2\,c\sqrt{\frac{rs}{r}}\,d[r]\,d[t] - \frac{c^2\,(r-rs)\,d[t]^2}{r} + r^2\,d[\theta]^2 + r^2\,d[\varphi]^2\,\text{Sin}[\theta]^2$$

$$\text{gUU} = \begin{pmatrix} -\frac{1}{c^2} & \frac{\sqrt{\frac{rs}{r}}}{c} & 0 & 0 \\ \frac{\sqrt{\frac{rs}{r}}}{c} & \frac{r-rs}{r} & 0 & 0 \\ 0 & 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & 0 & \frac{\text{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

```
gUU computed in 0.010322 sec
Gamma computed in 0.007061 sec
Riemann(dddd) computed in 0.009 sec
Riemann(Uddd) computed in 0.009821 sec
Ricci computed in 0.00045 sec
Weyl computed in 0.00003 sec
Ricci Flat
All tasks completed in 0.046369 seconds
In[8]:= RUddd[[1, 2, 1, 2]]
```

Out[8]= $\frac{rs}{r^3}$

```
Rdddd[[1, 2, 1, 2]]
0
d[t] ∧ d[t]
0
```

$$\dot r \dot t \;:\; + \cancel{\tfrac{1}{2}}\cancel{\left(1-\frac{r_s}{r}\right)}\cdot \cancel{2}\, c\sqrt{\frac{r_s}{r}}\,\frac{1}{\cancel{\left(1-\frac{r_s}{r}\right)}\cancel{c^2}}\,\cancel{c^2}\frac{r_s}{r^2} \quad \text{Ok}$$

$$\ddot r = \frac{c}{r_s}\left(\frac{r_s}{r}\right)^{5/2}\dot r \dot t + \frac{1}{2r_s}\left(\frac{r_s}{r}\right)^{2}\dot r^{2}$$

$$- \frac{c^2}{2r_s}\left(1-\frac{r_s}{r}\right)\left(\frac{r_s}{r}\right)^{2}\dot t^{2}$$

$$+ \left(1-\frac{r_s}{r}\right) r\dot\theta^{2} + \left(1-\frac{r_s}{r}\right) r\sin^2\theta\,\dot\phi^{2}$$

Ok

Also by direct comp. of $\Gamma^{r}_{\alpha\beta}$
(see Mathematica file) Ok

<u>Here</u>: coord. singul. of $g_{\bar t\bar t}$ at $r=r_s$

<u>Schwarzschild</u>: coord sing of $g_{\bar t\bar t}$ and g_{rr}.

Radial null $ds^2=0$

$$-c^2\left(1-\frac{r_s}{r}\right)d\bar{t}^2+2\sqrt{\frac{r_s}{r}}\,c\,d\bar{t}\,dr+dr^2=0$$

$$\dot{r}^2+2c\sqrt{\frac{r_s}{r}}\,\dot{r}-c^2\left(1-\frac{r_s}{r}\right)=0$$

$$\Rightarrow\quad \dot{r}=c\left(\pm 1-\sqrt{\frac{r_s}{r}}\right)<0 \text{ for } r<r_s$$

OK

$$\bar{t}=\bar{t}(r,t)$$

$$d\bar{t}=\frac{\partial\bar{t}}{\partial r}dr+\frac{\partial\bar{t}}{\partial t}dt=$$

$$=dt+\frac{1}{c}\sqrt{\frac{r_s}{r}}\left(1-\frac{r_s}{r}\right)^{-1}dr$$

$$ds^2=-c^2\left(1-\frac{r_s}{r}\right)\left(dt^2+\frac{2}{c}\cancel{\sqrt{\frac{r_s}{r}}\left(1-\frac{r_s}{r}\right)^{-1}}dt\,dr\right.$$

$$\left.+\frac{r_s}{c^2 r}\left(1-\frac{r_s}{r}\right)^{-2}dr^2\right)+dr^2+$$

$$+2c\cancel{\sqrt{\frac{r_s}{r}}\left(dt\,dr\right.}+\frac{1}{c}\sqrt{\frac{r_s}{r}}\left(1-\frac{r_s}{r}\right)^{-1}\left.dr^2\right)$$

$$+\ldots=-c^2\left(1-\frac{r_s}{r}\right)dt^2-\frac{r_s}{r}\left(1-\frac{r_s}{r}\right)^{-1}dr^2$$

$$+2\frac{r_s}{r}\left(1-\frac{r_s}{r}\right)^{-1}dr^2+dr^2+\ldots=$$

$$= -c^2\left(1-\frac{r_s}{r}\right)dt^2 + \frac{dr^2}{1-\frac{r_s}{r}} + \dots$$

OK

Q2 $\quad ds^2 = e^{2\varphi/c^2}\eta_{\alpha\beta}\,dx^\alpha dx^\beta$

Conformally flat metric; $\varphi = \varphi(x)$.

