University of Oxford

Physics Department

# GENERAL RELATIVITY AND COSMOLOGY

EXAM PAPER

2011

## SOLUTION NOTES

by Andrei Starinets

andrei.starinets@physics.ox.ac.uk

## Section V. (General relativity and cosmology)

1. The space time metric around the Earth is

$$ds^2 = -c^2\alpha(r)dt^2 + [\alpha(r)]^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2$$

with $\alpha(r) = 1 - \frac{2GM}{rc^2}$ and where $M$ is the mass of the Earth, $R$ is its radius and $\theta = 0$ corresponds to the North Pole. Geodesics in this space time satisfy:

$$\begin{aligned}
\frac{d}{d\lambda}\left(2c^2\alpha\dot{t}\right) &= 0 \\
\frac{d}{d\lambda}\left(2r^2\dot{\theta}\right) - 2r^2\sin\theta\cos\theta\dot{\phi}^2 &= 0 \\
\frac{d}{d\lambda}\left(2r^2\sin^2\theta\dot{\phi}\right) &= 0 \\
\frac{d}{d\lambda}\left(\frac{2\dot{r}}{\alpha}\right) - c^2\alpha'\dot{t}^2 + \frac{\alpha'}{\alpha^2}\dot{r}^2 + 2r\dot{\theta}^2 + 2r\sin^2\theta\dot{\phi}^2 &= 0
\end{aligned}$$

where $\lambda$ is an affine parameter and $\alpha' = \frac{d\alpha}{dr}$. Consider Satellite A with a circular orbit, $\dot{r} = 0$, around the Earth along a line of constant $\dot{\phi} = 0$. Show that we are allowed to choose $\lambda = t$ and that

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{GM}{r^3}\,.$$ [6]

Consider an observation station on the surface of the Earth at the North Pole. Find the proper time elapsed on Satellite A during an interval of time $\Delta t$ as compared to the proper time elapsed on the observation station. Show that the ratio between the two can be approximated by

$$\frac{\Delta\tau_{\text{Satellite A}}}{\Delta\tau_{\text{North Pole}}} \simeq 1 + \frac{GM}{c^2R} - \frac{3}{2}\frac{GM}{c^2r}\,.$$ [10]

Now compare the proper time of a geostationary Satellite B on a circular orbit with $\theta = \pi/2$ with that on an observation station sitting on the equator right below it. Show that the ratio between the two proper times can be approximated by

$$\frac{\Delta\tau_{\text{Satellite B}}}{\Delta\tau_{\text{Equator}}} = 1 + \frac{GM}{c^2R} - \frac{GM}{c^2r}\,.$$ [5]

Explain the difference between the two results. [4]

not quite right

**2.** Consider the metric

$$ds^2 = -c^2dt^2 + a^2(t)(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)\,.$$

Write down the Lagrangian for the geodesic equations in terms of an affine parameter $\lambda$. [3]

Show that the geodesic equations on this space time are

$$\begin{aligned}
\frac{d}{d\lambda}\left(-2c^2\dot{t}\right) &= +2a\frac{da}{dt}(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) \\
\frac{d}{d\lambda}\left(2a^2\dot{r}\right) &= 2a^2r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \\
\frac{d}{d\lambda}\left(2a^2r^2\dot{\theta}\right) &= 2a^2r^2\sin\theta\cos\theta\dot{\phi}^2 \\
\frac{d}{d\lambda}\left(2a^2r^2\sin^2\theta\dot{\phi}\right) &= 0
\end{aligned}$$

where over-dots correspond to derivatives taken with respect to $\lambda$. Use these geodesic equations to write down all the non-zero connection coefficients for this space time. [10]

Now consider a geodesic along the radial direction and set $\theta = \pi/2$ and $\phi = 0$. Show that the geodesic equations can be solved to give

$$\begin{aligned}
\dot{r} &= \frac{\alpha}{a^2} \\
c^2\dot{t}^2 &= \frac{\alpha^2}{a^2} + \beta
\end{aligned}$$

where $\alpha$ and $\beta$ are integration constants. Show that for a massless particle we have $\beta = 0$. [6]

Show that, if you choose $a(t) = \exp[H(t - t_0)]$ (where $H$ is constant), the solution to the geodesic equations of a massless particle, emitted at time $t_e$ from a distance $r_e$ from the origin, is

$$\begin{aligned}
\exp[H(t - t_0)] &= \frac{H}{c}(\alpha\lambda + \epsilon) \\
r &= \frac{c^2}{H^2}\left[-\frac{1}{\alpha\lambda + \epsilon} + \delta\right]\,.
\end{aligned}$$

Combine these expression to show that at $t = t_0$

$$H a(t_0)\, r_e = cz$$

with $1 + z = 1/R(t_e)$. [6]

$a(t_e)$

**3.** The metric for a closed universe is given by

$$ds^2 = -c^2dt^2 + a^2(t)\left[\frac{dr^2}{1-(r/R)^2} + r^2d\theta^2 + r^2\sin^2\theta d\phi^2\right]$$

where the scale factor satisfies the FRW equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho + \frac{\Lambda c^2}{3} - \frac{c^2}{(Ra)^2}\,.$$