With

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} g^{\sigma\mu}\left(\partial_\alpha g_{\beta\mu} + \partial_\beta g_{\alpha\mu} - \partial_\mu g_{\alpha\beta}\right)$$

and $g_{\alpha\beta} = e^{2\varphi/c^2}\eta_{\alpha\beta}$ we have

$$\Gamma^\sigma_{\alpha\beta} = \frac{1}{2} e^{-2\varphi/c^2}\eta^{\sigma\mu}\Big(\partial_\alpha\varphi\,\frac{2}{c^2}e^{2\varphi/c^2}\eta_{\beta\mu} + $$

$$+ \partial_\beta\varphi\,\frac{2}{c^2}e^{2\varphi/c^2}\eta_{\alpha\mu} - \partial_\mu\varphi\,\frac{2}{c^2}e^{2\varphi/c^2}\eta_{\alpha\beta}\Big) =$$

$$= \frac{1}{c^2}\left[\partial_\alpha\varphi\,\delta^\sigma_\beta + \partial_\beta\varphi\,\delta^\sigma_\alpha - \eta_{\alpha\beta}\partial^\sigma\varphi\right].$$

$\partial^\sigma\varphi = \eta^{\sigma\nu}\partial_\nu\varphi$

$$R_{\nu\beta} = \partial_\mu\Gamma^\mu_{\beta\nu} - \partial_\beta\Gamma^\mu_{\mu\nu} + \Gamma^\mu_{\mu\epsilon}\Gamma^\epsilon_{\nu\beta} -$$

$$-\Gamma^\mu_{\epsilon\beta}\Gamma^\epsilon_{\nu\mu}$$

With $\varphi/c^2 \ll 1$ can ignore $\Gamma\Gamma$ terms.

Then $\Gamma^\mu_{\beta\nu} = \frac{1}{c^2}\left[\partial_\beta\varphi\,\delta^\mu_\nu + \partial_\nu\varphi\,\delta^\mu_\beta - \eta_{\beta\nu}\partial^\mu\varphi\right]$

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{c^2}\left[\partial_\mu\varphi \; \delta^{\mu}_{\nu} + 4\,\partial_\nu\varphi - \partial_\nu\varphi\right] =$$

$$= \frac{4}{c^2}\,\partial_\nu\varphi$$

$$R_{\nu\beta} = \frac{1}{c^2}\left[2\partial^2_{\nu\beta}\varphi - \eta_{\beta\nu}\,\partial_\mu\partial^\mu\varphi - 4\,\partial^2_{\nu\beta}\varphi\right] + \dots$$

$$= \frac{1}{c^2}\left[-\eta_{\nu\beta}\,\partial_\mu\partial^\mu\varphi - 2\,\partial^2_{\nu\beta}\varphi\right] + \dots$$

$$R^{\alpha}{}_{\beta} = g^{\alpha\nu}R_{\nu\beta} = e^{-2\varphi/c^2}\,\eta^{\alpha\nu}\,R_{\nu\beta} =$$

$$= -e^{-2\varphi/c^2}\,\frac{1}{c^2}\left[\delta^{\alpha}_{\beta}\,\partial_\mu\partial^\mu\varphi + 2\,\partial^\alpha\partial_\beta\varphi\right] + \dots$$

$$R = R^{\alpha}{}_{\alpha} = -\frac{1}{c^2}\,e^{-2\varphi/c^2}\left[4\,\partial_\mu\partial^\mu\varphi + 2\,\partial_\mu\partial^\mu\varphi\right] + \dots =$$

$$= -\frac{1}{c^2}\,e^{-2\varphi/c^2}\;6\;\partial_\mu\partial^\mu\varphi + \dots$$

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = -\frac{1}{c^2}\,\eta_{\alpha\beta}\,\partial_\mu\partial^\mu\varphi - \frac{2}{c^2}\partial^2_{\alpha\beta}\varphi \; -$$

$$-\frac{1}{2} e^{2\varphi/c^2} \eta_{\alpha\beta} \left(-\frac{1}{c^2} e^{-\frac{2\varphi}{c^2}} 6 \partial_\mu \partial^\mu \varphi + \dots \right) =$$

$$= -\frac{2}{c^2} \partial^2_{\alpha\beta} \varphi + \frac{2}{c^2} \eta_{\alpha\beta} \partial_\mu \partial^\mu \varphi =$$

$$= \frac{2}{c^2} \left[\eta_{\alpha\beta} \partial_\mu \partial^\mu \varphi - \partial^2_{\alpha\beta} \varphi \right] = G_{\alpha\beta},$$

where higher order terms ... are omitted in the $\varphi/c^2 \ll 1$ approxim.

2. $$ds^2 = e^{2\varphi/c^2}\eta_{\alpha\beta}\,dx^\alpha dx^\beta$$

$$\mathcal{L} = e^{2\varphi/c^2}\eta_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}$$

$$\frac{d}{d\lambda}\left[e^{\frac{2\varphi}{c^2}}\eta_{\alpha\beta}\dot{x}^\alpha\delta^\beta_\gamma + e^{2\varphi/c^2}\eta_{\alpha\beta}\dot{x}^\beta\delta^\alpha_\gamma\right]$$

$$= \partial_\gamma\varphi\,\frac{2}{c^2}e^{2\varphi/c^2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$\eta_{\alpha\beta}\ddot{x}^\alpha\delta^\beta_\gamma + \eta_{\alpha\beta}\ddot{x}^\beta\delta^\alpha_\gamma +$$

$$+ \frac{2}{c^2}\partial_\mu\varphi\,\delta^\beta_\gamma\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\mu + \frac{2}{c^2}\partial_\mu\varphi\,\eta_{\alpha\beta}\dot{x}^\beta\dot{x}^\mu\delta^\alpha_\gamma$$

$$= \partial_\gamma\varphi\,\frac{2}{c^2}e^{2\varphi/c^2}\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu$$

$$\cancel{2}\,\eta_{\gamma\beta}\ddot{x}^\beta + \frac{1}{c^2}\partial_\mu\varphi\,\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\mu\delta^\beta_\gamma +$$

$$+\frac{1}{c^2}\partial_\mu\varphi\,\eta_{\alpha\beta}\dot{x}^\beta\dot{x}^\mu\delta^\alpha_\gamma -$$

$$-\frac{1}{c^2}\partial_\gamma\varphi\,\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0$$

$$\ddot{x}^{\sigma} + \frac{1}{c^2}\partial_\mu \varphi \; \eta_{\alpha\beta}\, \dot{x}^{\alpha} \dot{x}^{\mu}\, \eta^{\sigma\beta} +$$

$$+ \frac{1}{c^2}\partial_\nu \varphi \; \eta_{\alpha\mu}\, \dot{x}^{\mu} \dot{x}^{\nu}\, \delta^{\alpha}_{\gamma}\, \eta^{\sigma\gamma} -$$

$$- \frac{1}{c^2}\partial_\gamma \varphi \; \eta_{\mu\nu}\, \dot{x}^{\mu} \dot{x}^{\nu}\, \eta^{\sigma\gamma} = 0$$

$$\ddot{x}^{\sigma} + \Big[\frac{1}{c^2}\partial_\mu \varphi \, \delta^{\sigma}_{\nu} + \frac{1}{c^2}\partial_\nu \varphi\, \delta^{\sigma}_{\mu} -$$

$$- \frac{1}{c^2}\partial_\gamma \varphi\, \eta^{\sigma\gamma}\, \eta_{\mu\nu} \Big] \dot{x}^{\mu} \dot{x}^{\nu} = 0$$

$$\ddot{x}^{\sigma} + \underbrace{\frac{1}{c^2}\Big[\partial_\mu \varphi\, \delta^{\sigma}_{\nu} + \partial_\nu \varphi\, \delta^{\sigma}_{\mu} - \eta_{\mu\nu}\, \partial^{\sigma}\varphi \Big]}_{= \Gamma^{\sigma}_{\mu\nu}} \dot{x}^{\mu} \dot{x}^{\nu} = 0$$