Here $\Lambda$ is the cosmological constant and $\rho$ is the energy density of dust. Draw a sketch showing how the three distinct contributions on the right hand side of the FRW equation evolve with time and the order in which each of them dominates. [4]

Consider $\Lambda = 0$ and assume that today, at $t = t_0$, we have that $\rho$ is just very slightly larger than $\frac{3c^2}{8\pi G(Ra)^2}$. How will this Universe evolve in the future? Give a rough estimate, in terms of $t_0$, for how long you expect this Universe to last until $a$ collapses to 0. [6]

Considering a radial geodesic for a photon, find the distance travelled as function of physical time, $r(t)$, as a function of the comoving distance,

$$D_c \equiv \int_0^t \frac{cdt'}{a(t')}\,.$$

What is the furthest distance a photon can travel in this universe? [6]

Now consider $\Lambda \neq 0$. Use the Raychauduri equation to find a static solution of the FRW equations. Show that in this Universe we have a particular value of the energy density, $\rho_E$, given by

$$\rho_E \equiv \frac{\Lambda c^2}{4\pi G}\,,$$

and that the scale factor is given by

$$a = \sqrt{\frac{1}{R^2\Lambda}}\,.$$

[5]

Consider a universe for which $\rho$ is just slightly larger than $\rho_E$. What is $\ddot{a}$ and how does this universe evolve? Consider now a universe for which $\rho$ is just slightly smaller than $\rho_E$. What is $\ddot{a}$ and how does this universe evolve? Given what you found, do you think the static universe is stable? [4]

4. Consider a flat, homogenous and isotropic universe with scale factor $a(t)$ satisfying the FRW equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}(\rho_{\rm M} + \rho_{\rm R}) ,$$

where $\rho_{\rm M}$ is the energy density in dust and $\rho_{\rm R}$ is the energy density in radiation. How do $\rho_{\rm M}$ and $\rho_{\rm R}$ depend on $a$ ? Find $a$ as a function of $t$ at early times (when $\rho_{\rm R}$ dominates) and at late times (when $\rho_{\rm M}$ dominates) and ensure that that they match at equality, $t_{\rm eq}$, when $\rho_{\rm M} = \rho_{\rm R}$. (Pick your constant of integration such that $a = 1$ today, at $t_0$.) [6]

Find an expression for the horizon size when $t < t_{\rm eq}$. Show that it takes the form

$$r_h(z) = \frac{c}{H_0}\frac{\sqrt{1+z_{\rm eq}}}{(1+z)^2} ,$$

where the redshift is defined through $1 + z = \frac{1}{a}$, $z_{\rm eq}$ is the redshift at equality and $H_0$ is the Hubble constant. [7]

The number density of photons in the Universe is given by

$$n_\gamma \simeq 0.486\frac{k_B T_0}{\hbar c}\frac{1}{a^3} \simeq \frac{8.3 \times 10^8}{a^3}\,{\rm m}^{-3}$$

where $T_0 = 2.73\,{\rm K}$. Explain, using rough arguments, why this expression arises from the assumption of thermal equilibrium in the early Universe. What conditions in the early Universe ensure that there is thermal equilibrium? [5]

Find an expression for the total number of photons within one cosmological horizon, $N_\gamma$ as a function of redshift for early times, assuming $z > z_{\rm eq}$ or $t < t_{\rm eq}$. Show that $N_\gamma \to 0$ as $z \to \infty$. Find an expression for the value of $z$ when $N_\gamma = 1$ in terms of $N_\gamma(t_0)$. Explain why your result isn't compatible with the assumption of thermal equilibrium or with a homogeneous and isotropic Universe at very early times. [7]

$$n_\gamma = 0.244\left(\frac{k_B T_0}{\hbar c}\right)^3\frac{1}{a^3} \simeq \frac{4.1\cdot 10^8}{a^3}\,m^{-3}.$$

1. For a circular orbit, $r = r_0 = \text{const}$

$\Rightarrow \frac{d}{d\lambda}\left(2c^2 \alpha \dot{t}\right) = 0$ implies $\ddot{t} = 0 \Rightarrow$

$t = C_1 \lambda + C_2$ and the constants can be chosen so that $t = \lambda$. With $\dot{r} = 0$, $\dot{\phi} = 0$ and $\dot{t} = \frac{dt}{d\lambda} = 1$ the last eq. of motion gives

$$\dot{\theta}^2 = \left(\frac{d\theta}{d\lambda}\right)^2 = \left(\frac{d\theta}{dt}\right)^2 = \frac{c^2 \alpha'(r_0)}{2 r_0} = \frac{GM}{r_0^3}.$$

To find the proper time as a function of $t$, use $-c^2 d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu$, where the r.h.s. is specified by the coordinates $r, t, \theta, \varphi$ at the point of interest (e.g. the position of the satellite or the observation station).