OK.

$$\Gamma^{\mu}_{\beta\nu} = \frac{1}{c^2}\Big[\partial_\nu \varphi\, \delta^{\mu}_{\beta} + \partial_\beta \varphi\, \delta^{\mu}_{\nu} -$$

$$- \partial^{\mu}\varphi\; \eta_{\beta\nu} \Big]$$

$$\Gamma^{\mu}_{\mu\nu} = \frac{1}{c^2}\Big[\partial_\nu\varphi\,\delta^{\mu}_{\mu} + \partial_\mu\varphi\,\delta^{\mu}_{\nu} - $$
$$- \partial^{\mu}\varphi\,\eta_{\mu\nu}\Big]$$

$$\partial_\mu\Gamma^{\mu}_{\beta\nu} = \frac{1}{c^2}\Big[\partial_\beta\partial_\nu\varphi + \partial_\nu\partial_\beta\varphi - $$
$$- \partial_\mu\partial^{\mu}\varphi\;\eta_{\beta\nu}\Big]$$

$$\partial_\beta\Gamma^{\mu}_{\mu\nu} = \frac{1}{c^2}\Big[\partial_\beta\partial_\nu\varphi\;\delta^{\mu}_{\mu} + \partial_\beta\partial_\mu\varphi\,\delta^{\mu}_{\nu}$$
$$- \partial_\beta\partial^{\mu}\varphi\;\eta_{\mu\nu}\Big]$$

$$R_{\nu\beta} = \frac{1}{c^2}\Big[2\partial_\beta\partial_\nu\varphi - \partial_\mu\partial^{\mu}\varphi\;\eta_{\beta\nu} - $$
$$- \partial_\beta\partial_\nu\varphi\;\delta^{\mu}_{\mu} - \partial_\beta\partial_\nu\varphi + \partial_\beta\partial_\nu\varphi\Big]$$
$$+ O(1/c^4)$$

$$R_{\nu\beta} = -\frac{1}{c^2}\left[2\,\partial_\nu\partial_\beta\varphi + \partial_\mu\partial^\mu\varphi\;\eta_{\nu\beta}\right]$$

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2}R\,g_{\alpha\beta}$$

$$\eta^{\lambda\nu}R_{\nu\beta} = -\frac{1}{c^2}\left[2\,\partial^\lambda\partial_\beta\varphi + \partial_\mu\partial^\mu\varphi\;\delta_\beta^\lambda\right]$$

$$R = -\frac{1}{c^2}\left[2\,\partial^\beta\partial_\beta\varphi + 4\,\partial_\mu\partial^\mu\varphi\right] =$$

$$= -\frac{6}{c^2}\,\partial_\mu\partial^\mu\varphi\,.$$

$$G_{\alpha\beta} = -\frac{1}{c^2}\left[2\,\partial_\alpha\partial_\beta\varphi + \partial_\mu\partial^\mu\varphi\;\eta_{\alpha\beta}\right] +$$

$$+\frac{1}{2}\,g_{\alpha\beta}\,\frac{6}{c^2}\,\partial_\mu\partial^\mu\varphi \quad \approx$$

$$g_{\alpha\beta} \longrightarrow \eta_{\alpha\beta} + O(1/c^2)$$

$$\approx -\frac{1}{c^2}\left[2\,\partial_\alpha\partial_\beta\varphi - 2\,\partial_\mu\partial^\mu\varphi\;\eta_{\alpha\beta}\right]$$

$$= \frac{2}{c^2}\left[\eta_{\alpha\beta}\,\partial_\mu\partial^\mu\varphi - \partial_\alpha\partial_\beta\varphi\right] \quad \text{OK.}$$

$$ds^2 = e^{2\varphi/c^2}\left(-c^2 dt^2 + dr^2 + r^2 d\theta^2 + \right.$$
$$\left. + r^2 \sin^2\theta \, d\phi^2\right)$$

in spher. coord.

$$\mathcal{L} = e^{2\varphi/c^2}\left(-c^2 \dot{t}^2 + \dot{r}^2 + r^2\dot{\theta}^2 + \right.$$
$$\left. + r^2 \sin^2\theta \, \dot{\phi}^2\right)$$

If $\varphi = \varphi(r)$ only ; (eq: $\theta = \pi/2$)

then for cyclic var t and ϕ :

$$\frac{d}{d\lambda}\left(-2c^2 e^{2\varphi(r)/c^2}\, \dot{t}\right) = \frac{\partial \mathcal{L}}{\partial t} = 0$$

$$\boxed{e^{2\varphi/c^2}\dot{t} = \alpha = \text{const}}$$

$$\frac{d}{d\lambda}\left(e^{2\varphi/c^2} \cdot 2r^2\dot{\phi}\right) = \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

$$\boxed{e^{2\varphi/c^2} r^2 \dot{\phi} = l = \text{const}}$$

Photons: $ds^2 = 0$

$$e^{2\varphi/c^2}\left(-c^2\dot{t}^2 + \dot{r}^2 + r^2\dot{\phi}^2\right) = 0$$

$\Rightarrow \quad \dot{r}^2 = c^2\dot{t}^2 - r^2\dot{\phi}^2$ same as Mink.

$$\left(\frac{dr}{d\phi}\right)^2 = c^2\frac{\dot{t}^2}{\dot{\phi}^2} - r^2 =$$

$$= c^2\frac{d^2}{\ell^2}r^4 - r^2$$

(note $e^{2\varphi/c^2}$ cancels in ratio)

With $u = 1/r$ this is

$$\left(\frac{du}{d\phi}\right)^2 = \frac{c^2 d^2}{\ell^2} - u^2$$

$$\pm\int d\varphi = \int \frac{du}{\sqrt{a^2 - u^2}} + \text{const},$$

$a \equiv c\,d/\ell$

$\Rightarrow \pm\phi = \arcsin\frac{u}{a} + \text{const}$

$\Rightarrow \quad u = \pm a \sin(\phi - \phi_0)$

$r \sin(\phi - \phi_0) = \pm 1/a$

this is $\quad y = \pm 1/a$ in Cartesian coord. $\Rightarrow$ Light is not deflected $\forall\ \varphi(r)$.

3. $$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu = R^\mu{}_{\nu\alpha\beta} V^\nu$$

$$\nabla_\beta V^\mu = \partial_\beta V^\mu + \Gamma^\mu_{\beta\sigma} V^\sigma$$

$$\nabla_\alpha (\nabla_\beta V^\mu) = \partial_\alpha (\nabla_\beta V^\mu) + \Gamma^\mu_{\alpha\sigma} \nabla_\beta V^\sigma -$$

$$- \Gamma^\sigma_{\alpha\beta} \nabla_\sigma V^\mu =$$

$$= \cancel{\partial_\alpha \partial_\beta V^\mu} + \underline{\partial_\alpha (\Gamma^\mu_{\beta\sigma}) V^\sigma} + \Gamma^\mu_{\beta\sigma} \partial_\alpha V^\sigma$$

$$+ \Gamma^\mu_{\alpha\sigma} \partial_\beta V^\sigma + \Gamma^\mu_{\alpha\sigma} \Gamma^\sigma_{\beta\varkappa} V^\varkappa -$$

$$- \cancel{\Gamma^\sigma_{\alpha\beta} \partial_\sigma V^\mu} - \Gamma^\sigma_{\alpha\beta} \Gamma^\mu_{\sigma\rho} V^\rho$$

$$\nabla_\beta \nabla_\alpha V^\mu = \cancel{\partial_\beta \partial_\alpha V^\mu} + \underline{\partial_\beta (\Gamma^\mu_{\alpha\sigma}) V^\sigma} +$$

$$+ \Gamma^\mu_{\alpha\sigma} \partial_\beta V^\sigma + \Gamma^\mu_{\beta\sigma} \partial_\alpha V^\sigma + \Gamma^\mu_{\beta\sigma} \Gamma^\sigma_{\alpha\varkappa} V^\varkappa$$

$$- \cancel{\Gamma^\sigma_{\beta\alpha} \partial_\sigma V^\mu} - \Gamma^\sigma_{\beta\alpha} \Gamma^\mu_{\sigma\rho} V^\rho$$