For the satellite $A$, $r = r_0$, $\dot{\theta}^2 = GM/r_0^3$, $\dot{\phi} = 0 \Rightarrow -c^2 d\tau_A^2 = -c^2 \alpha(r_0) dt^2 + r_0^2 d\theta^2 =$

$$= -c^2 \alpha(r_0)\, dt^2 + \frac{GM}{r_0}\, dt^2 = -c^2\left(1 - \frac{3GM}{r_0 c^2}\right) dt^2$$

For the North Pole, $-c^2 d\tau_{NP}^2 = -c^2 \alpha(R_E) dt^2$

Thus,

$$\frac{\Delta\tau_A}{\Delta\tau_{NP}} = \sqrt{\frac{1-\frac{3GM}{c^2 r_0}}{1-\frac{2GM}{c^2 R_E}}} \approx 1 - \frac{3\,GM}{2\,c^2 r_0} + \frac{GM}{c^2 R_E},$$

since $GM/r_0 c^2 \ll 1$, $GM/R_E c^2 \ll 1$, and

$(1-x)^\alpha = 1 - \alpha x + \dots$.

Now consider satellite B on a geostationary circular orbit around the equator. Strictly speaking, the Schwarzschild metric does not describe a rotating spherically symmetric body (one needs the Kerr metric for that) but the difference is not very significant for relatively slow rotation. So we use Schwarzschild metric and two moving points: the satellite B at some $r = r_0$ and the observation station at $r = R_E$:

$$-c^2 d\tau_B^2 = -c^2\left(1 - \frac{3GM}{c^2 r_0}\right) dt^2$$

The observation point on the equator should have the same angular velocity as the satellite B, i.e. for this point $\dot{\theta}^2 = GM/r_B^3$. Then $-c^2 d\tau_{Eq}^2 =$

$$= -c^2\left(1-\frac{2GM}{c^2R_E}\right)dt^2 + R_E^2\, d\theta^2 =$$

$$= -c^2\left(1-\frac{2GM}{c^2R_E}\right)dt^2 + R_E^2\,\frac{GM}{r_B^3}\,dt^2 .$$

$$\frac{\Delta\tau_B}{\Delta\tau_{Eq}} = \frac{\sqrt{1-3GM/c^2r_B}}{\sqrt{1-\frac{2GM}{c^2R_E}-\frac{GM}{c^2R_E}\left(\frac{R_E}{r_B}\right)^3}} \simeq$$

$$\simeq 1 - \frac{3GM}{2c^2r_B} + \frac{GM}{c^2R_E} + \frac{GM}{c^2R_E}\,\frac{1}{2}\left(\frac{R_E}{r_B}\right)^3 + \dots =$$

$$= 1 + \frac{R_S}{2R_E}\left\{1 - \frac{3}{2}\,\frac{R_E}{r_B} + \frac{1}{2}\left(\frac{R_E}{r_B}\right)^3\right\} + \dots ,$$

where $R_S = \frac{2GM}{c^2} \simeq 8.87\cdot 10^{-3}$ м, $R_E \simeq 6400$ km, $r_B \sim 36000$ km. The last term is a $\sim 10^{-3}$ correction to the main result which is qualitatively the same as in case A.

The corresponding equation in the formulation of the problem is not correct.

Note that it would be incorrect to write „$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2$" in this case: we have $g_{\mu\nu}\,dx^\mu dx^\nu = -c^2 d\tau^2$ and so $g_{\mu\nu}\dfrac{dx^\mu}{d\lambda}\dfrac{dx^\nu}{d\lambda} = -c^2\left(\dfrac{d\tau}{d\lambda}\right)^2 \Rightarrow$

$$\mathcal{L} = g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = -c^2\left(\frac{d\tau}{d\lambda}\right)^2 = -c^2\left(\frac{d\tau}{dt}\right)^2 =$$

$$= -c^2\left(1 - \frac{3GM}{c^2 r_0}\right) \quad \text{for } r = r_0 .$$

2 $ds^2 = -c^2 dt^2 + a^2(t)\left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right)$

The coordinates: $x^0 = ct, \; r, \; \theta, \; \varphi$.

One can use the Lagrangian $\mathcal{L} = g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu$

and the Euler-Lagrange equations

$$\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{x}^\mu} = \frac{\partial \mathcal{L}}{\partial x^\mu}$$

to find the geodesics:

$$\mathcal{L} = -c^2 \dot{t}^2 + a^2(t)\dot{r}^2 + a^2(t) r^2 \dot{\theta}^2 + a^2 r^2 \sin^2\theta \, \dot{\varphi}^2$$

$$-2c^2 \ddot{t} = 2a \frac{da}{dt}\left(\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2\theta \, \dot{\varphi}^2\right)$$

$$\frac{d}{d\lambda}\left(2a^2 \dot{r}\right) = 2 r a^2 \left(\dot{\theta}^2 + \sin^2\theta \, \dot{\varphi}^2\right)$$

$$\frac{d}{d\lambda}\left(2\dot{\theta} a^2 r^2\right) = 2 \sin\theta \cos\theta \, a^2 r^2 \dot{\varphi}^2$$

$$\frac{d}{d\lambda}\left(2a^2 r^2 \sin^2\theta \, \dot{\varphi}\right) = 0$$

Since $\lambda$ is an affine parameter, the geodesics eq has the form

$$\ddot{x}^\alpha = -\Gamma^\alpha_{\beta\gamma} \dot{x}^\beta \dot{x}^\gamma$$

It will be more convenient to use $t$ rather than $x^0 = ct$ (this is fine as long as you are consistent with your declaration in all formulas.

Then $\Gamma^t_{rr} = \frac{a}{c^2}\frac{da}{dt} = \frac{a^2}{c^2}H(t)$

(with $x^0$, it would be $\Gamma^0_{11} = a\frac{da}{dx^0}$)

$$\Gamma^t_{\theta\theta} = \frac{a^2}{c^2}r^2 H(t) \qquad \Gamma^t_{\varphi\varphi} = \frac{a^2}{c^2}r^2\sin^2\theta\, H(t)$$

Here $H(t) = \frac{1}{a}\frac{da}{dt}$.

Next, $a^2\ddot{r} + 2a\frac{da}{d\lambda}\dot{r} = ra^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\varphi}^2)$

$\frac{da}{d\lambda} = \frac{da}{dt}\dot{t} \Rightarrow \Gamma^r_{tr} = \Gamma^r_{rt} = H(t)$

$$\Gamma^r_{\theta\theta} = -r \qquad \Gamma^r_{\varphi\varphi} = -r\sin^2\theta$$

The third equation,

$$\ddot{\theta} = \sin\theta\cos\theta\,\dot{\varphi}^2 - \frac{2}{r}\dot{\theta}\dot{r} - 2H\dot{\theta}\dot{t}$$

gives

$$\Gamma^\theta_{\varphi\varphi} = -\sin\theta\cos\theta \qquad \Gamma^\theta_{r\theta} = \Gamma^\theta_{\theta r} = 1/r$$

$$\Gamma^\theta_{\theta t} = \Gamma^\theta_{t\theta} = H$$

The last eq.,

$$\ddot{\varphi} = -2H\dot{t}\dot{\varphi} - \frac{2}{r}\dot{r}\dot{\varphi} - 2\cot\theta\,\dot{\varphi}\,\dot{\theta},$$

gives $\Gamma^{\varphi}_{t\varphi} = \Gamma^{\varphi}_{\varphi t} = H$, $\Gamma^{\varphi}_{r\varphi} = \Gamma^{\varphi}_{\varphi r} = \frac{1}{r}$,

$\Gamma^{\varphi}_{\varphi\theta} = \Gamma^{\varphi}_{\theta\varphi} = \cot\theta$.

There are 12 different non-vanishing connection coefficients:

$$\Gamma^{t}_{rr} = \frac{a^2 H}{c^2} \qquad \Gamma^{t}_{\theta\theta} = \frac{a^2 r^2}{c^2} H \qquad \Gamma^{t}_{\varphi\varphi} = \frac{a^2 r^2 \sin^2\theta}{c^2} H$$

$$\Gamma^{r}_{\theta\theta} = -r \qquad \Gamma^{r}_{\varphi\varphi} = -r\sin^2\theta \qquad \Gamma^{r}_{rt} = H$$

$$\Gamma^{\theta}_{\varphi\varphi} = -\sin\theta\cos\theta \qquad \Gamma^{\theta}_{r\theta} = 1/r \qquad \Gamma^{\theta}_{\theta t} = H$$

$$\Gamma^{\varphi}_{t\varphi} = H \qquad \Gamma^{\varphi}_{r\varphi} = 1/r \qquad \Gamma^{\varphi}_{\theta\varphi} = \cot\theta$$

For a radial geodesic, $\dot{\theta} = 0$, $\dot{\varphi} = 0$, the two non-trivial e.o.m. are

$$\frac{d}{d\lambda}(a^2\dot{r}) = 0 \Rightarrow \dot{r} = \frac{\alpha}{a^2}, \quad \alpha = \text{const}$$

and $$\frac{d}{d\lambda}\dot{t} = -\frac{a}{c^2}\frac{da}{dt}\dot{r}^2 = -\frac{a}{c^2}\frac{da}{dt}\frac{\alpha^2}{a^4}$$

Since $\frac{da}{dt} = \frac{da}{d\lambda}\frac{1}{\dot t}$, the eq. can be written

as $$\dot t \ddot t = -\frac{\alpha^2}{c^2}\frac{da}{d\lambda}\frac{1}{a^3}$$

or $$\frac{1}{2}\frac{d}{d\lambda}\left(\dot t^2\right) = -\frac{\alpha^2}{c^2}\frac{d}{d\lambda}\left(-\frac{1}{2a^2}\right)$$

$$\Rightarrow \frac{d}{d\lambda}\left(\dot t^2 - \frac{\alpha^2}{c^2a^2}\right) = 0$$

$$\Rightarrow \dot t^2 = \frac{\alpha^2}{c^2a^2} + \frac{\beta}{c^2}, \quad \beta = \text{const}.$$

For a massless particle, $ds^2 = 0 \Rightarrow$

$-c^2 dt^2 + a^2 dr^2 = 0 \Rightarrow \dot t^2 = \frac{a^2}{c^2}\dot r^2$

$\Rightarrow \beta = 0$ in the eq. above.