$$(\nabla_\alpha \nabla_\beta - \nabla_\beta \nabla_\alpha) V^\mu =$$

$$= \left(\partial_\alpha \Gamma^\mu_{\beta\sigma} - \partial_\beta \Gamma^\mu_{\alpha\sigma}\right) V^\sigma +$$

$$+ \left(\Gamma^\mu_{\alpha\sigma} \Gamma^\sigma_{\beta\varkappa} - \Gamma^\mu_{\beta\sigma} \Gamma^\sigma_{\alpha\varkappa}\right) V^\varkappa +$$

$$+ \cancel{\left(\Gamma^\sigma_{\beta\alpha} \Gamma^\mu_{\sigma\rho} - \Gamma^\sigma_{\alpha\beta} \Gamma^\mu_{\sigma\rho}\right) V^\rho} =$$

$$= \left(\partial_\alpha \Gamma^\mu_{\beta\sigma} - \partial_\beta \Gamma^\mu_{\alpha\sigma} + \Gamma^\mu_{\alpha\varkappa} \Gamma^\varkappa_{\beta\sigma} - \right.$$

$$\left. - \Gamma^\mu_{\beta\varkappa} \Gamma^\varkappa_{\alpha\sigma}\right) V^\sigma =$$

$$= R^\mu{}_{\sigma\alpha\beta} V^\sigma . \qquad \text{OK}$$

Killing vector U^μ:

$$\nabla_\mu U_\nu + \nabla_\nu U_\mu = 0$$

$$W^\mu \equiv U^\nu \nabla_\nu V^\mu - V^\nu \nabla_\nu U^\mu$$

$$\nabla_\mu W^\nu + \nabla_\nu W^\mu = 0 \quad (?)$$

$$\nabla_\mu \left(U^\rho \nabla_\rho V^\nu\right) + \nabla_\nu \left(U^\sigma \nabla_\sigma V^\mu\right) -$$

$$- \nabla_\mu \left(V^\rho \nabla_\rho U^\nu\right) - \nabla_\nu \left(V^\varkappa \nabla_\varkappa U^\mu\right)$$

$$= \boxed{\cancel{\left(\nabla_\mu U^\rho\right)\left(\nabla_\rho V^\nu\right)}}\checkmark + U^\rho \nabla_\mu\left(\nabla_\rho V^\nu\right) +$$

$$+ \underline{\cancel{\left(\nabla_\nu U^\sigma\right)\left(\nabla_\sigma V^\mu\right)}} + U^\sigma \nabla_\nu\left(\nabla_\sigma V^\mu\right) -$$

$$- \underline{\cancel{\left(\nabla_\mu V^\rho\right)\left(\nabla_\rho U^\nu\right)}} - V^\rho \nabla_\mu\left(\nabla_\rho U^\nu\right) -$$

$$- \cancel{\left(\nabla_\nu V^\varkappa\right)\left(\nabla_\varkappa U^\mu\right)}\checkmark - V^\varkappa \nabla_\nu\left(\nabla_\varkappa U^\mu\right)$$

using that U^μ, V^μ are Killing vectors

$$= U^\rho \nabla_\mu\left(\nabla_\rho V^\nu\right) + U^\rho \nabla_\nu\left(\nabla_\rho V^\mu\right) -$$

$$- V^\rho \nabla_\mu\left(\nabla_\rho U^\nu\right) - V^\rho \nabla_\nu\left(\nabla_\rho U^\mu\right) =$$

$$= \underline{U^\rho \nabla_\rho \nabla_\mu V^\nu} + \underline{U^\rho \nabla_\rho \nabla_\nu V^\mu} -$$

$$- \underline{V^\rho \nabla_\rho \nabla_\mu U^\nu} - \underline{V^\rho \nabla_\rho \nabla_\nu U^\mu} +$$

$$+ u^\rho R^\nu_{\;\gamma\mu\rho} V^\gamma + u^\rho R^\mu_{\;\gamma\nu\rho} V^\gamma -$$

$$- V^\rho R^\nu_{\;\gamma\mu\rho} u^\gamma - V^\rho R^\mu_{\;\gamma\nu\rho} u^\gamma =$$

$$= \underline{u^\rho R^\nu_{\;\gamma\mu\rho} V^\gamma} + \underline{u^\rho R^\mu_{\;\gamma\nu\rho} V^\gamma} -$$

$$- \underline{u^\rho R^\nu_{\;\rho\mu\gamma} V^\gamma} - \underline{u^\rho R^\mu_{\;\rho\nu\gamma} V^\gamma} =$$

$= 0$ since $R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}$.

OK

$$T_{\mu\nu} = \nabla_\mu \nabla_\nu \phi - g_{\mu\nu} \nabla^\sigma \nabla_\sigma \phi$$

$$\nabla^\mu T_{\mu\nu} - ?$$

$$T_{\mu\nu} = \nabla_\mu (\partial_\nu \phi) - g_{\mu\nu} \nabla^\sigma (\partial_\sigma \phi)$$

$$\nabla^\mu T_{\mu\nu} = \nabla^\mu (\nabla_\mu \partial_\nu \phi) - \nabla_\nu (\nabla^\sigma \partial_\sigma \phi) =$$

$$= \nabla^\mu (\nabla_\mu \partial_\nu \phi) - \nabla^\sigma (\nabla_\nu \partial_\sigma \phi) -$$

$$- R^\sigma{}_{\varkappa\nu\sigma} \partial^\varkappa \phi =$$

$$= \underbrace{\nabla^\mu (\nabla_\mu \partial_\nu \phi) - \nabla^\mu (\nabla_\nu \partial_\mu \phi)}_{\nabla^\mu ([\nabla_\mu \nabla_\nu] \phi) = 0,\ \text{if no torsion}} +$$

$$+ \underbrace{R^\sigma{}_{\varkappa\sigma\nu} \partial^\varkappa \phi}$$

$$R_{\varkappa\nu} \partial^\varkappa \phi = R_\nu{}^\sigma \nabla_\sigma \phi .$$

$$\nabla^\mu T_{\mu\nu} = R_\nu{}^\sigma \nabla_\sigma \phi \qquad \text{OK.}$$

3-6

$$\nabla^\mu T_{\mu\nu} = 0$$

$$\nabla^\nu \left(T_{\mu\nu} V^\mu\right) = \overbrace{\left(\nabla^\nu T_{\mu\nu}\right)}^{=0} V^\mu +$$

$$+ \underbrace{T_{\mu\nu}}_{\text{sym}} \underbrace{\nabla^\nu V^\mu}_{\text{antisym.}} = 0$$

$$\left(\nabla^\nu V^\mu = -\nabla^\mu V^\nu\right)$$

OK

4. $$\frac{\dot{a}^2}{a^2} \equiv H^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2} + \frac{c^2\Lambda}{3}$$

$$\begin{pmatrix}\Lambda = 0\\ k = 0\end{pmatrix} \qquad \dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + P/c^2\right) = 0$$

$$P = w\rho c^2 : \quad w = -1 \ (\text{infl.})$$
$$w = 1/3 \ (\text{rad})$$
$$w = 0 \ (\text{matter})$$

1) $w = -1$: $\rho = \text{const} \Rightarrow H^2 = \frac{8\pi G}{3}\rho = \text{const}$

$$a(t) = e^{Ht} - 1$$
$$a_1 = a(t_1) = e^{Ht_1} - 1$$

2) $w = 1/3$: $\dot{\rho}/\rho = -4\,\dot{a}/a$

$$\Rightarrow \rho = \rho_0^R/a^4 \qquad \frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho_0^R}{3a^4} = H^2$$

$$\Rightarrow a(t) \sim \sqrt{t}$$

3) $w = 0 \Rightarrow \rho = \rho_0^M/a^3 \Rightarrow H^2 = \frac{8\pi G\rho_0^M}{3a^3}$

$$\Rightarrow a(t) \sim t^{2/3}$$

Deceleration parameter

$$q = -\frac{a\ddot{a}}{\dot{a}^2}$$

$$q = \frac{1}{2}\frac{8\pi G\rho}{3H^2}(1+3w) \cancel{- \frac{\Lambda c^2}{3H^2}}$$