Now choosing $a(t) = \exp(H(t-t_0))$, where $H = \text{const}$, we find $(\beta = 0)$:

$$\dot t^2 = \frac{\alpha^2}{c^2} e^{-2H(t-t_0)}$$

$$\Rightarrow e^{H(t-t_0)} dt = \pm\frac{\alpha}{c} d\lambda$$

$$\pm e^{H(t-t_0)} = \frac{\alpha H}{c}\lambda + \text{const} \equiv \frac{H}{c}(\alpha\lambda + \varepsilon).$$

Also, $\dot{r} = \frac{\alpha}{a^2} = \alpha e^{-2H(t-t_0)} =$

$$= \frac{\alpha c^2}{H^2} \frac{1}{(\alpha\lambda + \varepsilon)^2} \Rightarrow$$

$$\Rightarrow r = \frac{\alpha c^2}{H^2} \int \frac{d\lambda}{(\alpha\lambda + \varepsilon)^2} = \frac{\alpha c^2}{H^2}\left(-\frac{1}{\alpha\lambda+\varepsilon}\frac{1}{\alpha}\right) +$$

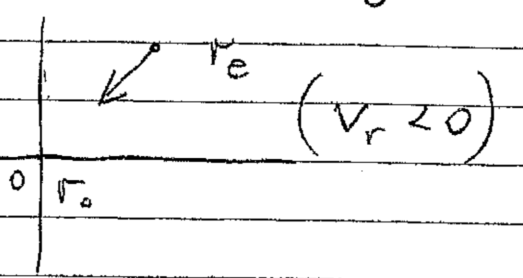
$$+ \text{const} \equiv \frac{c^2}{H^2}\left(-\frac{1}{\alpha\lambda+\varepsilon} + \delta\right).$$

One has to be careful with the choice of the sign. Assuming we want $\lambda$ to increase with $t$ increasing (this is a matter of convenience), we have to choose

$$-e^{H(t-t_0)} = \frac{H}{c}(\alpha\lambda + \varepsilon).$$

Here $\alpha < 0$ to ensure the radial velocity of the photon has the correct sign:

$\dot{r} = \frac{\alpha}{a^2} < 0$



Then for $r_e = r(t_e)$ we have

$$r_e = \frac{c}{H} e^{-H(t_e - t_0)} + r(t_0) - c/H$$

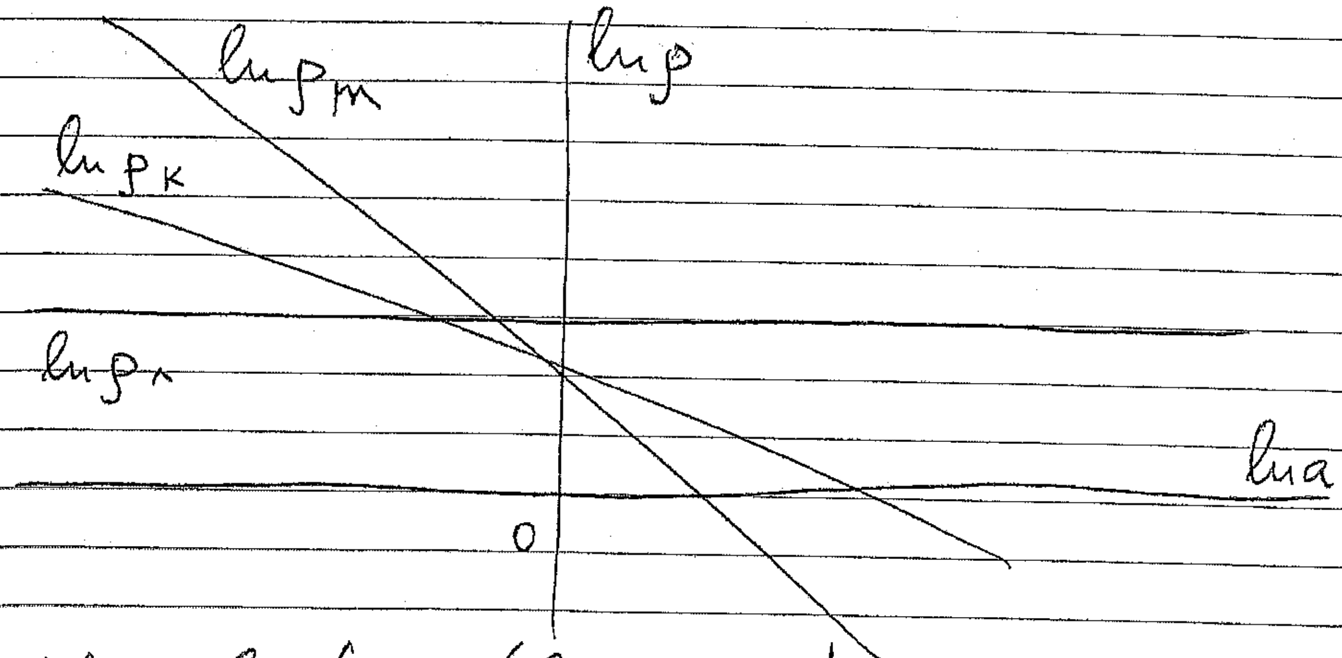
or $$H(r_e - r_0) = c\left(\frac{1}{a(t_e)} - 1\right) = cz.$$

Note that $a(t_0) = 1$ here and $r_0 = 0$ is the origin, so $H a(t_0) r_e = cz$.

3. For dust (cold matter) we have $\rho_M = \rho_0 a_0^3 / a^3$. The three contributions on the r.h.s. can be written as $\rho_M + \rho_\Lambda + \rho_k =$
$= \frac{\Omega_M}{a^3} + \Omega_\Lambda + \frac{\Omega_k}{a^2}$ showing their explicit ~~dependence on $a(t)$~~. Then

$$\ln \rho_M = \ln \Omega_M - 3 \ln a$$
$$\ln \rho_k = \ln \Omega_k - 2 \ln a$$
$$\ln \rho_\Lambda = \ln \Omega_\Lambda$$

It is convenient to sketch $\ln \rho$ vs $\ln a$ with e.g. $a(t_0) = 1$ and $\ln a_0 = 0$




At early time ($\ln a \to -\infty$) $\rho_M > \rho_k > \rho_\Lambda$, at late time $\rho_\Lambda$ dominates.

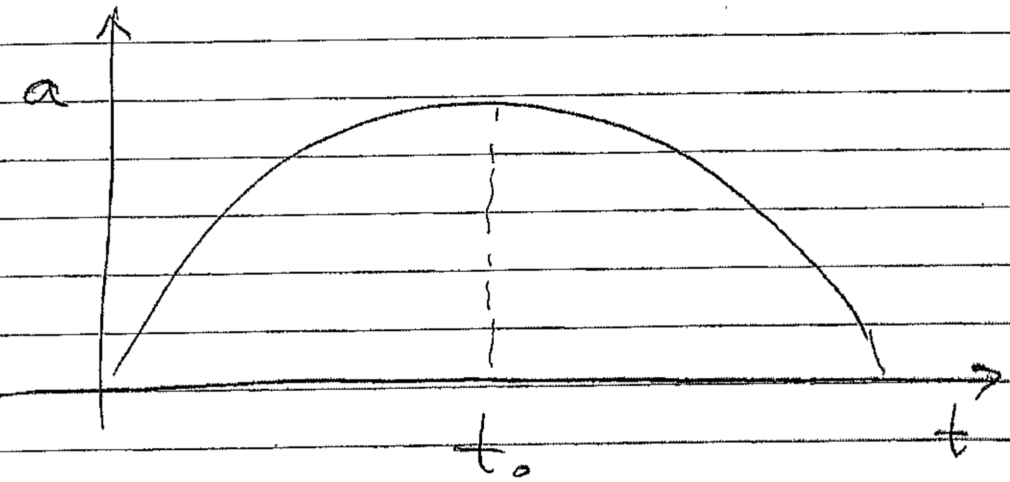
For $\Lambda = 0$ the FRW eq. becomes

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\rho - \frac{c^2}{R^2 a^2},$$

$\rho = \rho_0 / a^3$. If at $t = t_0$ $\frac{8\pi G \rho}{3} > \frac{c^2}{R^2 a^2(t_0)}$ only slightly, the subsequent expansion will lead to $\dot{a} = 0$. Since $2\dot{a}\ddot{a} =$

$= -\frac{8\pi G}{3}\rho_0 \frac{1}{a^2}$ and $\dot{a} > 0$ during expansion,

$\ddot{a} < 0 \Rightarrow \dot{a} = 0$ is a maximum. Then the Universe collapses during the time $\sim t_0$, so the lifetime of such a Universe $\sim 2t_0$:



Photon's radial geodesic:

$$c^2 dt^2 = a^2 \frac{dr^2}{1 - r^2/R^2}$$

$$\Rightarrow \quad c \int_0^t \frac{dt}{a} = \int_{r_0}^{r(t)} \frac{dr}{\sqrt{1 - r^2/R^2}} = D_c$$