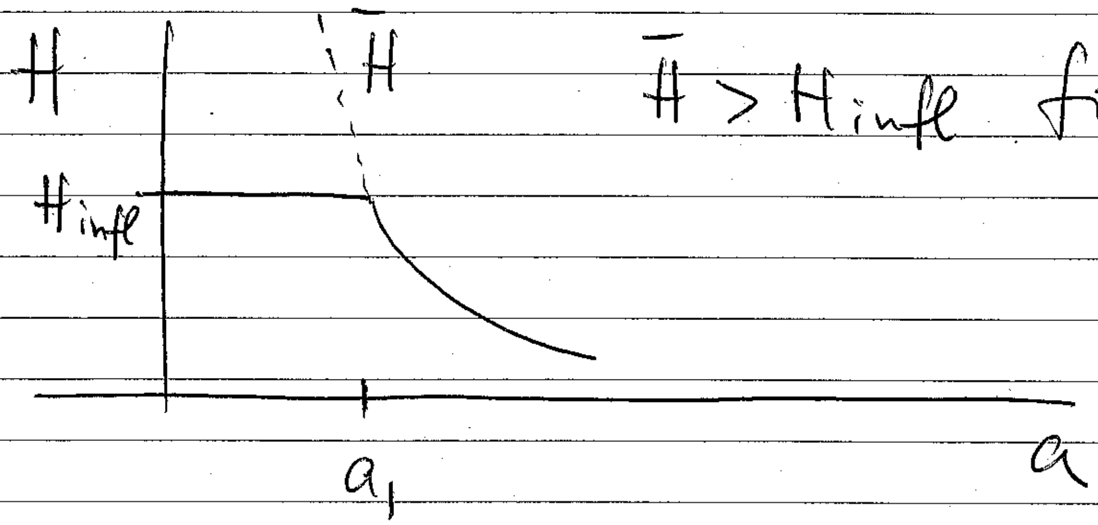
1) $w=-1$: $q = \frac{1}{2}(-3+1) = -1$

2) $w = 1/3$: $q = 1$

3) $w = 0$ $\quad q = 1/2$

Inflat. phase : $H(a) = \text{const}$

Rad : $H \sim 1/a^2$



$\bar{H} > H_{infl}$ for $a < a_1$.

<u>Age</u>: $\frac{\dot{a}}{a} = H \rightarrow \frac{da}{aH} = dt$

$$T = \int_{a_{in}}^{a_0} \frac{da}{aH(a)}$$ with different $H(a)$ ~~between~~

a_{in} and a_1, a_1 and a_2, a_2 and $a_0 = 1$.

$$T_{infl} = \int_{a_{in}}^{a_1} \frac{da}{aH} = \frac{1}{H} \ln a_1/a_{in} \rightarrow \infty$$

for $a_{in} \rightarrow 0$

$$T_{non\text{-}infl} = \int_{a_{in}}^{a_1} \frac{da\, a^2}{a\, \overline{H}} = \frac{1}{2\overline{H}} \left(a_1^2 - a_{in}^2\right)$$

(finite)

But maybe need to clarify a_{in} here.

<u>Horizons</u>:

Particle horizon $d_H = c\, a(t_*) \int_0^{t_*} \frac{dt}{a}$

1) $a \sim e^{Ht}$

$$d_H = c e^{Ht_*} \int_0^{t_*} e^{-Ht} dt \sim \frac{c}{H} e^{Ht_*}.$$

2) $a \sim \sqrt{t}$

$$d_H = 2c\sqrt{t_*}\left(\sqrt{t_*} - \sqrt{t_0}\right) \sim 2ct_*.$$

3) $d_H \sim 3ct_*^{2/3}\left(t_*^{1/3} - t_0^{1/3}\right) \sim 3ct_*.$

$\left(a \sim t^{2/3}\right)$

So $d_{phys} = a(t)\, d_{com.}$ comparing to d_H:

1) $\frac{d_{phys}}{d_H} \sim \frac{H a \cdot d_c}{c e^{Ht}} \sim \frac{H}{c} d_c \sim \text{const}$

2) $\frac{d_{phys}}{d_H} \sim \frac{\sqrt{t}\, d_c}{2ct} \sim \frac{1}{\sqrt{t}} \to 0$

3) $\frac{d_{phys}}{d_H} \sim \frac{t^{2/3} d_c}{3ct} \sim \frac{1}{t^{1/3}} \to 0$

In 2) and 3) d_H becomes much larger

than the phys scale with time
revealing disconnected patches
In infl. scenario $d_{phys} \sim d_H$
all the time, everything stays
connected.