With $x = r/R$, $\quad R \int_{r_0/R}^{r(t)/R} \frac{dX}{\sqrt{1-X^2}} = D_c$

$$\int \frac{dx}{\sqrt{1-X^2}} = \int dz \quad \text{with } \cancel{z}\ X = \sin z,$$
$$dX = \cos z \, dz$$

$$\Rightarrow \int \frac{dX}{\sqrt{1-X^2}} = \arcsin X + \text{const}$$

Let $r_0 = 0$, then $R \arcsin \frac{r}{R} = D_c$,

$$r(t) = R \sin D_c/R \quad \Rightarrow \quad r_{max} = R.$$

Now consider $\Lambda \neq 0$. Einstein's eqs. with $\Lambda \neq 0$, $\quad R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$, with the FRW ansatz $ds^2 = -c^2 dt^2 + a^2(t)\left(dr^2 + r^2 d\Omega^2\right)$ for the metric and $T_{\mu\nu} = \frac{P}{c^2} g_{\mu\nu} + \left(\rho + P/c^2\right) u_\mu u_\nu$ of a perfect fluid give the system of ODEs:

$$\frac{\dot a^2}{a^2} = \frac{8\pi G}{3}\rho + \frac{c^2\Lambda}{3} - \frac{kc^2}{a^2}$$

$$2a\ddot a + \dot a^2 + kc^2 = -\frac{8\pi G}{c^2} P a^2 + c^2 \Lambda a^2$$

$$\dot\rho + 3\frac{\dot a}{a}\left(\rho + P/c^2\right) = 0$$

Only 2 of these eqs. are independent. The two indep. ODEs for 3 var. $(a, \rho, P)$ should be supplemented by the eq. of state for matter $P = P(\rho)$.

The first and the second eqs. can be combined to give $\frac{\ddot a}{a} = \frac{c^2\Lambda}{3} - \frac{4\pi G}{3}(\rho + P)$ sometimes known as Raychauduri eq.

In our case $P = 0$ (the matter is dust), $k = 1/R^2$. Static solution corresponds to

$\dot{a} = 0, \quad \ddot{a} = 0 \quad \Rightarrow \rho = \rho_E = \dfrac{c^2 \Lambda}{4\pi G}$

from Raychauduri eq. and

$a^2 = \dfrac{1}{R^2 \Lambda}$ (from the FRW eq)

The solution is known as Einstein static universe. With $P = 0$, the Raychauduri eq. can be written as

$$\ddot{a} + \frac{4\pi G}{3}(\rho - \rho_E) a = 0$$

For $\rho > \rho_E$ $\ddot{a}/a < 0 \Rightarrow$ oscillatory solution, for $\rho < \rho_E$ - monotonic, any small fluctuation from $\rho = \rho_E$ leads to qualitatively different behavior $\Rightarrow$ suspect instability of the static solution.

4. $$\frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3}\left(\rho_M + \rho_R\right)$$

$$\rho_M = \rho_{0M}\, a_0^3 / a^3 \qquad \rho_R = \rho_{0R}\, a_0^4 / a^4$$

The FRW eq. can be written in the form

$$\frac{\dot{a}^2}{a^2} = H_0^2\left(\frac{\Omega_M}{a^3} + \frac{\Omega_R}{a^4}\right) \qquad (*)$$

with obvious identifications.

In the radiation-dominated Universe:

$$\frac{\dot{a}^2}{a^2} = H_0^2\,\frac{\Omega_R}{a^4}$$

$$\Rightarrow a_R(t) = (2H_0)^{1/2}\,\Omega_R^{1/4}\left(t - t_{*R}\right)^{1/2}$$

In the matter-dominated one:

$$\frac{\dot{a}^2}{a^2} = H_0^2\,\frac{\Omega_M}{a^3}$$

$$a_M(t) = \left(\frac{3H_0}{2}\right)^{2/3}\Omega_M^{1/3}\left(t - t_{*M}\right)^{2/3}.$$

Here $t_{*R}$ and $t_{*M}$ are the integration constants. Requiring that $a_M(t_0) = 1$, we find $t_{*M} = t_0 - \frac{2}{3H_0 \Omega_M^{1/2}}$

$$\Rightarrow a_M(t) = \left(\frac{3H_0}{2}\right)^{2/3} \Omega_M^{1/3} \left(t - t_0 + \frac{2}{3H_0 \Omega_M^{1/2}}\right)^{2/3}.$$

At $t = t_{eq}$ we have $\frac{\Omega_M}{a_{eq}^3} = \frac{\Omega_R}{a_{eq}^4}$

$$\Rightarrow a_{eq} = a(t_{eq}) = \Omega_R / \Omega_M .$$

$$a_R(t_{eq}) = \frac{\Omega_R}{\Omega_M} = (2H_0)^{1/2} \Omega_R^{1/4} (t_{eq} - t_{*R})^{1/2}$$

$$\Rightarrow t_{*R} = t_{eq} - \frac{1}{2} \frac{\Omega_R^{3/2}}{\Omega_M^2 H_0}$$

$$\Rightarrow a_R(t) = (2H_0)^{1/2} \Omega_R^{1/4} \left(t - t_{eq} + \frac{\Omega_R^{3/2}}{2H_0 \Omega_M^2}\right)^{1/2}$$

The condition $a_M(t_{eq}) = a_R(t_{eq})$ relates $t_{eq}$ and $t_0$.

<u>Remark</u>: obviously, this is a crude approximation – the actual smooth solution obeying $a(t_0) = 1$, $a(0) = 0$ can be found by integrating the eq. $(*)$ exactly.

The horizon $d_H = c a_R(t) \int_{t_{*R}}^{t} \frac{dt}{a_R(t)} =$

$$= c\left(t - t_{*R}\right)^{1/2} \int_{t_{*R}}^{t} \frac{dt}{\left(t - t_{*R}\right)^{1/2}} =$$

$= 2c\left(t - t_{*R}\right)$ can be related to redshift

using $1+z = 1/a$ : $a^2 = (1+z)^{-2} =$

$$= 2H_0 \Omega_R^{1/2}\left(t - t_{*R}\right) = \frac{H_0}{c} \Omega_R^{1/2} d_H$$

$$\Rightarrow d_H = \frac{c}{H_0 (1+z)^2 \Omega_R^{1/2}} = \frac{c}{H_0} \frac{\sqrt{1+z_{eq}}}{(1+z)^2 \Omega_M^{1/2}},$$

since $a_{eq} = \frac{\Omega_R}{\Omega_M} = \frac{1}{1+z_{eq}}$.

Since here $\Omega_M + \Omega_R = 1$ and $\Omega_R \ll 1$,

we have $d_H = \frac{c}{H_0} \frac{\sqrt{1+z_{eq}}}{(1+z)^2}$.

In equilibrium, the number density of photons is found by integrating the Planck distribution

$$dn_\omega = \frac{1}{\pi^2 c^3} \frac{\omega^2 d\omega}{e^{\hbar\omega/kT} - 1}$$

over all frequencies: $n_\gamma = \int_0^\infty dn_\omega =$

$$= \frac{2\zeta(3)}{\pi^2} \left(\frac{kT}{\hbar c}\right)^3 \approx 0.244 \left(\frac{kT}{\hbar c}\right)^3.$$

Assuming that the number of photons in the Universe is approximately conserved (after recombination), we have $n_\gamma^0 a_0^3 =$ $= n_\gamma a^3 \Rightarrow n_\gamma \simeq 0.244 \left(\frac{kT_0}{\hbar c}\right)^3 \frac{1}{a^3} \approx$

$$\approx \frac{4.1 \cdot 10^8}{a^3} m^{-3}.$$

After recombination, photons are not in thermal equilibrium with matter in the expanding Universe but the shape of the Planckian spectrum is preserved since $h\nu = h\nu_0 (1+z)$ and

$$dn_\nu = \frac{8\pi}{c^3} \frac{\nu^2 d\nu}{e^{h\nu/kT} - 1} = \frac{8\pi}{c^3} \frac{\nu_0^2 d\nu_0 (1+z)^3}{e^{h\nu_0(1+z)/kT} - 1}$$

$$\Rightarrow n_\gamma \sim \frac{1}{a^3} \sim (1+z)^3 \quad \text{and}$$

$$\frac{1+z}{T} = \frac{1}{T_0} \Rightarrow aT = a_0 T_0 = \text{const.}$$

$$\Rightarrow T \sim 1/a.$$

Thus, if photons were in thermal equilibrium in the early Universe at the time of recombination, we should observe (and we do) a blackbody-type spectrum of relic photons now at $T_0 = T/(1+z) \simeq 2.73\,K$.

To have thermal equilibrium in the early Universe, the mean free time $\tau_{mft} \sim \frac{1}{n\sigma c}$ in the system should be much shorter than the characteristic expansion rate $\tau \sim \sim \frac{1}{H} \sim \frac{a}{\dot{a}}$. For $a \sim t^{1/2}$, this implies

$\frac{t^{3/2}}{\sigma(T)} \ll t$ which is true for sufficiently small $t$ provided $\sigma(T)$ is not too small.

Although $\sigma(T)$ can be indeed small for superhigh $T$ due to asymptotic freedom, for moderate energy ($\sim 1$ MeV) this is not so $\Rightarrow$ can have thermal eq. in hot $T \sim 1/a$ and dense $n \sim 1/a^3$ medium.

$$N_\gamma = \frac{4}{3}\pi d_H^3 \cdot n_\gamma ,$$

where $n_\gamma = 0.244 \left(\frac{kT_0}{\hbar c}\right)^3 \frac{1}{a^3}$ and

$$d_H = \frac{c}{H_0} \frac{\sqrt{1+z_{eq}}}{(1+z)^2} \text{ as computed above.}$$

$$\Rightarrow N_\gamma = \bar{c} \left(\frac{kT_0}{\hbar c}\right)^3 \left(\frac{c}{H_0}\right)^3 \frac{(1+z_{eq})^{3/2}}{(1+z)^3} = N_{\gamma,0} \frac{1}{(1+z)^3}.$$

Clearly, $N_\gamma \to 0$ for $z \to \infty$.

$N_\gamma = 1$ for $1 + z_* = N_{\gamma,0}^{1/3}$. If there is only one „photon" per horizon volume, „photons" cannot interact $\Rightarrow$ cannot come to equilibrium (here „photon" stands for any rel. particle with $v \sim c$) in the early Universe.